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Wilbraham-Gibbs phenomenon

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[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/04a_Wilbraham-Gibbs.pdf]

The fact that Fourier series *overshoot* by about 9% at discontinuities was discovered by H. Wilbraham in 1848, but his paper was overlooked, and the phenomenon was rediscovered by J.W. Gibbs in 1899. [1] We recall a precise statement and proof, and give explanatory graphics. [2]

It suffices to consider the sawtooth function $s(x)$ defined by $s(x) = \frac{\pi}{2} - \frac{x}{2}$ on $[0, 2\pi]$ and extended to $\mathbb{R}$ by $2\pi\mathbb{Z}$ periodicity. [3] The normalization of the sawtooth is chosen so that the Fourier expansion is

$$s(x) \;=\; \sum_{0\neq n\in\mathbb{Z}} \frac{e^{inx}}{2in} \;=\; \sum_{n\geq 1} \frac{\sin nx}{n}$$

The Wilbraham-Gibbs constant is

$$C \;=\; \frac{\displaystyle\int_0^\pi \frac{\sin t\; dt}{t} \;-\; \text{limit value at } 0^+}{\text{total jump}} \;=\; \frac{\displaystyle\int_0^\pi \frac{\sin t\; dt}{t} \;-\; \frac{\pi}{2}}{\pi} \;\sim\; 1.089$$

Since $\lim_{x\to 0^+} s(x) = \pi/2$, an *overshoot* for small $x > 0$ is a value above $\pi/2$.

[0.0.1] Theorem: For sufficiently large N, at $x = \frac{\pi}{N}$

$$\sum_{1\leq n\leq N} \frac{\sin nx}{n} \;\geq\; \frac{\pi}{2} + C\cdot\pi \;>\; \frac{\pi}{2} + 1.089\cdot\pi$$

Proof: Given N,

$$\sum_{1\leq n\leq N} \frac{\sin nx}{n} \;=\; \sum_{1\leq n\leq N} x\cdot\frac{\sin nx}{nx} \;\longrightarrow\; \int_0^\pi \frac{\sin t\; dt}{t} \qquad\qquad (\text{as } N\to\infty \text{ with } x = \tfrac{\pi}{N}\to 0^+)$$

The integral is strictly bigger than the limiting value $\pi/2$, as numerical computation discloses, so this gives an overshoot. ///

[0.0.2] Remark: That the Wilbraham-Gibbs constant is greater than 1, that is, that the integral is greater than $\pi/2$, is visible from a human-accessible computation:

$$\int_0^\pi \frac{\sin t\; dt}{t} \;\geq\; \int_0^3 \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}}{t}\, dt \;=\; \int_0^3 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!}\, dt \;=\; 3 - \frac{3^3}{3!\cdot 3} + \frac{3^5}{5!\cdot 5} - \frac{3^7}{7!\cdot 7}$$

$$= \; 3 - \frac{27}{18} + \frac{81}{200} - \frac{243}{560\cdot 7} \;>\; 3 - \frac{3}{2} + \frac{2}{5} - \frac{1}{3\cdot 7} \;\geq\; 1.85 \;=\; \frac{3.7}{2} \;>\; \frac{\pi}{2}$$

