(*August 29, 2013*)

Weierstrass and Hadamard products

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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1. Weierstrass products
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Apart from factorization of polynomials, perhaps the oldest product expression is Euler's

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big)$$

Granting this, Euler equated the power series coefficients of z^2, evaluating $\zeta(2)$ for the first time:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The Γ-function factors:

$$\int_0^\infty e^{-t}\, t^z \, \frac{dt}{t} = \Gamma(z) = \frac{1}{z\, e^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-z/n}}$$

where the Euler-Mascheroni constant γ is essentially defined by this relation. The integral (Euler's) converges for $\mathrm{Re}(z) > 0$, while the product (Weierstrass') converges for all complex z except non-positive integers. Granting this, the Γ-function is visibly related to *sine* by

$$\frac{1}{\Gamma(z)\cdot\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big) = -\frac{z}{\pi} \cdot \sin \pi z$$

because the exponential factors are *linear*, and can *cancel*.

Linear exponential factors are exploited in Riemann's *explicit formula* [Riemann 1859], derived from equality of the *Euler product* and *Hadamard product* [Hadamard 1893] for the zeta function $\zeta(s) = \sum_n \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s) = \frac{e^{a+bs}}{s-1} \cdot \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{\rho s} \cdot \prod_{n=1}^{\infty} \Big(1 + \frac{s}{2n}\Big) e^{-s/2n}$$

where the product expansion of $\Gamma(\frac{s}{2})$ is visible, corresponding to *trivial zeros* of $\zeta(s)$ at negative even integers, and ρ ranges over all other, *non-trivial* zeros, known to be in the *critical strip* $0 < \mathrm{Re}(s) < 1$.

The hard part of the proof (below) of Hadamard's theorem is adapted from [Ahlfors 1953/1966], with various rearrangements. A somewhat different argument is in [Lang 1993]. We recall some standard (folkloric?) proofs of supporting facts about harmonic functions, starting from scratch.

1. *Weierstrass products*

Given a sequence of complex numbers z_j with no accumulation point in $\mathbb{C}$, we will construct an entire function with zeros exactly the z_j. This is essentially elementary.

[1.1] Basic construction

Taylor-MacLaurin polynomials of $-\log(1-z)$ will play a role: let

$$p_n(z) \;=\; z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^n}{n}$$

We will show that there is a sequence of integers n_j giving an *absolutely convergent infinite product* vanishing precisely at the z_j, with vanishing at $z = 0$ accommodated by a suitable leading factor z^m:

$$z^m \prod_j \Big(1 - \frac{z}{z_j}\Big)\, e^{p_{n_j}(z/z_j)} \;=\; z^m \prod_j \Big(1 - \frac{z}{z_j}\Big)\, \exp\Big(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^{n_j}}{n_j z_j^{n_j}}\Big)$$

Absolute convergence of $\sum_j \log(1 + a_j)$ implies absolute convergence of the infinite product $\prod_j (1 + a_j)$. Thus, we show that

$$\sum_j \Big|\log\Big(1 - \frac{z}{z_j}\Big) + p_{n_j}\Big(\frac{z}{z_j}\Big)\Big| \;<\; \infty$$

Fix a large radius R, keep $|z| < R$, and ignore the finitely-many z_j with $|z_j| < 2R$, so in the following we have $|z/z_j| < \frac{1}{2}$. Using the power series expansion of *log*,

$$\Big|\log(1 - \frac{z}{z_j}) - p_n(\frac{z}{z_j})\Big| \;\le\; \frac{1}{n+1}\cdot\Big|\frac{z}{z_j}\Big|^{n+1} + \frac{1}{n+2}\cdot\Big|\frac{z}{z_j}\Big|^{n+2} + \ldots \;\le\; \frac{1}{n+1}\cdot\frac{|z/z_j|^{n+1}}{1 - |z/z_j|} \;\le\; 2\cdot\frac{|z/z_j|^{n+1}}{n+1}$$

Thus, we want a sequence of positive integers n_j such that

$$\sum_{|z_j| \ge 2R} \frac{|z/z_j|^{n_j+1}}{n_j + 1} \;<\; \infty \qquad\qquad (\text{with } |z| < R)$$

