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Dirichlet series from automorphic forms

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Even before Riemann's observations on $\zeta(s)$ (1858), and certainly after Hecke's work on L-functions of number fields and L-functions attached to holomorphic modular forms (1920's-1930's) broadened the scope of the idea, it has been understood that at least *holomorphic* modular forms give rise to *Dirichlet series*

$$D(s) = \sum_{n=1}^{+\infty} \frac{c_n}{n^s} \qquad \text{(for } s \in \mathbb{C}\text{, coefficients } c_n \in \mathbb{C}\text{)}$$

with *meromorphic continuations* and *functional equations*, via integrals (*Mellin transforms*)

$$Mf(s) = \int_0^\infty f(y)\, y^s \, \frac{dy}{y}$$

The Euler integral

$$\Gamma(s) = \int_0^\infty e^{-y}\, y^s \, \frac{dy}{y}$$

for the gamma function $\Gamma(s)$ yields [1] the useful identity

$$\Gamma(s)\, \frac{1}{n^s} = \int_0^\infty e^{-ny}\, y^s \, \frac{dy}{y}$$

Applying this to a sum

$$f(y) = \sum_{n=1}^{\infty} c_n\, e^{-2\pi n y}$$

gives (assuming convergence)

$$(2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{c_n}{n^s} = \int_0^\infty e^{-2\pi n y}\, y^s \, \frac{dy}{y} = \int_0^\infty f(y)\, y^s \, \frac{dy}{y}$$

It was only with Maaß' work in the late 1940's and 1950's that a perfectly analogous connection between *waveforms* and Dirichlet series came to light. (Mellin transforms are Fourier transforms in different coordinates.)

This integral trick assumes greater significance when the function f is known to have strong decay properties both at 0 and at ∞, since then the Mellin transform is *entire* in s. One way to ensure such rapid decay is via eigenfunction properties in the context of automorphic forms. [2]

- The archetype Mellin transform: zeta from theta
- Abstracting to holomorphic modular forms
- Variation: waveforms
- Appendix: proofs of Poisson summation

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[1] The identity follows by changing variables, replacing y by ny.

[2] Further, *Converse Theorems*, developed by Hecke, Weil, Jacquet-Langlands, Piatetski-Shapiro, and many others, assert very roughly that *any* Dirichlet series (perhaps with Euler product) with meromorphic continuation and functional equation comes from an automorphic form of some sort. This very naive claim (roughly Hecke's form) is false outside the very simplest cases, as Weil already illustrated by his converse theorem, whose hypothesis needs the meromorphic continuation of a larger family of so-called *twists* of the original Dirichlet series. Consideration of much larger *families* of twists play a role in all modern converse theorems.

1. *The archetype Mellin transform: zeta from theta*

The first instance of proof of analytic properties of Dirichlet series

$$\text{Dirichlet series } = \sum_{n\geq 1} \frac{c_n}{n^s}$$

from properties of modular forms came at latest [3] with Riemann's 1858 discussion of the connection between the complex zeros of $\zeta(s)$ and the error term in a (at that time *putative*) prime number theorem. [4] Jacobi's **theta function** [5] is

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad\qquad (z \in \mathfrak{H})$$

The connection between $\theta(z)$ and $\zeta(s) = \sum_n \frac{1}{n^s}$ is the following integral representation, [6] but the impact of the relation is the *meromorphic continuation* and *functional equation* proven just after.