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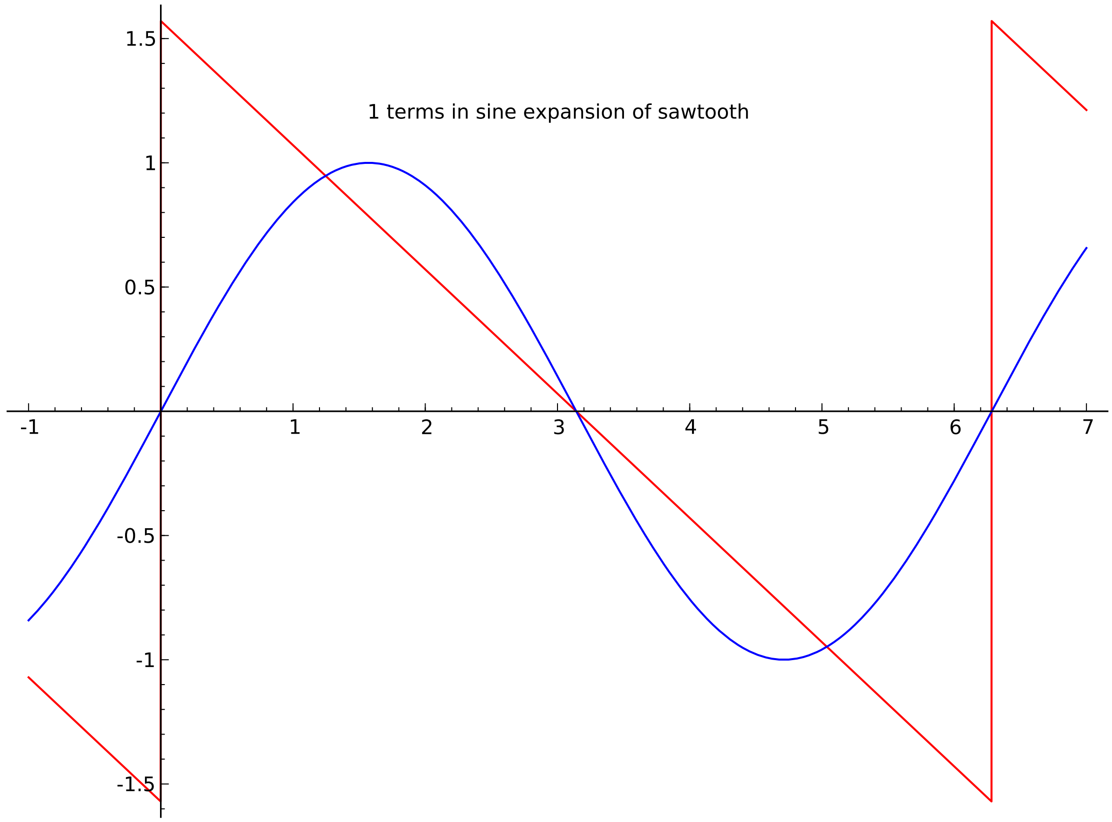
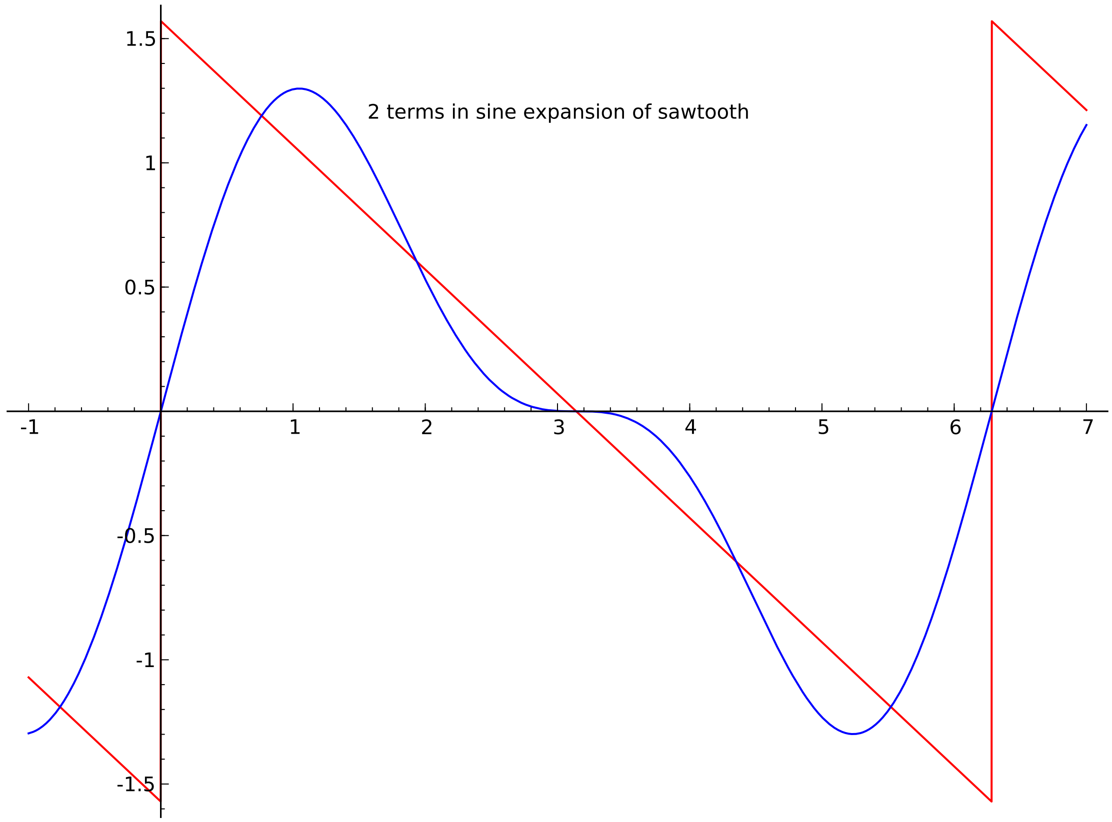
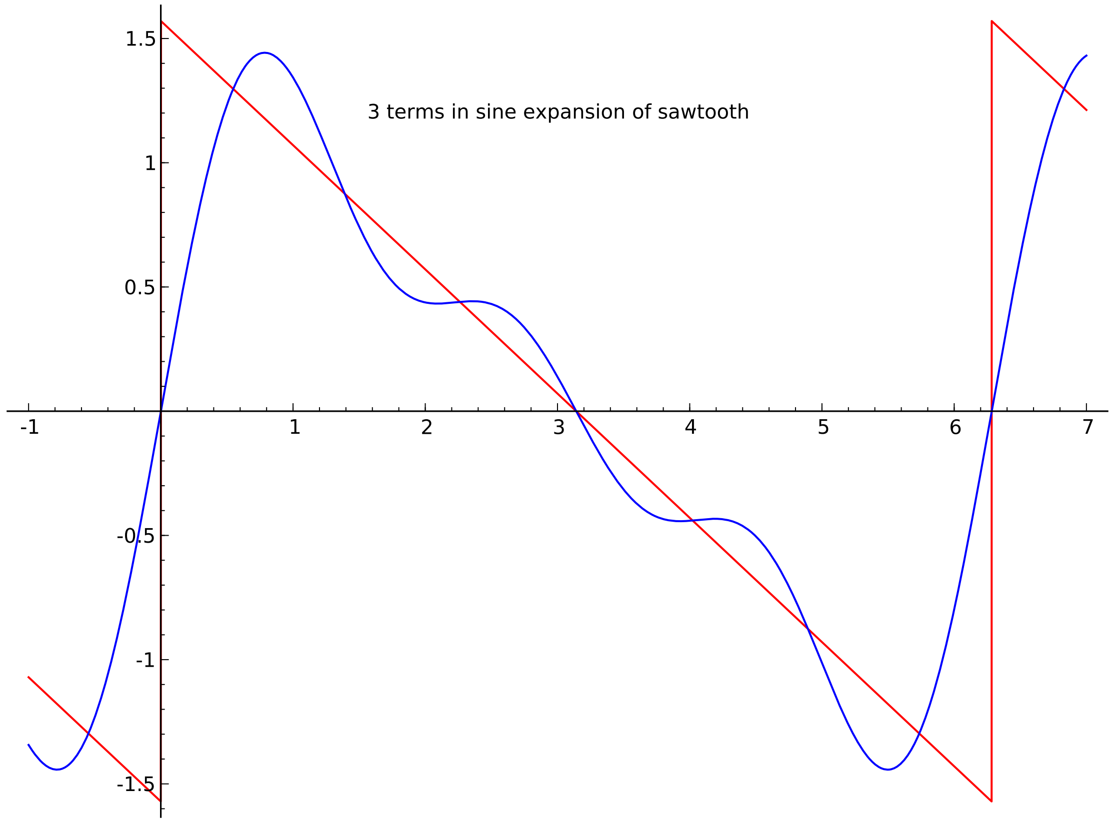
[1] Discovered by Henry Wilbraham (1848), rediscovered by J. Willard Gibbs (1899): see https://en.wikipedia.org/wiki/Gibbs_phenomenon (uploaded 11 August 2013)