Of course, the choice of n_j's must be compatible with enlarging R, but this is easily arranged. For example, $n_j = j - 1$ succeeds:

$$\sum_j \Big|\frac{z}{z_j}\Big|^j \;=\; \sum_{|z_j| < 2R} \Big|\frac{z}{z_j}\Big|^j + \sum_{|z_j| \ge 2R} \Big|\frac{z}{z_j}\Big|^j \;\le\; \sum_{|z_j| < 2R} \Big|\frac{z}{z_j}\Big|^j + \sum_j 2^{-j}$$

Since $\{z_j\}$ is discrete, the sum over $|z_j| < 2R$ is finite, so we have convergence, and convergence of the infinite product with $n_j = j$:

$$\prod_j \Big(1 - \frac{z}{z_j}\Big)\, e^{p_j(z/z_j)} \;=\; \prod_j \Big(1 - \frac{z}{z_j}\Big)\, \exp\Big(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^j}{jz^j}\Big)$$

[1.2] Canonical products and genus

Given entire f with zeros $z_j \neq$ and a zero of order m at 0, ratios

$$\varphi(z) \;=\; \frac{f(z)}{z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)}}$$

with *convergent* infinite products are *entire*, and *do not vanish*. Non-vanishing entire φ has an entire *logarithm*:

$$g(z) \;=\; \log \varphi(z) \;=\; \int_0^z \frac{\varphi'(\zeta)\, d\zeta}{\varphi(\zeta)}$$

Thus, non-vanishing entire φ is expressible as

$$\varphi(z) \;=\; e^{g(z)} \qquad\qquad (\text{with } g \textit{ entire})$$

Thus, the most general entire function with prescribed zeros is of the form

$$f(z) \;=\; e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)} \qquad\qquad (\text{with } g \text{ entire})$$

Naturally, with fixed f, altering the n_j necessitates a corresponding alteration in g.

We are most interested in zeros $\{z_j\}$ allowing a *uniform* integer h giving convergence of the infinite product in an expression

$$f(z) \;=\; e^{g(z)} \cdot z^m \cdot \prod_j \Big(1 - \frac{z}{z_j}\Big)\, e^{p_h(z/z_j)} \;=\; z^m \prod_j \Big(1 - \frac{z}{z_j}\Big) \exp\Big(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{hz^h}\Big)$$

When f admits a product expression with a uniform h, a product expression with *minimal* uniform h is a *canonical product* for f.

When, further, the leading factor $e^{g(z)}$ for f has $g(z)$ a *polynomial*, the *genus* of f is the maximum of h and the *degree* of g.

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2. *Poisson-Jensen formula*

Jensen's formula and the Poisson-Jensen formula are essential in the difficult half of the Hadamard theorem (below) comparing *genus* of an entire function to its *order of growth*.

The logarithm $u(z) = \log |f(z)|$ of the absolute value $|f(z)|$ of a non-vanishing holomorphic function f on a neighborhood of the unit disk is *harmonic*, that is, is annihilated by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$: expand

$$\Delta \log |f(z)| \;=\; \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\tfrac{1}{2}\log f(z) + \tfrac{1}{2}\log \overline{f}\,(z)\Big)$$

Conveniently, the two-dimensional Laplacian is the product of the Cauchy-Riemann operator and its conjugate. Since $\log f$ is holomorphic and $\log \overline{f}$ is *anti*-holomorphic, both are annihilated by the product of the two linear operators. This verifies that $\log |f(z)|$ is harmonic.