[1.0.1] Claim: For $\mathrm{Re}(s) > 1$

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \int_0^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y}$$

Proof: Starting from the integral, compute directly

$$\int_0^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y} = \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y}\, y^{s/2}\, \frac{dy}{y} = \sum_{n\geq 1}\int_0^\infty e^{-\pi n^2 y}\, y^{s/2}\, \frac{dy}{y}$$

$$= \pi^{-s/2}\sum_{n\geq 1}\frac{1}{n^s}\int_0^\infty e^{-y}\, y^{s/2}\,\frac{dy}{y} = \pi^{-s/2}\Gamma(\frac{s}{2})\sum_{n\geq 1}\frac{1}{n^s} = \pi^{-s/2}\Gamma(\frac{s}{2})\,\zeta(s)$$

by replacing y by $y/(\pi n^2)$. ///

[1.0.2] Remark: The leading factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ should *not* be construed as objectionable in any way, but, rather, as something that really does *belong* with $\zeta(s)$. The $\pi^{-s/2}\Gamma(\frac{s}{2})$ is called the **gamma factor** for $\zeta(s)$. In the context of the *Euler product* [7]

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}}$$

[3] The integral representation of ζ in terms of Jacobi's theta function was apparently known earlier than Riemann, and Riemann's innovation was to emphasize the role of the complex zeros of ζ.

[4] Forms of the basic Prime Number Theorem were conjectured by Euler, Legendre, Gauss, and others. The assertion is that the number $\pi(x)$ of primes less than a real number x is asymptotic to $x/\log x$, meaning that $\lim_{x\to\infty}\pi(x)/(\frac{x}{\log x}) = 1$. This was proven in 1896 by Hadamard and by de la Vallée Poussin, independently.

[5] The term *theta series* in a classical setting refers approximately to Fourier series involving squares and sums of squares, for example.

[6] The choice to use $y^{s/2}$ in the integral, as opposed to the possibly more natural y^s, is a convenience that one would discover after doing the computation first with y^s, and seeing that $\zeta(2s)$ came out.

[7] This Euler product expansion of $\zeta(s)$ follows from unique factorization in $\mathbb{Z}$, and expansion of expressions $1/(1-p^{-s})$ in geometric series $1+p^{-s}+p^{-2s}+p^{-3s}+\ldots$.

the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime* [8] ∞.

The proof of the following archetypical result is also an archetype.

[1.0.3] Theorem: The function $\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$ has a meromorphic continuation to $s \in \mathbb{C}$, with poles only at $s = 0$ and $s = 1$, which are *simple*. There is the functional equation

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \pi^{-(1-s)/2}\,\Gamma(\frac{1-s}{2})\,\zeta(1-s) \qquad \text{(invariance under } s \leftrightarrow 1-s\text{)}$$

[1.0.4] Remark: It is useful to know that the gamma function $\Gamma(s)$ has no zeros which might mask poles of $\zeta(s)$.

Proof: Use the integral representation

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \int_0^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y}$$

Observe that as $y \to +\infty$ the function $\frac{\theta(iy)-1}{2}$ is of rapid decay, by the estimate

$$\frac{\theta(iy)-1}{2} = \sum_{n\ge 1} e^{-\pi n^2 y} \le e^{-\pi y/2} \sum_{n\ge 1} e^{-\pi n^2/2} \qquad \text{(for } y \ge 1\text{)}$$

Thus, since the integral from 1 (not 0) to $+\infty$ is nicely convergent for *all* values of s,

$$\int_1^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y} = \text{entire in } s$$

The main trick (known to Riemann and before) is to convert the other part of the integral, from 0 to 1, into a similar integral from 1 to $+\infty$, up to more elementary terms admitting direct analysis. [9] This conversion will follow from the *Jacobi identity*

$$\theta(iy) = y^{-1/2} \cdot \theta(-1/iy)$$

proven below, and by making the change of variables $y \to 1/y$. The minor book-keeping complication is that the function whose Mellin transform we have is not exactly $\theta(z)$. Thus, we must do a little computation

$$\frac{\theta(-1/iy)-1}{2} = \frac{y^{1/2}\theta(y)-1}{2} = y^{1/2}\frac{\theta(y)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}$$