[2] Graphs made using SAGE Mathematical Software, Version 5.10, http://www.sagemath.org PDF images imbedded in PlainTeX via J. Shipman's device via miniltx (”Mini-LaTeX”), found 11 Aug 2013 at http://infohost.nmt.edu/tcc/help/dtp/tex/pdfimages.html Screen capture of Shipman's device at the end of this document.

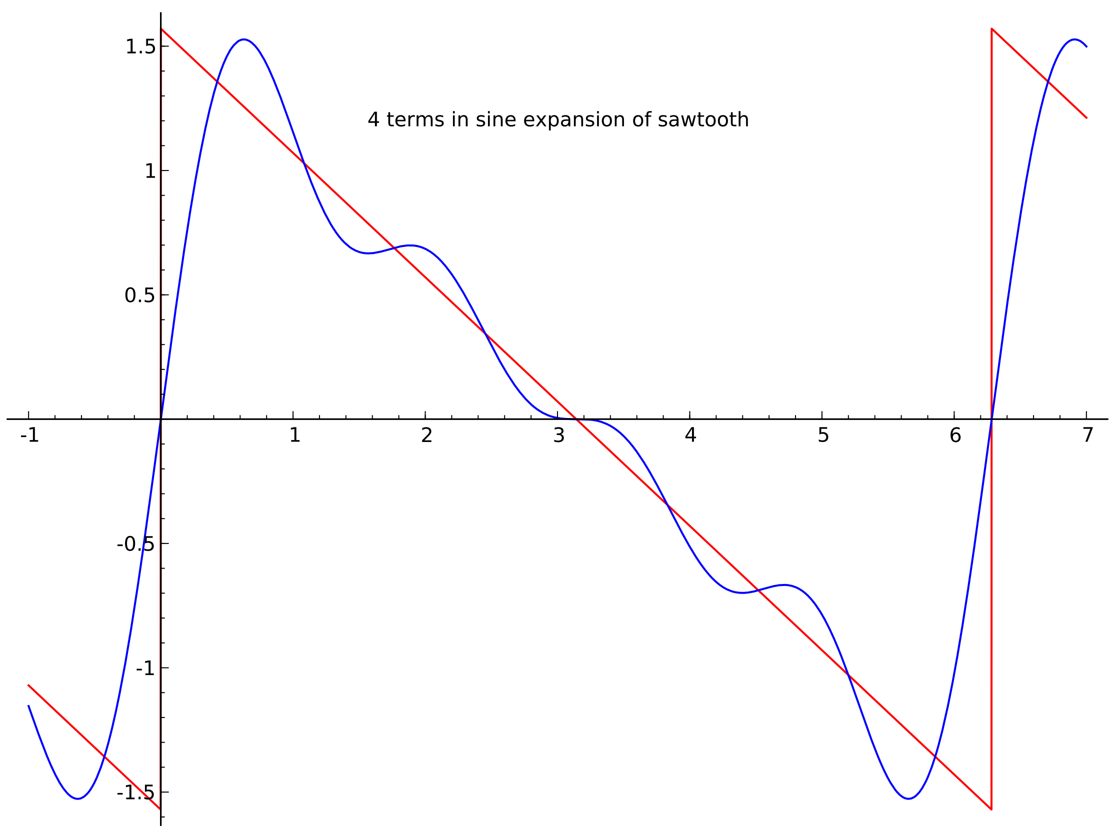
[3] Finite linear combinations of translates of the sawtooth can be added to a finitely-piecewise-C^1 function with finitely-many jump discontinuities to eliminate the jump discontinuities, etc.