Thus, $\log |f(z)|$ satisfies the mean-value property for harmonic functions:

$$\log |f(0)| \;=\; u(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{i\theta})|\, d\theta$$

Next, let f have zeros ρ_j in $|z|<1$ but none on the unit circle. We manufacture a holomorphic function F from f but without zeros in $|z|<1$, and with $|F|=|f|$ on $|z|=1$, by the standard ruse

$$F(z) \;=\; f(z)\cdot \prod_j \frac{1-\overline{\rho}_j z}{z-\rho_j}$$

Indeed, for $|z|=1$, the numerator of each factor has the same absolute value as the denominator:

$$|z-\rho_j| \;=\; |\frac{1}{z}-\overline{\rho}_j| \;=\; \frac{1}{|z|}\cdot|1-\overline{\rho}_j z| \;=\; |1-\overline{\rho}_j z|$$

For simplicity, suppose no ρ_j is 0. Applying the mean-value identity to $\log|F(z)|$ gives

$$\log|f(0)| - \sum_j \log|\rho_j| \;=\; \log|F(0)| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta$$

and then the basic *Jensen's formula*

$$\log|f(0)| - \sum_j \log|\rho_j| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta \qquad\qquad (\text{for } |\rho_j|<1)$$

The Poisson-Jensen formula is obtained by replacing 0 by an arbitrary point z inside the unit disk. This is obtained by replacing f by $f\circ\varphi_z$, where φ_z is a *linear fractional transformation* mapping $0\to z$ and stabilizing [1] the unit disk:

$$\varphi_z = \begin{pmatrix} 1 & z \\ \overline{z} & 1\end{pmatrix} \;:\; w \;\longrightarrow\; \frac{w+z}{\overline{z}w+1}$$

This replaces the zeros ρ_j by $\varphi_z^{-1}(\rho_j) = \frac{\rho_j - z}{-\overline{z}\rho_j+1}$. Instead of the mean-value property expressing $f(0)$ as an integral over the circle, use the *Poisson formula* (see appendix) for $f(z)$. This gives the basic *Poisson-Jensen formula*

$$\log|f(z)| - \sum_j \log\Big|\frac{\rho_j - z}{-\overline{z}\rho_j+1}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\cdot\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \qquad (\text{for } |z|<1,\ |\rho_j|<1)$$

More generally, for holomorphic f on a neighborhood of a disk of radius $r>0$ with zeros ρ_j in that disk, apply the previous to $f(r\cdot z)$ with zeros ρ_j/r in the unit disk:

$$\log|f(r\cdot z)| - \sum_j \log\Big|\frac{\rho_j/r - z}{-\overline{z}\rho_j/r+1}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(r\cdot e^{i\theta})|\cdot\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \qquad (\text{for } |z|<1)$$

Replacing z by z/r gives

$$\log|f(z)| - \sum_j \log\Big|\frac{\rho_j/r - z/r}{-\overline{z}\rho_j/r^2+1}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\cdot\frac{1-|z/r|^2}{|z/r-e^{i\theta}|^2}\,d\theta \qquad (\text{for } |z|<r)$$

[1] To verify that such maps stabilize the unit disk, expand the natural expression:

$$1-\Big|\frac{w+z}{\overline{z}w+1}\Big|^2 \;=\; |\overline{z}w+1|^{-2}\cdot\Big(|\overline{z}w+1|^2 - |w+z|^2\Big) \;=\; |\overline{z}w+1|^{-2}\cdot\Big(|zw|^2 + \overline{z}w + z\overline{w} + 1 - |w|^2 - \overline{z}w - z\overline{w} - |z|^2\Big)$$

$$=\; |\overline{z}w+1|^{-2}\cdot\Big(|zw|^2 + 1 - |w|^2 - |z|^2\Big) \;=\; |\overline{z}w+1|^{-2}\cdot(1-|z|^2)\cdot(1-|w|^2) \;>\; 0$$

which rearranges slightly to the general *Poisson-Jensen formula*

$$\log|f(z)| - \sum_j \log\Big|\frac{\rho_j - z}{-\overline{z}\rho_j/r + r}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2}\, d\theta \qquad \text{(for } |z| < r,\ |\rho_j| < r)$$

The case $z = 0$ is the general *Jensen formula* for arbitrary radius r:

$$\log|f(0)| - \sum_j \log\Big|\frac{\rho_j}{r}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta \qquad \text{(with } |\rho_j| < r)$$

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3. *Hadamard products*

The *order* of an entire function f is the smallest positive real λ, if it exists, such that, for every $\varepsilon > 0$,

$$|f(z)| \;\le\; e^{|z|^{\lambda+\varepsilon}} \qquad \text{(for all sufficiently large } |z|)$$

The connection to infinite products is:

[3.0.1] Theorem: *(Hadamard)* The *genus* h and order λ are related by $h \le \lambda < h + 1$. In particular, one is *finite* if and only the other is finite.

