(*September 11, 2013*)

Fields of elliptic functions

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/03a_fields_elliptic.pdf]

[0.0.1] Theorem: *Every* elliptic function (with lattice Λ) is expressible in terms of the corresponding $\wp$ and $\wp'$. That is, for lattice Λ, the field of meromorphic Λ-periodic functions is exactly the collection of rational expressions in $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$. Further, all *even* Λ-periodic functions are rational expressions in $\wp_\Lambda(z)$.

Incidental to the proof of the theorem, we have

[0.0.2] Claim: Let f be a Λ-periodic meromorphic function. For a fixed choice of basis ω_1, ω_2 for Λ, let F be the corresponding *fundamental domain* as above. Let $z_1, \ldots, z_m$ be the zeros of f in F, and let $p_1, \ldots, p_n$ be the poles, both including multiplicities. [1] Then $m = n$. Further,

$$\sum_i z_i - \sum_j p_j = 0 \bmod \Lambda$$

Proof: Integrating f/f' around the boundary of F (make minor adaptations in case a zero or pole happens to be exactly on that path) computes $2\pi i(m-n)$, by Cauchy's residue theorem. On the other hand, by periodicity of f/f', and since we integrate on opposite edges of the parallelogram F in opposite directions, this integral is 0. Thus, $m = n$.

Similarly, integrate $z \cdot f'/f$ around the boundary of F. On one hand, by Cauchy's residue theorem this computes

$$2\pi i \cdot (\sum_i z_i - \sum_j p_j)$$

This time, since the function with the factor of z thrown in is *not* periodic, the integral is not 0. However, there is still some cancellation. The integral is

$$-\omega_2 \int_0^{\omega_1} \frac{f'}{f} + \omega_1 \int_0^{\omega_2} \frac{f'}{f}$$

One may easily overlook the fact that the two integrals are *integer* multiples of $2\pi i$, which follows from [2]

$$\int_0^{\omega_i} \frac{f'}{f} = \int_0^{\omega_i} \frac{d \log f}{d\zeta}$$

and the fact that $f(0) = f(\omega_i)$. That is, as ζ goes from 0 to ω_i, the function $(\log f)(\zeta)$ traces out a closed path circling 0 some *integer* number of times, say k_i. Then the integral is

$$-\omega_2 \cdot 2\pi i k_1 + \omega_1 \cdot 2\pi i k_2 \in 2\pi i \cdot \Lambda$$

Cancelling the factor of $2\pi i$, equating the two outcomes gives

$$\sum_i z_i - \sum_j p_j \in \Lambda$$

---

[1] Usually *including multiplicities* means that for a zero z_o of order ℓ the point z_o is included ℓ times on the list of zeros. That is, this list is a *multiset*, not an ordinary set, since ordinary sets (by their nature) do not directly keep track of multiple occurrences of the same element.

[2] This is an instance of the *Argument Principle*.

as claimed. ///

Proof: Let f be a Λ-periodic meromorphic function on $\mathbb{C}$. We can break f into odd and even pieces by

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$$

For f *odd*, the function $\wp' \cdot f$ is *even*, so it suffices to prove that every *even* elliptic function is rational in $\wp$.

The previous claim has immediate implications for the values of $\wp$, which we use to form an expression in $\wp$ that will duplicate the zeros and poles of the given *even* f. Generally, for *even* f, since $f(-z) = f(z)$, for $2z_o \notin \Lambda$ and $f(z_o) = 0$, then $f(-z_o) = 0$ and z_o and $-z_o$ are distinct modulo Λ. For $2z_o \in \Lambda$, the *oddness* (and periodicity) of f' yields

$$f'(z_o) = -f'(-z_o) = -f'(-z_o + 2z_o) = -f'(z_o)$$

so $f'(z_o) = 0$, and the order of the zero z_o is at least 2.

In particular, by the previous claim, since $\wp(z) - \wp(a)$ has the obvious double pole on Λ, it has exactly two zeros, whose sum is 0 modulo Λ. Obviously a itself is a 0, and for $a \notin \frac{1}{2}\Lambda$ the unique (mod Λ) other zero is $-a$. And for $a \in \frac{1}{2}\Lambda$ it is a *double* zero of $\wp(z) - \wp(a)$.

Thus, for a zero $z_o \notin \Lambda$ of f, the order of vanishing of $\wp(z - \wp(z_o)$ at *all* its zeros is *at most* that of f at those zeros. Thus, by comparison to $f(z)$, the function

$$\frac{f(z)}{\wp(z) - \wp(z_o)}$$

has lost two zeros (either z_o and $-z_o$ or a double zero at z_o). The double pole of $\wp(z) - \wp(z_o)$ at 0 makes $f(z)/(\wp(z) - \wp(z_o))$ have order of vanishing at 0 two more than that of $f(z)$. No new poles are introduced by such an alteration, nor any zeros off Λ. Thus, since there are only finitely-many zeros (modulo Λ), after finitely-many such modifications we have a function $g(z)$ with *no* zeros off Λ.

Next, we get rid of *poles* of $g(z)$ off Λ by a similary procedure, repeatedly *multiplying* by factors $\wp(z) - \wp(z_o)$. Thus, for some list of points z_i not in Λ, with positive and negative integer exponents e_i,

$$f(z) \cdot \prod_i \big(\wp(z) - \wp(z_i)\big)^{e_i}$$

has no poles or zeros off Λ. From the previous discussion, this expression has *no* zeros or poles at all, and then is constant. ///

[0.0.3] Remark: There is at least one other way to construct doubly-periodic functions directions, due to Jacobi, who expressed doubly-periodic functions as ratios of *entire* functions (*theta functions*) which are genuinely singly-periodic with periods (for example) $\mathbb{Z}$, and nearly (but not quite) periodic in another direction. (Indeed, we saw just above that entire functions that are genuinely doubly-periodic are constant!)