1 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7

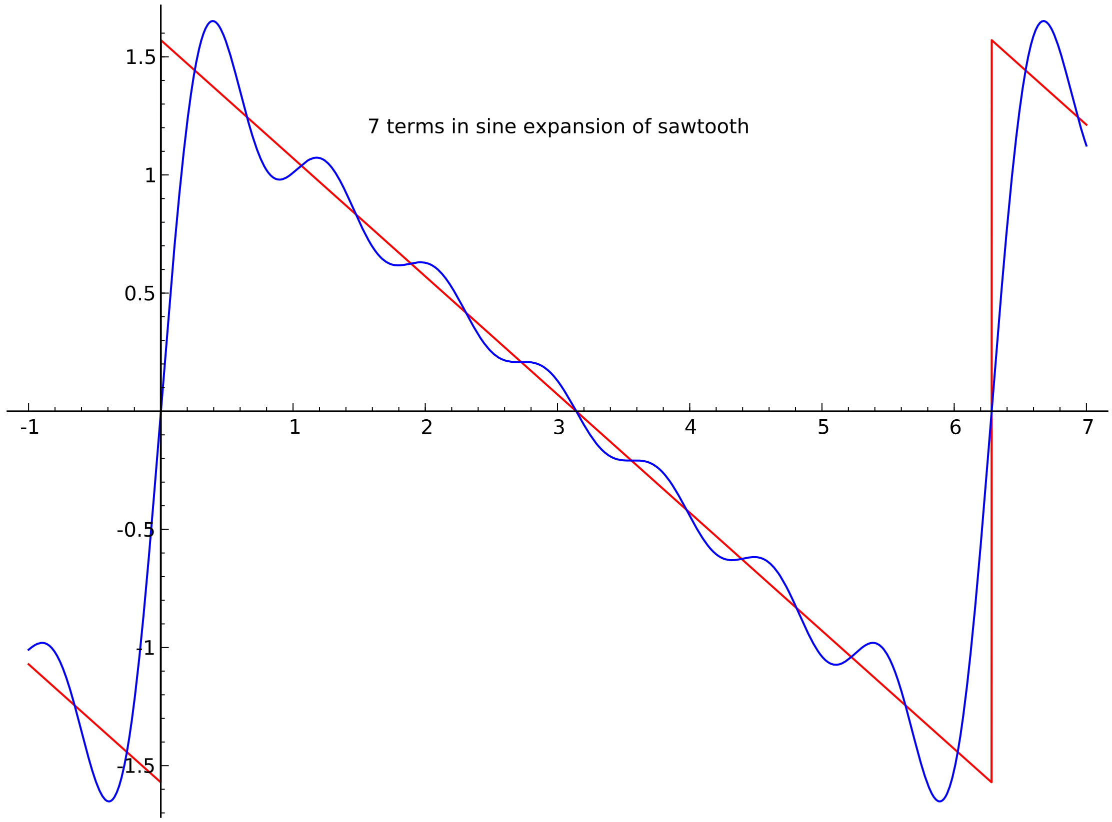

2 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7

