(*December 25, 2013*)

Fourier series, Weyl equidistribution

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- Dirichlet's pigeon-hole principle, approximation theorem
- Kronecker's approximation theorem
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1. *Dirichlet's pigeon-hole principle, approximation theorem*

The *pigeon-hole* principle[1] formulated by Dirichlet by 1834, observes that when $N+1$ things are partitioned into N disjoint subsets, there is at least one subset containing at least 2 things. The archetypical application is the following:

[1.0.1] Theorem: *(Dirichlet)* For every real α and every integer $N \geq 1$, there are integers p, q with $1 \leq q \leq N$ such that $|q\alpha - p| \leq \frac{1}{N}$.

Proof: For each m in the range $1 \leq m \leq N+1$, choose $n = n_m$ so that $m\alpha - n_m \in [0, 1)$ is the *fractional part*[2] of $m\alpha$. The $N+1$ choices of m produce $N+1$ numbers $m\alpha - n$ in $[0, 1)$. Dividing the interval into N subintervals of length $\frac{1}{N}$, by the pigeon-hole principle some subinterval contains both $m\alpha - n$ and $m'\alpha - n'$ for some $1 \leq m' < m \leq N+1$. That is,

$$\frac{1}{N} \;\geq\; |(m\alpha - n) - (m'\alpha - n')| \;=\; |(m - m')\alpha - (n - n')|$$

so $1 \leq q = m - m' \leq N$ and $p = n - n'$ meet the requirement of the theorem. ///

[1.0.2] Remark: This result admits many elaborations and strengthenings, with similar proofs.

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2. *Kronecker's approximation theorem*

The following is a special case of Kronecker's 1884 generalization of Dirichlet's approximation results, in which Kronecker illustrates a different causal mechanism:

[2.0.1] Theorem: The collection of integer multiples $n\alpha$ of *irrational* real α is *dense* in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Proof: The assertion is equivalent to $\mathbb{Z}\alpha + \mathbb{Z}$ being dense in $\mathbb{R}$. The topological closure Γ of $\mathbb{Z}\alpha + \mathbb{Z}$ in $\mathbb{R}$ is still a subgroup of $\mathbb{R}$. The topologically-closed subgroups of $\mathbb{R}$ are *classifiable*: they are exactly $\{0\}$, free $\mathbb{Z}$-modules $\mathbb{Z}\beta$ on a single generator $\beta \neq 0$, and the whole $\mathbb{R}$. Granting this classification for a moment, if Γ is *not* the whole, then $\Gamma = \mathbb{Z} \cdot \beta$ for some β, and certainly $\mathbb{Z}\alpha + \mathbb{Z} \subset \mathbb{Z}\beta$. Thus, $\mathbb{Z} \subset \mathbb{Z}\beta$, so $1 = n\beta$ for some integer n, and $\beta = \frac{1}{n}$. Similarly, $1 \cdot \alpha = m\beta = \frac{m}{n}$ for some n, so α is rational.

Now we classify topologically-closed subgroups $\Gamma \neq \{0\}$ of $\mathbb{R}$. Since $\Gamma \neq \{0\}$ and is closed under additive inverses, Γ contains *positive* elements.

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[1] Dirichlet used *Schubfachprinzip*, which is *drawer principle*.

[2] The *fractional part* of a real number α, sometimes denoted $\{\alpha\}$ or $\langle\alpha\rangle$ or α mod 1, in an elementary context is $\langle\alpha\rangle = \alpha - \lfloor\alpha\rfloor$, where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal α. More to the point, the fractional part is really the *image* of α in the quotient $\mathbb{R}/\mathbb{Z}$.

In the case that there is a *least* positive element γ_o, we claim that $\Gamma = \mathbb{Z}\cdot\gamma_o$. Indeed, given another $0 < \gamma \in \Gamma$, by the archimedean property of the real numbers there is an integer n such that $n\gamma_o \le \gamma < (n+1)\gamma_o$. In fact, $n\gamma_o = \gamma$, or else $0 < \gamma - n\gamma_o < \gamma_o$, contradicting the minimality of γ_o.