Then

$$\int_0^1 \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y} = \int_1^\infty \frac{\theta(-1/iy)-1}{2}\, y^{-s/2}\, \frac{dy}{y} = \int_1^\infty \Big(y^{1/2}\frac{\theta(y)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}\Big)\, y^{-s/2}\, \frac{dy}{y}$$

$$= \int_1^\infty \frac{\theta(y)-1}{2}\, y^{-s/2}\, \frac{dy}{y} + \int_1^\infty \Big(\frac{y^{(1-s)/2}}{2} - \frac{y^{-s/2}}{2}\Big)\, \frac{dy}{y} = \int_1^\infty \frac{\theta(y)-1}{2}\, y^{-s/2}\, \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

$$= (\text{entire}) + \frac{1}{s-1} - \frac{1}{s}$$

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[8] As remarked upon earlier, an insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime ∞ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes p and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* ∞. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there's little point in fretting about this.

[9] It is not obvious that $\frac{\theta(iy)-1}{2}$ has any property that would ensure this. However, in the context of 1850 properties of functions akin to $\theta(z)$ had been much studied by Jacobi and others, so such possibilities would have been known to Riemann and others.

The elementary expressions $1/(s-1)$ and $1/s$ certainly have meromorphic continuations to $\mathbb{C}$, with explicit poles. Thus, together with the first integral from 1 to ∞, we have

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \int_1^\infty \frac{\theta(iy)-1}{2}\,(y^{s/2}+y^{(1-s)/2})\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

The integral on the right-hand side is entire, and the two elementary summands have obvious extensions. This gives the meromorphic continuation of $\pi^{-s/2}\,\Gamma\frac{s}{2})\,\zeta(s)$. Further, the right-hand side is visibly symmetrical under $s \to 1-s$, which gives the functional equation. ///

[1.0.5] Remark: Attempting to avoid the gamma factor leads to an unsymmetrical and unenlightening form of the functional equation.

[1.0.6] Remark: The fact that $\Gamma(s/2)$ has no zeros assures that the meromorphic continuation of $\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$ masks no poles of $\zeta(s)$.

Now we prove Jacobi's functional equation for $\theta(z)$

[1.0.7] Claim: $\theta(-1/iy) = y^{1/2}\cdot\theta(iy)$

Proof: This underlying symmetry itself follows from a more fundamental fact, the *Poisson summation formula*[10]

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad \text{(for Schwartz functions } f\text{, with Fourier transform } \widehat{f}\text{)}$$

[... iou ...]

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[1.0.8] Remark: Meromorphic continuation and functional equations for Dirichlet L-functions are readily contrived in analogy to the above. However, for more general *number fields*,[11] a proof in this style requires much more effort. Hecke accomplished this, but the Tate-Iwasawa *adelic* reformulation removes many illusory difficulties, allowing the general case to be no harder than the simplest.

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2. *Abstracting to holomorphic modular forms*

The argument using $\theta(z)$ to prove the meromorphic continuation and functional equation of $\zeta(s)$ only used a few *qualitative* features of $\theta(z)$, which admit considerable abstraction. Here we take the first obvious steps.

Consider holomorphic functions f on the upper half-plane which are *periodic* in $x = \mathrm{Re}(z)$, namely

$$f(z+1) = f(z)$$

Thus, f has a Fourier expansion in $x = \mathrm{Re}(z)$, whose coefficients are functions of $y = \mathrm{Im}(z)$

$$f(x+iy) = \sum_{n\in\mathbb{Z}} c_n(y)\,e^{2\pi inx}$$

[10] There are at least two different-looking proofs for the Poisson summation formula. One often given reduces the assertion to the convergence of Fourier series. There is also a more direct proof using tempered distributions. Both of these are given in the appendix.

[11] By definition, a *number field* k is a finite field extension k of $\mathbb{Q}$. The *algebraic integers* $\mathfrak{o}$ of k consist of all $\alpha \in k$ satisfying a monic equation $f(\alpha) = 0$ with *integer* coefficients.