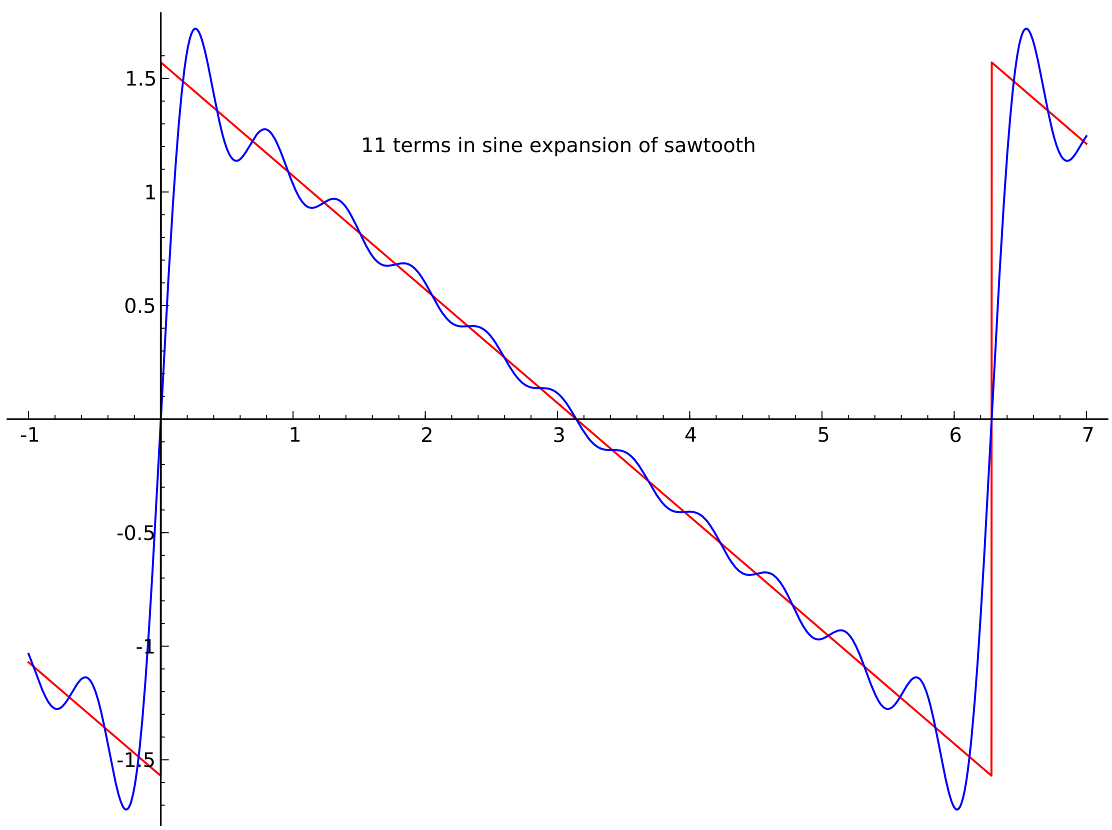


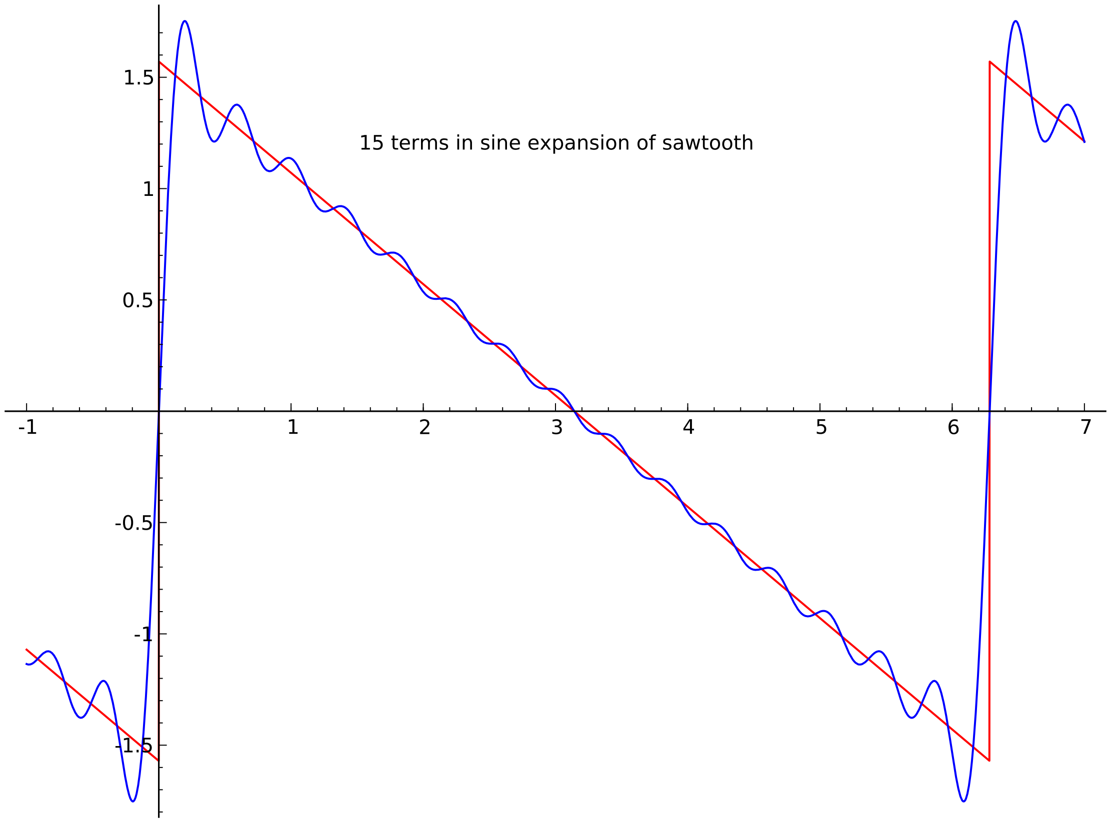
3 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7