Proof: First, the easier half. For f of finite genus h expressed as

$$f(z) \;=\; e^{g(z)} \cdot z^m \prod_j \Big(1 - \frac{z}{z_j}\Big) e^{p_h(z/z_j)} \;=\; e^{g(z)} \cdot z^m \prod_j \Big(1 - \frac{z}{z_j}\Big) \exp\Big(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{hz^h}\Big)$$

the leading exponential has polynomial g of degree at most h, so $e^{g(z)}$ is of *order* at most h. The order of a product is at most the maximum of the orders of the factors, so it suffices to prove that the order of the infinite product is at most $h + 1$.

The assumption that h is the genus of f is equivalent to

$$\sum_j \frac{1}{|z_j|^{h+1}} \;<\; \infty$$

We use this to directly estimate the infinite product, showing that it has order of growth λ at most $h + 1$.

We need an estimate on $F_h(w) = (1 - w)e^{p_h(w)}$ applicable for all w, not merely for $|w| < 1$. We collect some inequalities. There is the basic

$$\log|F_h(w)| \;=\; \log|(1 - w)e^{p_{h-1}(w)} \cdot e^{w^h/h}| \;\le\; \log|F_{h-1}(w)| + \frac{|w|^h}{h} \qquad \text{(for all } w)$$

As before, for $|w| < 1$,

$$\log|F_h(w)| \;\le\; \frac{1}{h+1} \cdot |w|^{h+1} + \frac{1}{h+2} \cdot |w|^{h+2} + \ldots \;\le\; |w|^{h+1} \cdot \frac{1}{1 - |w|} \qquad \text{(for } |w| < 1)$$

This gives $(1 - |w|) \cdot \log|F_h(w)| \le |w|^{h+1}$ for $|w| < 1$. Adding to the latter the basic relation multiplied by $|w|$ gives

$$\log|F_h(w)| \;\le\; |w| \cdot \log|F_{h-1}(w)| + \big(1 + \frac{1}{h}\big)|w|^{h+1} \qquad \text{(for } |w| < 1)$$

In fact, the latter inequality also holds for $|w| \geq 1$ and $\log|F_{h-1}(w)| \geq 0$, from the basic relation. For $\log|F_{h-1}(w)| < 0$ and $|w| \geq 1$, from the basic relation,

$$\log|F_h(w)| \;\leq\; \log|F_{h-1}(w)| + \frac{|w|^h}{h} \;\leq\; \frac{|w|^h}{h} \;\leq\; \big(1+\frac{1}{h}\big)|w|^{h+1} \qquad (\text{for } \log|F_{h-1}(w)| < 0 \text{ and } |w| \geq 1)$$

Now prove $\log|F_h(w)| \ll_h |w|^{h+1}$, by induction on h. For $h = 0$, from $\log|x| \leq |x| - 1$,

$$\log|1-w| \;\leq\; |1-w| - 1 \;\leq\; 1 + |w| - 1 \;=\; |w|$$

Assume $\log|F_{h-1}(w)| \ll_h |w|^h$. For $|w| < 1$, we reach the desired conclusion by

$$\log|F_h(w)| \;\leq\; |w| \cdot \log|F_{h-1}(w)| + \big(1+\frac{1}{h}\big)|w|^{h+1} \;\ll_h\; |w| \cdot |w|^h + \big(1+\frac{1}{h}\big)|w|^{h+1} \qquad (\text{for } |w| < 1)$$

For $|w| \geq 1$ and $\log|F_{h-1}(w)| > 0$, from the basic relation

$$\log|F_h(w)| \;\leq\; \log|F_{h-1}(w)| + \frac{|w|^h}{h} \;\ll_h\; |w|^h + \frac{|w|^h}{h} \;\ll_h\; |w|^{h+1} \qquad (\text{for } |w| \geq 1 \text{ and } \log|F_{h-1}(w)| > 0)$$

For $\log|F_{h-1}(w)| \leq 0$ and $|w| \geq 1$, from the basic relation we already have

$$\log|F_h(w)| \;\leq\; \log|F_{h-1}(w)| + \frac{|w|^h}{h} \;\leq\; \frac{|w|^h}{h} \;\ll_h\; |w|^{h+1} \qquad (\text{for } |w| \geq 1 \text{ and } \log|F_{h-1}(w)| < 0)$$

This proves $\log|F_h(w)| \ll_h |w|^{h+1}$ for all w.

