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01. Euler and $\zeta(s)$

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[0.1] Summing series

Infinite sums that *telescope* can be understood in simpler terms. The archetype is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots \;=\; \Big(\frac{1}{1}-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\Big(\frac{1}{3}-\frac{1}{4}\Big)+\ldots \;=\; 1$$

Other subtler examples analyze-able via trigonometric functions give more interesting answers, but these are still elementary, the archetype being [1]

$$\frac{1}{1}-\frac{1}{3}-\frac{1}{5}+\ldots \;=\; \Big(\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\Big)\Big|_{x=1} \;=\; \int_0^x \frac{dt}{1+t^2}\;\Big|_{x=1} \;=\; \arctan 1 \;=\; \frac{\pi}{4}$$

As late as 1735, no one knew how to express in simpler terms

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$$

This was the *Basel problem*, after the town in Switzerland home to the Bernoullis, collectively a dominant force in European mathematics at the time.

L. Euler solved the problem, winning him useful notoriety at an early age, but more notably introducing larger ideas, as follows.

Given non-zero numbers $a_1,\ldots,a_n$, a polynomial with constant term 1 having these numbers as roots is

$$\Big(1-\frac{x}{a_1}\Big)\Big(1-\frac{x}{a_2}\Big)\ldots\Big(1-\frac{x}{a_n}\Big) \;=\; 1-(\frac{1}{a_1}\ldots+\frac{1}{a_n})x+(\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}+\ldots)x^2+\ldots+(-1)^n\frac{x^n}{a_1\ldots a_n}$$

Imagine that $\frac{\sin \pi x}{\pi x}$ has an analogous product expansion, in terms of its zeros at $\pm1,\pm2,\pm3,\ldots$, up to a normalizing constant needing determination: using the power series expansion of $\sin x$,

$$\frac{(\pi x)-\frac{(\pi x)^3}{3!}+\ldots}{\pi x} \;=\; \frac{\sin \pi x}{\pi x} \;=\; C\cdot\prod_{n=1}^{\infty}\big(1-\frac{x^2}{n^2}\big) \;=\; C\cdot\Big(x-x^3\sum_n\frac{1}{n^2}+\ldots\Big)$$

Assuming this works, equating constant terms gives $C=1$, and equating coefficients of x^2 gives

$$\frac{\pi^2}{6} \;=\; \sum\frac{1}{n^2}$$

Slightly messier manipulations yield all values $\sum\frac{1}{n^{2k}}$.

Euler did not manage to prove that the sine function had such a product expansion for some years. Nevertheless, *just this heuristic*, without the eventual proof, was what won him considerable notoriety, in part because it suggested an *underlying mechanism*. Further, everyone believed the heuristic because, the *numerical* plausibility was easy to check, once observed.

[1] There is some delicacy about convergence, but this is secondary.

[0.2] The Euler product

In effect, the Basel problem was about explaining the values $\zeta(2), \zeta(4), \ldots$, of the simplest *zeta function*

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad\qquad \text{(real } s\text{?, complex } s\text{?)}$$

Euler made another striking observation about $\zeta(s)$: it has an *Euler product* factorization/expansion, coming from *unique factorization* in $\mathbb{Z}$:

$$\sum_n \frac{1}{n^s} \;=\; \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \frac{1}{12^s} + \ldots\ldots$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + \frac{1}{7^s} + \frac{1}{2^{3s}} + \frac{1}{3^{2s}} + \frac{1}{2^s \cdot 5^s} + \frac{1}{11^s} + \frac{1}{2^{2s} \cdot 3^s} + \ldots$$

$$= \Big(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \ldots\Big)\Big(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \ldots\Big)\Big(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \ldots\Big) \ldots \;=\; \prod_{\text{primes } p} \frac{1}{1 - \frac{1}{p^s}}$$

from factoring *uniquely* into prime powers and massively regrouping in terms of geometric series for each prime:

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots \;=\; \frac{1}{1 - \frac{1}{p^s}}$$

In short,

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{\text{primes } p} \frac{1}{1 - \frac{1}{p^s}}$$

That is, we have an expression involving *just primes* equated to an expression *not* overtly involving primes, suggesting the relevance of the zeta function to prime numbers. Whether or not we care greatly about prime numbers, the *connection* between primes and behavior of $\zeta(s)$ is striking.

[0.3] Infinitude of primes

The ancient Greek mathematicians proved the infinitude of primes *qualitatively*, but experimentation quickly suggests that the estimates on asymptotic density arising from that classical argument are laughably bad.

The first serious *qualitative* result about primes was Euler's, using the Euler product expansion of $\zeta(s)$, proving

$$\sum_{\text{all primes } p} \frac{1}{p} \;=\; \infty$$

Comparison with $\int_1^\infty x^{-s}\,dx$ for real $s > 1$ proves that $\zeta(s) \to +\infty$ as $s \to 1^+$:

$$\frac{1}{s-1} \;=\; \int_1^\infty x^{-s}\,dx \;\le\; \zeta(s) \qquad\qquad \text{(for real } s > 1\text{)}$$

On the other hand, from $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$,

$$\log \zeta(s) \;=\; \sum_p -\log\big(1 - \frac{1}{p^s}\big) \;=\; \sum_p \Big(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \ldots\Big)$$

The terms after the initial $1/p^s$ do not contribute to any blow-up as $s \to 1^+$, by crudely estimating primes by positive integers:

$$\sum_p \sum_{\ell \geq 2} \frac{1}{\ell p^{\ell s}} \;\leq\; \sum_{n\geq 2}\sum_{\ell \geq 2} \frac{1}{\ell n^{\ell s}} \;\leq\; \sum_{\ell \geq 2} \frac{1}{\ell} \int_1^\infty \frac{dx}{x^{\ell s}}$$

$$= \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell s - 1} \;\leq\; \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell - 1} \;<\; \infty \qquad\qquad \text{(uniformly for } s \geq 1)$$

Thus, for some finite constant C,

$$\log \frac{1}{s-1} \;<\; \log \zeta(s) \;\leq\; C \;+\; \sum_p \frac{1}{p^s} \qquad\qquad \text{(for } s > 1)$$

Thus, the blow-up of $\log \zeta(s)$ as $s \to 1^+$ forces the divergence [2] of $\sum_p 1/p$.

This argument by itself did not attempt quantify *how fast* $\sum_{p<x} \frac{1}{p}$ blows up as a function of $x \to \infty$, although a similar idea can make progress there.

A related question of at least whimsical interest is about the prime-counting function [3]

$$\pi(x) \;=\; \sum_{p' < x} 1$$

That is, the *weights* in the counting are not $1/p^s$, not $1/p$, but just 1. About 160 years after Euler's first work on zeta, in 1896 Hadamard and de la Vallée-Poussin (independently) proved

$$\pi(x) \;\sim\; \frac{x}{\log x} \qquad\qquad \text{(as } x \to +\infty)$$

meaning that $\pi(x)/\big(x/\log x\big) \to 1$ as $x \to +\infty$. The question of refining the *error term* in this asymptotic is very much open currently.

Riemann's 1859 paper showed that the connection between prime numbers and the zeta function is far deeper than the discussion above may suggest.

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[2] For real $s > 1$, certainly $\sum_p p^{-s} \leq \sum_p 1/p$, so the blow-up of the left-hand side as $s \to 1^+$ implies divergence of the right.

[3] Slightly unfortunate notation $\pi(x)$, but it is traditional.