In the case that there is *no* least positive $\gamma_o \in \Gamma$, let $\gamma_1 > \gamma_2 > \ldots > 0$ be an infinite sequence of positive elements of Γ. The infimum γ_o of this sequence is in Γ, since Γ is closed. Replacing γ_j by $\gamma_j - \gamma_o$, we can assume that $\gamma_j \to 0$. Given real β, there is integer n such that $n\gamma_j \le \beta < (n+1)\gamma_j$ by archimedean-ness, so $\mathbb{Z}\gamma_j$ contains elements within $|\gamma_j|$ of any real number. Since $\gamma_j \to 0$, given $\varepsilon > 0$ there is $|\gamma_j| < \varepsilon$, so every real number is within $\varepsilon > 0$ of $\mathbb{Z}\gamma_j$. Thus, the topologically closed subgroup Γ must be $\mathbb{R}$. ///

A similar argument would prove Kronecker's multi-dimensional version:

[2.0.2] Theorem: *(Kronecker)* An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of real numbers is linearly independent over $\mathbb{Q}$ if and only if the topological closure of the collection $\{N \cdot \alpha : N = 1, 2, 3, \ldots\}$ of multiples of α is *dense* in the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. ///

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3. *Weyl equidistribution*

The idea of a sequence of real numbers $\alpha_1, \alpha_2, \ldots$ being *equidistributed* modulo $\mathbb{Z}$, that is, in $\mathbb{R}/\mathbb{Z}$, is a *quantitative* strengthening of a merely *qualitative* density assertion.

A sample equidistribution requirement is that the number of $\langle \alpha_n \rangle$ with $1 \le n \le N$ in any subinterval $[a, b]$ of $[0, 1]$ is asymptotically $N \cdot |b - a|$ as $N \to \infty$.

[3.1] One-dimensional equidistribution

With various formulations of *integral*, the stronger notion of equidistribution is equivalent to

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{\ell=1}^{N} f(\alpha_\ell) \;=\; \int_0^1 f(x)\, dx \qquad \text{(for every } \mathbb{Z}\text{-periodic } f \in C^\infty(\mathbb{R}))$$

[3.1.1] Theorem: *(Weyl)* A sequence $\{\alpha_\ell\}$ is equidistributed modulo $\mathbb{Z}$ if and only if

$$\lim_N \frac{1}{N} \sum_{\ell=1}^{N} e^{2\pi i n \alpha_\ell} \;=\; 0 \qquad \text{(for every } n \neq 0)$$

For example, for irrational real α, the sequence $\alpha, 2\alpha, 3\alpha, \ldots$ is equidistributed modulo $\mathbb{Z}$.

Proof: The function $f(x) = e^{2\pi i n x}$ is smooth and $\mathbb{Z}$-periodic, so the assumption of equidistribution of $\{\alpha_\ell\}$ implies

$$\lim_N \frac{1}{N} \sum_{\ell=1}^{N} e^{2\pi i n \alpha_\ell} \;=\; 0 \qquad \text{(for every } n \neq 0)$$

On the other hand, *smooth* $\mathbb{Z}$-periodic f has Fourier expansion $\sum_n \widehat{f}(n)\, e^{2\pi i n x}$ converging absolutely to $f(x)$ for all x. From the *uniform* estimate

$$\Big| \frac{1}{N} \sum_{\ell=1}^{N} e^{2\pi i n \alpha_\ell} \Big| \;\le\; 1 \qquad \text{(uniformly in } n \in \mathbb{Z})$$

and noting $\widehat{f}(0) = \int_0^1 f(x)\, dx$,

$$\frac{1}{N} \sum_{\ell=1}^{N} f(\alpha_\ell) \;=\; \frac{1}{N} \sum_{\ell=1}^{N} \sum_n \widehat{f}(n) e^{2\pi i n \alpha_\ell} \;=\; \frac{1}{N} \sum_n \widehat{f}(n) \sum_{\ell=1}^{N} e^{2\pi i n \alpha_\ell} \;=\; \widehat{f}(0) + \sum_{n\neq 0} \widehat{f}(n) \cdot \frac{1}{N} \cdot \sum_{\ell=1}^{N} e^{2\pi i n \alpha_\ell}$$

For any cut-off b, estimate $\frac{1}{N}\sum_{\ell=1}^N f(\alpha_\ell) - \int_0^1 f(x)\,dx$ by

$$\sum_{n\neq 0} |\widehat{f}(n)| \cdot \frac{1}{N} \cdot \Big|\sum_{\ell=1}^N f(\alpha_\ell)\Big| \;\le\; \sum_{0<|n|\le b} |\widehat{f}(n)| \cdot \frac{1}{N} \cdot \Big|\sum_{\ell=1}^N f(\alpha_\ell)\Big| + \sum_{b<|n|} |\widehat{f}(n)| \cdot 1$$