Estimate the infinite product:

$$\log\Big|\prod_j (1-\frac{z}{z_j}) \cdot e^{p_h(z/z_j)}\Big| \;=\; \sum_j \log\Big|(1-\frac{z}{z_j}) \cdot e^{p_h(z/z_j)}\Big| \;\ll_h\; \sum_j \Big|\frac{z}{z_j}\Big|^{h+1} \;<\; \infty$$

since $\sum 1/|z_j|^{h+1}$ converges. Thus, such an infinite product has *growth order* $\lambda \leq h+1$.

Now the difficult half of the proof. Let $h \leq \lambda < h+1$. Jensen's formula will show that the zeros z_j are sufficiently spread out for convergence of $\sum 1/|z_j|^{h+1}$. Without loss of generality, suppose $f(0) \neq 0$. From

$$\log|f(0)| - \sum_j \log\Big|\frac{z_j}{r}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta \qquad (\text{with } |z_j| < r)$$

certainly

$$\sum_{|z_j|<r/2} \log 2 \;\leq\; \sum_{|\rho_j|<r/2} -\log\Big|\frac{\rho_j}{r}\Big| \;\ll_\varepsilon\; -\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} r^{\lambda+\varepsilon}\, d\theta \;\ll\; r^{\lambda+\varepsilon} \qquad (\text{for every } \varepsilon > 0)$$

With $\nu(r)$ the number of zeros inside the disk of radius r, this gives

$$\lim_{r\to+\infty} \frac{\nu(r)}{r^{\lambda+\varepsilon}} \;=\; 0 \qquad (\text{for all } \varepsilon > 0)$$

Order the zeros by absolute value: $|z_1| \leq |z_2| \leq \ldots$ and for simplicity suppose no two have the same size. Then $j = \nu(|z_j|) \ll_\varepsilon |z|^{\lambda+\varepsilon}$. Thus,

$$\sum \frac{1}{|z_j|^{h+1}} \;\ll_\varepsilon\; \sum \frac{1}{(j^{\frac{1}{\lambda+\varepsilon}})^{h+1}} \;=\; \sum \frac{1}{j^{\frac{h+1}{\lambda+\varepsilon}}}$$

The latter converges for $\frac{h+1}{\lambda+\varepsilon} > 1$, that is, for $\lambda + \varepsilon < h+1$. When $\lambda < h+1$, there is $\varepsilon > 0$ making such an equality hold.

It remains to show that the entire function $g(z)$ in the leading exponential factor is of degree at most $h+1$, by showing that its $(h+1)^{th}$ derivative is 0.

In the Poisson-Jensen formula

$$\log|f(z)| - \sum_{|z_j|<r} \log\Big|\frac{z_j - z}{-\overline{z}z_j/r + r}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2}\, d\theta \qquad (\text{for } |z| < r)$$

application of $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ annihilates the anti-holomorphic parts, returning us to an equality of holomorphic functions, as follows. The effect on the integrand is

$$2\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \;=\; 2\frac{-\overline{z}}{(z - re^{i\theta})(\overline{z} - re^{-i\theta})} - \frac{r^2 - |z|^2}{(z - re^{i\theta})^2(\overline{z} - re^{-i\theta})}$$

$$=\; 2\frac{-|z|^2 + \overline{z}re^{i\theta} - r^2 + |z|^2}{(z - re^{i\theta})^2(\overline{z} - re^{-i\theta})} \;=\; 2\frac{re^{i\theta}}{(re^{i\theta} - z)^2}$$