One may intuit that the Fourier expansion of a *holomorphic* function should have the whole variable z in the exponent. In any case, the Cauchy-Riemann equation [12]

$$(\partial x + i\partial y)\,\big(c_n(y)\,e^{2\pi i n x}\big) = 0$$

gives a differential equation for $c_n(y)$

$$2\pi n \cdot c_n(y) + c_n'(y) = 0$$

Up to constant multiples, this first-order linear constant-coefficient equation has a unique solution $c_n(y) = e^{-2\pi n y}$. Thus, for numerical constants c_n, we have a *complex* Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n z}$$

Next, we impose the moderate-growth condition

$$|f(x+iy)| \le C\cdot y^N \qquad \text{(for some } C,\ N, \text{ as } y\to+\infty, \text{ for all } x\in\mathbb{R})$$

[2.0.1] Claim: The moderate-growth condition implies that negative-index Fourier coefficients are 0.

Proof: The usual integral formula for Fourier coefficients gives

$$|c_n\, e^{-2\pi n y}| = \Big|\int_0^1 e^{-2\pi i n x}\, f(x+iy)\,dx\Big| \le \int_0^1 |f(x+iy)|\,dx \le \int_0^1 C\cdot y^N\,dx = C\cdot y^N$$

Since for $n<0$ exponentials $e^{-2\pi n y}$ blow up for $y\to+\infty$, it must be that $c_n = 0$ for $n<0$. ///

Last, we require a symmetry with respect to the inversion $z\to -1/z$ similar to that satisfied by $\theta(z)$, namely [13]

$$f(-1/z) = z^k\cdot f(z) \qquad \text{(with positive integer } k \text{ (the } \textit{weight}\text{))}$$

Such f is a *holomorphic modular form*, a type of *automorphic form*. When $c_0 = 0$ then f is called a holomorphic **cuspform**.

[2.0.2] Theorem: For $f(z) = \sum_{n\ge 0} c_n\, e^{2\pi i n z}$ satisfying $f(-1/z) = z^k\, f(z)$ and not identically 0, necessarily $0 < k \in 2\mathbb{Z}$, and the associated Dirichlet series

$$D_f(s) = \sum_{n>0} \frac{c_n}{n^s}$$

has a meromorphic continuation and functional equation. Specifically, with a suitable gamma factor $(2\pi)^{-s}\Gamma(s)$,

$$(2\pi)^{-s}\,\Gamma(s)\,D_f(s) = (\text{entire}) + \frac{c_0}{s-k} - \frac{c_0}{s}$$

is multiplied by i^k under $s\to k-s$. The Dirichlet series is *entire* when f is a *cuspform*.

[12] The requirement that a function be complex-differentiable immediately implies that it is annihilated by this operator.

[13] The square root in the functional equation for $\theta(z)$ causes some difficulties that are not our point here. Also, in fact, the two elements $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ generate the whole group $SL_2(\mathbb{Z})$, but we do not use that fact here, and one should not rely upon ascertaining explicit generators for groups. While for certain delicate purposes it *is* useful to understand something about generators for congruence subgroups, and some results are known, such things will not enter our discussion.

Proof:

[... iou ...]

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3. *Variation: waveforms*

Mellin transforms of waveforms *also* give Dirichlet series with gamma factors, and with meromorphic continuation and functional equation. An temporary notion of *waveform* f sufficient for this discussion is that f is a $\Delta^{\mathfrak{H}}$-eigenfunction

$$\Delta^{\mathfrak{H}} f = \alpha(\alpha-1) \qquad \text{(for some complex } \alpha\text{)}$$

on the upper half-plane with the invariance property

$$f(z+1) = f(z)$$

And we do assume (as with holomorphic modular forms) that f is *of moderate growth*. The ramifications of the moderate growth hypothesis are less obvious for waveforms than for holomorphic modular forms. We grant for this discussion that moderate growth of such a waveform implies [14] that the waveform is expressible as