Since the Fourier series of f converges absolutely, given $\varepsilon > 0$ there is large-enough b so that $\sum_{|n|>b} |\widehat{f}(n)| < \varepsilon$. With that b, since $\frac{1}{N}\sum_{1\le\ell\le N} e^{2\pi i n\alpha_\ell} \to 0$ for each fixed $n \neq 0$, and since there are only finitely-many n with $0 < |n| \le b$, for large-enough N

$$\sum_{0<|n|\le b} |\widehat{f}(n)| \cdot \frac{1}{N} \cdot \Big|\sum_{\ell=1}^N f(\alpha_\ell)\Big| \;<\; \varepsilon$$

Thus,

$$\Big|\frac{1}{N}\sum_{\ell=1}^N f(\alpha_\ell) \;-\; \widehat{f}(0)\Big| \;<\; 2\varepsilon$$

That is, $\frac{1}{N}\sum_{\ell=1}^N f(\alpha_\ell) \longrightarrow \int_0^1 f(x)\,dx$, and Weyl's criterion suffices for equidistribution.

In the example of integer multiples of an irrational number, by summing a geometric series, with $n \neq 0$,

$$= \;\frac{1}{N}\sum_{\ell=1}^N e^{2\pi i n\cdot\ell\alpha} \;=\; \frac{1 - e^{2\pi i n(N+1)\alpha}}{1 - e^{2\pi i n\alpha}}$$

The irrationality of α and $n \neq 0$ assure that the denominator does not vanish. Thus,

$$\frac{1}{N}\sum_{\ell=1}^N e^{2\pi i n\cdot\ell\alpha} \;\le\; \frac{1}{N} \cdot \frac{2}{|1 - e^{2\pi i n\alpha}|} \;\to\; 0 \qquad \text{(for each fixed } n \neq 0)$$

proving equidistribution of $\{\ell\alpha\}$. ///

[3.1.2] Remark: The proof only used absolute convergence pointwise of the Fourier series of f to f. Infinite-differentiability of f assures this, but much less is needed. On the other hand, more than just continuity of f is needed, since the Fourier series of typical continuous functions do not converge to them pointwise.

[3.2] Higher-dimension analogues

Fourier series of functions f on $\mathbb{R}^n/\mathbb{Z}^n$ can be written

$$\sum_{\xi\in\mathbb{Z}^n} \widehat{f}(\xi)\, e^{2\pi i\xi\cdot x}$$

where $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n$, and

$$\widehat{f}(\xi) \;=\; \int_0^1 \ldots \int_0^1 e^{-2\pi i\xi\cdot x}\, f(x)\, dx$$

The notion of *equidistribution* modulo $\mathbb{Z}^n$ of a sequence $\{\alpha_\ell\}$ in $\mathbb{R}^n$ is the same:

$$\lim_N \frac{1}{N}\sum_{1\le\ell\le N} f(\alpha_\ell) \;=\; \int_0^1 \ldots \int_0^1 f(x)\, dx \qquad \text{(for all } \mathbb{Z}^n\text{-periodic smooth } f \text{ on } \mathbb{R}^n)$$

Assuming we know that Fourier series of a nice $\mathbb{Z}^n$-periodic function f on $\mathbb{R}^n$ converges absolutely to f pointwise, the same argument proves

[3.2.1] Theorem: *(Weyl)* A sequence $\{\alpha_\ell\}$ in $\mathbb{R}^n$ is equidistributed modulo $\mathbb{Z}^n$ if and only if

$$\lim_N \frac{1}{N} \sum_{1\le \ell\le N} e^{2\pi i \xi\cdot\alpha_\ell} \;=\; 0 \qquad\qquad \text{(for all } 0\neq\xi\in\mathbb{Z}^n)$$

For example, for real numbers $\beta_1,\ldots,\beta_n$, the sequence $\alpha_\ell = \ell\cdot(\beta_1,\ldots,\beta_n)$ is equidistributed modulo $\mathbb{Z}^n$ if and only if $1,\beta_1,\ldots,\beta_n$ are linearly independent over $\mathbb{Q}$. ///

[3.2.2] Remark: Weyl's criterion for equidistribution can be applied to *compact topological groups* K of various sorts, in place of $\mathbb{R}^n/\mathbb{Z}^n$, especially compact *Lie groups*, because of the spectral decomposition of $L^2(K)$ analogous to Fourier series on $\mathbb{R}^n/\mathbb{Z}^n$.

[3.2.3] Remark: Weyl treated a more serious problem, that of equidistribution modulo $\mathbb{Z}$ of sequences $\alpha_\ell = P(\ell)$ with polynomial P.

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