Thus,

$$\frac{f'(z)}{f(z)} - \sum_{|z_j|<r} \frac{1}{z - z_j} + \sum_{|z_j|<r} \frac{\overline{z}_j}{\overline{z}_j z - r^2} \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^2}\, d\theta$$

Further differentiation h times in z gives

$$\Big(\frac{f'(z)}{f(z)}\Big)^{(h)} \;=\; \sum_{|z_j|<r} \frac{(-1)^h h!}{(z - z_j)^{h+1}} - \sum_{|z_j|<r} \frac{(-1)^h h! \cdot \overline{z}_j^{h+1}}{(\overline{z}_j z - r^2)^{h+1}} + \frac{(-1)^h (h+1)!}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}}\, d\theta$$

We claim that the second sum and the integral go to 0 as $r \to +\infty$.

Regarding the integral, Cauchy's integral formula gives

$$\int_0^{2\pi} \frac{re^{i\theta}}{(z - re^{i\theta})^{h+2}}\, d\theta \;=\; 0$$

Thus, letting M_r be the maximum of $|f|$ on the circle of radius r, and taking $|z| < r/2$, up to sign the integral is

$$\int_0^{2\pi} \log\Big(\frac{M_r}{|f(re^{i\theta})|}\Big) \cdot \frac{2re^{i\theta}\, d\theta}{(z - re^{i\theta})^{h+2}} \;\ll\; \frac{1}{r^{h+1}}\int_0^{2\pi} \log\Big(\frac{M_r}{|f(re^{i\theta})|}\Big)\, d\theta \;\ll_\varepsilon\; \frac{r^{\lambda+\varepsilon}}{r^{h+1}} \cdot \int_0^{2\pi} -\log|f(re^{i\theta})|\, d\theta$$

Jensen's formula gives

$$\frac{1}{2\pi}\int_0^{2\pi} -\log|f(re^{i\theta})|\, d\theta \;\le\; -\log|f(0)|$$

Thus, for $\lambda + \varepsilon < h+1$ the integral goes to 0 as $r \to +\infty$.

For the second sum, again take $|z| < r/2$, so for $|z_j| < r$

$$\Big|\frac{\overline{z}_j^{h+1}}{(\overline{z}_j z - r^2)^{h+1}}\Big| \;\le\; \frac{|z_j^{h+1}|}{(r^2 - |z_j| \cdot \frac{r}{2})^{h+1}} \;\ll\; \frac{|z_j^{h+1}|}{r^{h+1}(r - |z_j|)^{h+1}} \;\ll\; \frac{1}{r^{h+1}}$$

We already showed that the number $\nu(r)$ of $|z_j| < r$ satisfies $\lim \nu(r)/r^{h+1} = 0$. Thus, this sum goes to 0 as $r \to +\infty$. Thus, taking the limit,

$$\Big(\frac{f'}{f}\Big)^{(h)} \;=\; (-1)^h h! \sum_j \frac{1}{(z - z_j)^{h+1}}$$

Returning to $f(z) = e^{g(z)} \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_h(z/z_j)}$, taking logarithmic derivative gives

$$\frac{f'}{f} \;=\; g' + \sum_j \Big(\frac{1}{z - z_j} + \frac{p_h'(z/z_j)}{z_j} \Big)$$

and taking h further derivatives gives

$$\Big(\frac{f'}{f}\Big)^{(h)} \;=\; g^{(h+1)} + \sum_j \frac{(-1)^h h!}{(z - z_j)^{h+1}}$$

Since the h^{th} derivative of f'/f is the latter sum, $g^{(h+1)} = 0$, so g is a polynomial of degree at most h.
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4. *Appendix: harmonic functions*

We recall the *mean-value property* and *Poisson's formula* for harmonic functions. The *Laplacian* is

$$\Delta \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and continuously twice-differentiable functions u with $\Delta u = 0$ are *harmonic*.