$$f(x+iy) = ay^s + by^{1-s} + \sum_{0\neq n\in\mathbb{Z}} c_n\, \varphi_1(|n|y)\, e^{2\pi inx}$$

with φ_1 a slightly renormalized version of the earlier φ_1, and with the earlier s replaced by α, namely

$$\varphi_1(y) = 2\pi^{-\alpha}\Gamma(\alpha)\int_{\mathbb{R}} e^{-2\pi int}\,\frac{y^\alpha}{(x^2+y^2)^\alpha}\,dt = y^{\frac{1}{2}}\int_0^\infty e^{-\pi(\frac{1}{t}+t)y}\,t^{\alpha-\frac{1}{2}}\,\frac{dt}{t}$$

We suppose that f is a *cuspform*, meaning that the 0^{th} Fourier coefficient $ay^s + by^{1-w}$ is just 0. We also assume the invariance [15]

$$f(-1/z) = f(z)$$

Finally, a less critical but convenient assumption is that

$$f(x-iy) = f(x+iy)$$

This assures that $c_{-n} = c_n$, by uniqueness of Fourier expansions, comparing the Fourier expansions of the two sides of $f(x-iy) = f(x+iy)$.

[14] As discussed earlier, to *prove* that all the Fourier components are as indicated uses the fact that, up to constants, $\varphi_1(y)$ is the only moderate-growth solution of the differential equation arising from the eigenfunction condition for the n^{th} Fourier component:

$$\Delta^{\mathfrak{H}}(u(y)\,e^{2\pi inx}) = \alpha(\alpha-1)\cdot u(y)\,e^{2\pi inx}$$

[15] As in the discussion of *holomorphic* modular forms, the fact that the inversion $z \to -1/z$ and translation $z \to z+1$ generate $SL_2(\mathbb{Z})$ is not used.

[3.0.1] Theorem: For waveform $f(z) = \sum_{\neq n \in \mathbb{Z}} c_n \, \varphi_1(|n|y) \, e^{2\pi i n x}$ with eigenvalue $\alpha(\alpha-1)$ and $f(x-y) = f(x+iy)$ the associated Dirichlet series

$$D_f(s) = \sum_{n>0} \frac{c_n \sqrt{|n|}}{n^s}$$

has a meromorphic continuation and functional equation. Specifically, adding a suitable gamma factor we have

$$\pi^{-s} \, \Gamma\left(\frac{s+\alpha-\frac{1}{2}}{2}\right) \Gamma\left(\frac{s-\alpha+\frac{1}{2}}{2}\right) D_f(s) = \text{entire}$$

is invariant under $s \to 1-s$. The Dirichlet series is *entire.*

[3.0.2] Remark: The insertion of the $\sqrt{|n|}$ has the immediate effect of making the functional equation correspond to $s \to 1-s$ rather than to $s \to -s$ as it would be without the $\sqrt{|n|}$. Less superficially, with hindsight, the renormalized coefficients $c_n\sqrt{|n|}$ behave better than the original c_n's. For this and other reasons one must be careful about varying normalizations in the Fourier expansions of waveforms.

Proof: With hindsight, we use a Mellin transform with $y^{s-\frac{1}{2}}$ rather than y^s. Also, since $c_{-n} = c_n$, we could divide by 2 to avoid a factor of 2 in the result, but, in fact, a convenient 2 arises anyway, so we don't need this extra division by 2. Otherwise compute directly. First, we claim that, for $\mathrm{Re}(s)$ large enough to assure convergence,

$$\int_0^\infty y^{s-\frac{1}{2}} \, f(iy) \, \frac{dy}{y} = \pi^{-s} \, \Gamma\left(\frac{s+\alpha-\frac{1}{2}}{2}\right) \Gamma\left(\frac{s-\alpha+\frac{1}{2}}{2}\right) D_f(s)$$

