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Phragmén-Lindelöf Theorems

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E. Phragmén, E. Lindelöf, *Sur une extension d'un principe classique de l'analyse*, Acta Math. **31** (1908), 381-406 proved the theorem here.

The maximum modulus principle [1] in complex analysis does not apply to *unbounded* regions. That is, holomorphic functions on an unbounded region may be bounded by 1 on the edges but be severely unbounded in the interior.

A simple example is $f(z) = e^{e^z}$: for z real and going to $+\infty$ this function blows up. Indeed,

$$\left| e^{e^{x+iy}} \right| = e^{\mathrm{Re}\left(e^{x+iy}\right)} = e^{e^x \cdot \cos y}$$

Thus, for fixed $y = \mathrm{Im}\, z$ with $\cos y > 0$, the function blows up as $x = \mathrm{Re}\, z \to +\infty$. On the other hand, for $\cos y = 0$ the function is *bounded.* Thus, on the strip $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, the function e^{e^z} is bounded on the edges but blows up as $x \to +\infty$.

This example suggests *growth conditions* under which a bound of 1 on the edges implies the same bound throughout the strip.

[1.0.1] Theorem: Let f be a holomorphic function on the horizontal half-strip

$$\{z \,:\, -\frac{\pi}{2} \le y \le \frac{\pi}{2} \text{ and } 0 \le x\}$$

If

$$|f(z)| \ll e^{e^{C\cdot \mathrm{Re}\, z}} \qquad \text{(for some constant } 0 \le C < 1)$$

then $|f(z)| \le 1$ on the edges of the half-strip implies $|f(z)| \le 1$ in the interior, as well.

Proof: Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed D in the range

$$C < D < 1$$

The function

$$F_\varepsilon(z) = f(z)/e^{\varepsilon e^{D\cdot z}} \qquad \text{(for } \varepsilon > 0)$$

is bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in y as $x \to +\infty$, for fixed $\varepsilon > 0$. This uniform decay in the interior uses the modification with D. Thus, on a rectangle

$$R_T = \{z \,:\, -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ and } 0 \le x \le T\}$$

for sufficiently large $T > 0$ depending upon ε, the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, *for each fixed* z_o in the half-strip,

$$|f(z_o)| \le e^{\varepsilon\cdot e^{D \mathrm{Re}\, z_o}} \qquad \text{(for all } \varepsilon > 0)$$

We can let $\varepsilon \to 0^+$, giving $|f(z_o)| \le 1$. ///

[1] The *maximum modulus principle* in complex analysis is that a holomorphic function f on a bounded region in $\mathbb{C}$ with $|f(z)| \le 1$ on the edges is bounded by 1 in the interior, as well.

[1.0.2] Remark: Analogous theorems on strips of other widths follow by using $e^{c\cdot e^z}$ with suitable constants c.

An analogous theorem on a full strip, rather than half-strip, follows by using a function like $e^{\cosh z}$ in place of e^{e^z}, as follows.

[1.0.3] Theorem: Let f be a holomorphic function on the full horizontal strip

$$\{z \;:\; -\frac{\pi}{2} \;\le\; \mathrm{Im}\, z \;\le\; \frac{\pi}{2}\}$$

If

$$|f(z)| \;\ll\; e^{\cosh C\cdot \mathrm{Re}\, z} \qquad\qquad \text{(for some constant } 0 \le C < 1)$$

then $|f(z)| \le 1$ on the edges of the strip implies $|f(z)| \le 1$ in the interior, as well.

Proof: Again, reduce to the maximum modulus principle. Fix D in the range $C < D < 1$. The function

$$F_\varepsilon(z) \;=\; f(z)/e^{\varepsilon \cosh Dz} \qquad\qquad \text{(for } \varepsilon > 0)$$

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in y as $x \to \pm\infty$, for fixed $\varepsilon > 0$. Thus, on a rectangle

$$R_T \;=\; \{z \;:\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ and } -T \le x \le T\} \qquad\qquad \text{(for large } T > 0, \text{ depending upon } \varepsilon)$$

the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, *for each fixed* z_o in the half-strip,

$$|f(z_o)| \;\le\; e^{\varepsilon \,\cosh D \mathrm{Re}\, z_o} \qquad\qquad \text{(for all } \varepsilon > 0)$$

We can let $\varepsilon \to 0^+$, giving $|f(z_o)| \le 1$. ///

The details of various adjustments can be made to disappear by strengthening the hypotheses:

[1.0.4] Corollary: Let f be a holomorphic function on a strip or half-strip, with a bound

$$|f(z)| \;\ll\; e^{|z|^A} \qquad\qquad \text{(for some } A > 0)$$

If $|f(z)| \le 1$ on the edges of the (half-)strip, then $|f(z)| \le 1$ in the interior, as well. ///

[1.0.5] Remark: Further variations are easily possible, by additional adjustments of functions. For example, *polynomial growth* of a function f on the edges of a strip or half-strip can be accommodated by considering $f(z)/(z - z_o)^M$ for z_o outside the strip, and large M.