[4.0.1] **Theorem:** *(Mean-value property)* For harmonic u on a neighborhood of the unit disk,

$$u(0) \;=\; \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta$$

Proof: Consider the rotation-averaged function

$$v(z) \;=\; \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} \cdot z)\, d\theta \qquad\qquad (\text{for } |z| \le 1)$$

Since the Laplacian Δ is rotation-invariant, v is a rotation-invariant harmonic function. In polar coordinates, for rotation-invariant functions $v(z) = f(|z|)$, the Laplacian is

$$\Delta v \;=\; \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f(\sqrt{x^2+y^2}) \;=\; \frac{\partial}{\partial x}\big(\frac{x}{|z|} f'(|z|)\big) + \frac{\partial}{\partial y}\big(\frac{y}{|z|} f'(|z|)\big)$$

$$=\; \frac{1}{|z|} f' - \frac{x^2}{|z|^3} f' + \frac{x^2}{|z|^2} f'' + \frac{1}{|z|} f' - \frac{y^2}{|z|^3} f' + \frac{y^2}{|z|^2} f'' \;=\; f'' + \frac{1}{|z|} f'$$

The ordinary differential equation $f'' + f'/r = 0$ on an interval $(0, R)$ is an equation of *Euler type*, meaning expressible in the form $r^2 f'' + Brf' + Cf = 0$ with constants B, C. In general, such equations are solved by letting $f(r) = r^\lambda$, substituting, dividing through by r^λ, and solving the resulting *indicial equation* for λ:

$$\lambda(\lambda - 1) + A\lambda + B \;=\; 0$$

Distinct roots λ_1, λ_2 of the indicial equation produce linearly independent solutions r^{λ_1} and r^{λ_2}. However, as in the case at hand, a repeated root λ produces a second solution $r^\lambda \cdot \log r$.

Here, the indicial equation is $\lambda^2 = 0$, so the general solution is $a + b \log r$.

When $b \neq 0$, the solution $a + b\log r$ blows up as $r \to 0^+$. Since $f(0) = v(0) = u(0)$ is finite, it must be that $b = 0$. Thus, a *rotation-invariant* harmonic function on the disk is *constant*. Thus, its average over a circle is its central value. This proves the mean-value theorem for harmonic functions. ///

[4.0.2] Remark: The solutions $a + b\log r$ do indeed exhaust the possible solutions: given $f'' + f'/r = 0$ on $(0, R)$,

$$\frac{\partial}{\partial r}\big(r \cdot f'\big) \;=\; r \cdot f'' + f' \;=\; r \cdot \big(-f'/r\big) + f' \;=\; 0$$

Thus, $r \cdot f'$ is *constant*, and so on.

With some computation, from mean-value property we will obtain

[4.0.3] Theorem: *(Poisson's formula)* For u harmonic on a neighborhood of the unit disk,

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \qquad\qquad (\text{for } |z|<1)$$

Proof: Composition with holomorphic maps preserves harmonic-ness. With φ_z the linear fractional transformation given by matrix $\varphi_z \sim \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix}$, the mean-value property for $u \circ \varphi_z$ gives

$$u(z) \;=\; (u \circ \varphi_z)(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} (u \circ \varphi_z)(e^{i\theta})\,d\theta$$

Linear fractional transformations stabilizing the unit disk map the unit circle to itself. Replace $e^{i\theta}$ by $e^{i\theta'} = \varphi_z^{-1}(e^{i\theta})$

$$u(z) \;=\; (u \circ \varphi_z)(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta'$$

Computing the change of measure will yield the Poisson formula. This is computed by

$$ie^{i\theta'} \cdot \frac{\partial\theta'}{\partial\theta} \;=\; \frac{\partial}{\partial\theta}e^{i\theta'} \;=\; \frac{ie^{i\theta}}{1-\overline{z}e^{i\theta}} + \frac{i\overline{z}e^{i\theta}(e^{i\theta}-z)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{ie^{i\theta} - i\overline{z}e^{2i\theta} + i\overline{z}e^{2i\theta} - ie^{i\theta}|z|^2}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{ie^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2}$$

Thus,

$$\frac{\partial\theta'}{\partial\theta} \;=\; \frac{1}{e^{i\theta'}}\frac{ie^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{1-\overline{z}e^{i\theta}}{e^{i\theta}-z} \cdot \frac{e^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{1-|z|^2}{|z-e^{i\theta}|^2}$$

giving the asserted integral. ///

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