The first part of this is easy, assuming as we do that the Fourier expansion of f has non-zero Fourier coefficients all expressible in terms of the special function [16] φ_1, and that $c_{-n} = c_n$. Namely, assuming convergence,

$$\int_0^\infty y^{s-\frac{1}{2}} \, f(iy) \, \frac{dy}{y} = \int_0^\infty y^{s-\frac{1}{2}} \sum_{n\neq 0} c_n \varphi_1(|n|y) \, \frac{dy}{y} = 2\sum_{n>0} c_n \int_0^\infty y^{s-\frac{1}{2}} \, \varphi_1(ny) \, \frac{dy}{y}$$

$$= 2\sum_{n>0} \frac{c_n}{n^{s-\frac{1}{2}}} \left(\cdot \int_0^\infty y^{s-\frac{1}{2}} \, \varphi_1(y) \, \frac{dy}{y} \right)$$

At this point, we will be happy to show that

$$\int_0^\infty y^{s-\frac{1}{2}} \, \varphi_1(y) \, \frac{dy}{y} = \frac{1}{2} \, \pi^{-s} \, \Gamma\left(\frac{s+\alpha-\frac{1}{2}}{2}\right), \Gamma\left(\frac{s-\alpha+\frac{1}{2}}{2}\right)$$

which will yield the desired outcome. To this end, use the modifed integral expression for φ_1, namely

$$\varphi_1(y) = y^{\frac{1}{2}} \int_0^\infty e^{-\pi(\frac{1}{t}+t)y} \, t^{\alpha-\frac{1}{2}} \, \frac{dt}{t}$$

to obtain

$$\int_0^\infty y^{s-\frac{1}{2}} \, \varphi_1(y) \, \frac{dy}{y} = \int_0^\infty \int_0^\infty y^s \, t^{\alpha-\frac{1}{2}} \, e^{-\pi(t+\frac{1}{t}y)} \, \frac{dy}{y} \, \frac{dt}{t}$$

[16] Again, modulo some elementary features which are not of interest to us, since we make no use of classical formulaic facts, this φ_1 is essentially a *Bessel* function, and/or a *Whittaker function.* The latter terminology embraces a larger family of classical functions, and for this and other reasons is a more convenient catch-phrase.

(by interchanging order of integration). Replacing y by $y \cdot (t/\pi)$ yields

$$\pi^{-s} \int_0^\infty \int_0^\infty y^s \, t^{s+\alpha-\frac{1}{2}} \, e^{-t^2 y} \, e^{-y} \, \frac{dy}{y} \, \frac{dt}{t}$$

Replacing t by $\sqrt{t}$ turns this into

$$\frac{1}{2} \, \pi^{-s} \int_0^\infty \int_0^\infty y^s \, t^{\frac{s+\alpha-\frac{1}{2}}{2}} \, e^{-ty} \, e^{-y} \, \frac{dy}{y} \, \frac{dt}{t}$$

Then replace t by t/y to obtain

$$\frac{1}{2} \, \pi^{-s} \int_0^\infty \int_0^\infty y^{\frac{s-\alpha+\frac{1}{2}}{2}} \, t^{\frac{s+\alpha-\frac{1}{2}}{2}} \, e^{-t} \, e^{-y} \, \frac{dy}{y} \, \frac{dt}{t} = \frac{1}{2} \, \pi^{-s} \, \Gamma\left(\frac{s+\alpha-\frac{1}{2}}{2}\right), \Gamma\left(\frac{s-\alpha+\frac{1}{2}}{2}\right)$$

as claimed. [17]

Having given an integral representation of the Dirichlet series (with gamma factor) in terms of f, we use the invariance $f(i/y) = f(iy)$ to prove the analytic continuation and functional equation.

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4. *Appendix: proofs of Poisson summation*

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[17] As often in these arguments, at first the equality holds only where things converge nicely, but then for all values by the identity principle from complex analysis.