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01b. Product expansion of $\sin x$

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Euler did eventually *prove* the product expansion

$$\sin \pi x \;=\; \pi x \cdot \prod_{n\ge 1} \Big(1 - \frac{x^2}{n^2}\Big)$$

The question does not mention complex numbers at all, but the simplest verification of this product expansion is a standard application of basic *complex analysis*, at the level of *Liouville's theorem* and *Laurent expansions* near poles, providing yet another powerful motivation for understanding basic complex analysis. [1]

[0.1] Partial fraction expansion of $\frac{\pi^2}{\sin^2 \pi x}$

We claim that there is a *partial fraction expansion*

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

To see this, first note that both sides have double poles exactly at integers. The Laurent expansion of the right-hand side near $n \in \mathbb{Z}$ begins

$$\frac{1}{(z-n)^2} + \text{holomorphic}$$

The left-hand side is periodic, so it suffices to see the Laurent expansion near 0:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{\pi^2}{\big(\pi z + \frac{(\pi z)^3}{3!} + \ldots\big)^2} \;=\; \frac{1}{z^2} \cdot \frac{1}{\big(1 - \frac{\pi^2 z^2}{3!} + \ldots\big)^2} \;=\; \frac{1}{z^2} \cdot \Big(1 + \frac{\pi^2 z^2}{3!} + \ldots\Big)^2 \;=\; \frac{1}{z^2} + \text{holomorphic}$$

This Laurent expansion matches that of the partial fraction expansion. Thus,

$$f(z) \;=\; \frac{\pi^2}{\sin^2 \pi z} \;-\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

has no poles in $\mathbb{C}$, so is *entire*. On the real line, after cancellation of poles, f *continuous*. The periodicity $f(z+1) = f(z)$ is visible, so f is *bounded* on the real line. In fact, since f is bounded on any region $\{x+iy : 0 \le x \le 1,\; |y| \le N\}$, the periodicity gives the boundedness of $f(z)$ on every band $|y| \le N$ containing $\mathbb{R}$.

Both parts of $f(z)$ go to 0 as $|y| \to \infty$. Thus, f is bounded and entire, so constant, by *Liouville's theorem.* Since $f(z) \to 0$ as $|y| \to \infty$, this constant is 0, giving the desired equality.

[0.2] Partial fraction expansion of $\pi \cot \pi x$

Next, regroup slightly, to improve convergence:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{1}{z^2} + \sum_{n\ge 1} \Big(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}\Big)$$

[1] Weierstraß and Hadamard product expansions apply to general *entire* functions, but with more overhead than needed here.

The left-hand side is the derivative of $-\pi \cot \pi z$, and with the improved convergence the right-hand side is the obvious termwise derivative, so up to a constant C,

$$\pi \cot \pi z \;=\; C + \frac{1}{z} + \sum_{n \geq 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

The identity

$$\frac{1}{z-n} + \frac{1}{z+n} \;=\; \frac{(z+n)+(z-n)}{z^2-n^2} \;=\; \frac{2z}{z^2-n^2}$$

certifies that convergence is uniform and absolute, *and* that the summands are *odd* functions of z. Everything but the constant C is *odd* as a function of z, so $C = 0$. Thus,

$$\pi \cot \pi z \;=\; \frac{1}{z} + \sum_{n \geq 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

[0.3] Product expansion of $\sin \pi x$

Also, $\pi \cot \pi z$ is the logarithmic derivative of $\sin \pi z$:

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{\pi \cos \pi z}{\sin \pi z} \;=\; \pi \cot \pi z$$

Thus,

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{1}{z} + \sum_{n \geq 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

We intend to integrate. First, anticipating our goal, note that

$$\frac{d}{dz} \log \big(1 - \frac{z}{n}\big) \;=\; \frac{-\frac{1}{n}}{1 - \frac{z}{n}} \;=\; \frac{1}{z-n}$$

Thus, integrating, for some constant C,

$$\log(\sin \pi z) \;=\; C + \log z + \sum_{n \geq 1} \Big(\log \big(1 - \frac{z}{n}\big) + \log \big(1 - \frac{z}{n}\big) \Big) \;=\; C + \log z + \sum_{n \geq 1} \log \Big(1 - \frac{z^2}{n^2}\Big)$$

Exponentiating,

$$\sin \pi z \;=\; e^C \cdot z \cdot \prod_{n \geq 1} \Big(1 - \frac{z^2}{n^2}\Big)$$

Looking at the power series at $z = 0$, we see that $e^C = \pi$, so

$$\sin \pi z \;=\; \pi z \cdot \prod_{n \geq 1} \Big(1 - \frac{z^2}{n^2}\Big)$$