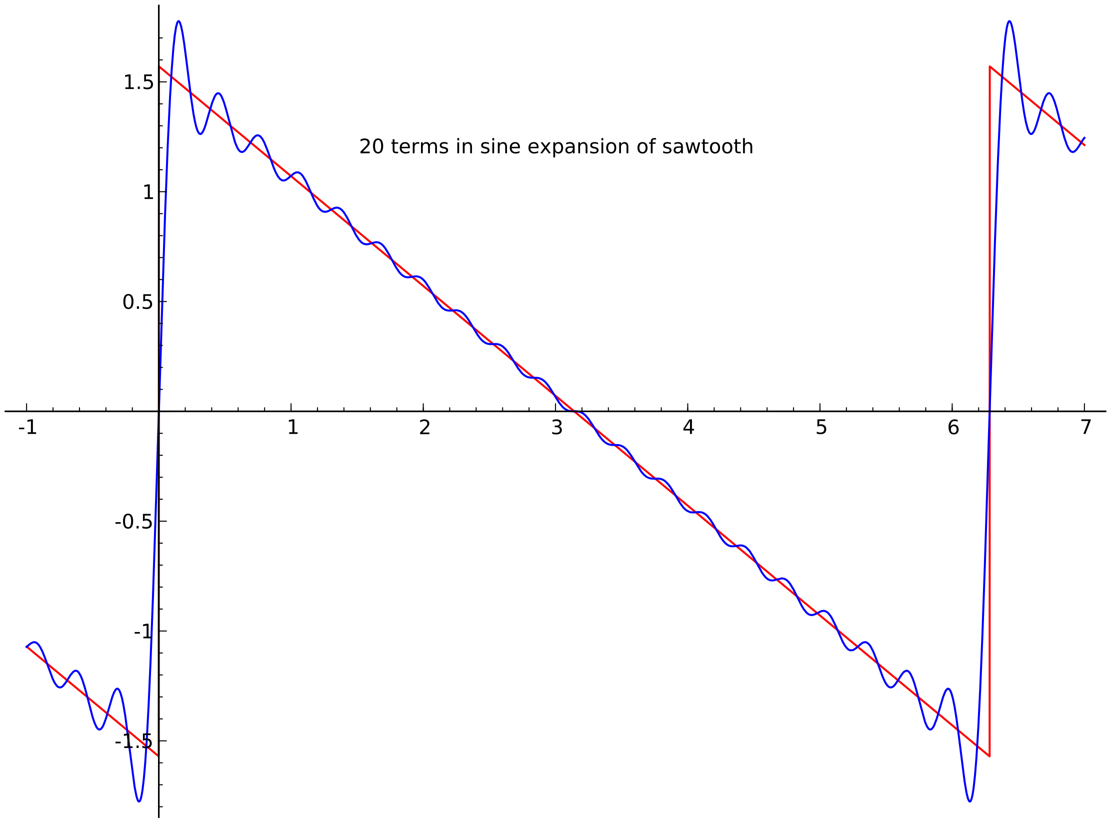


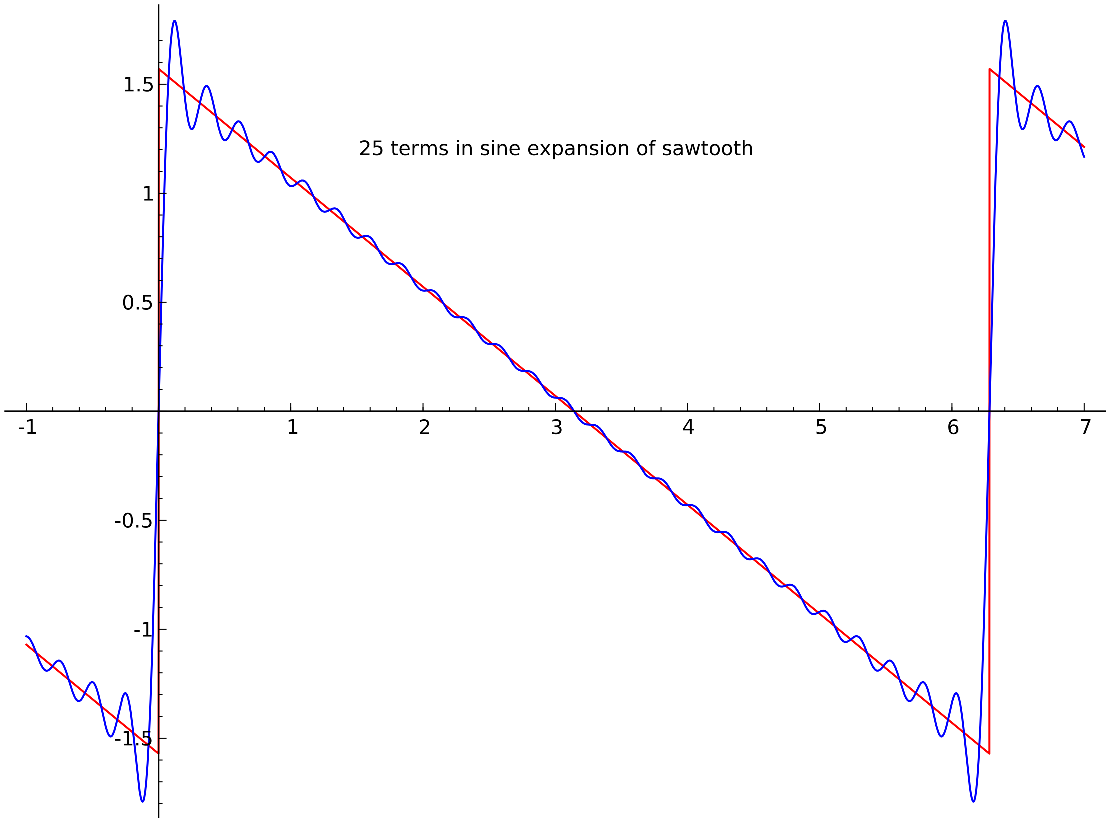
4 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7

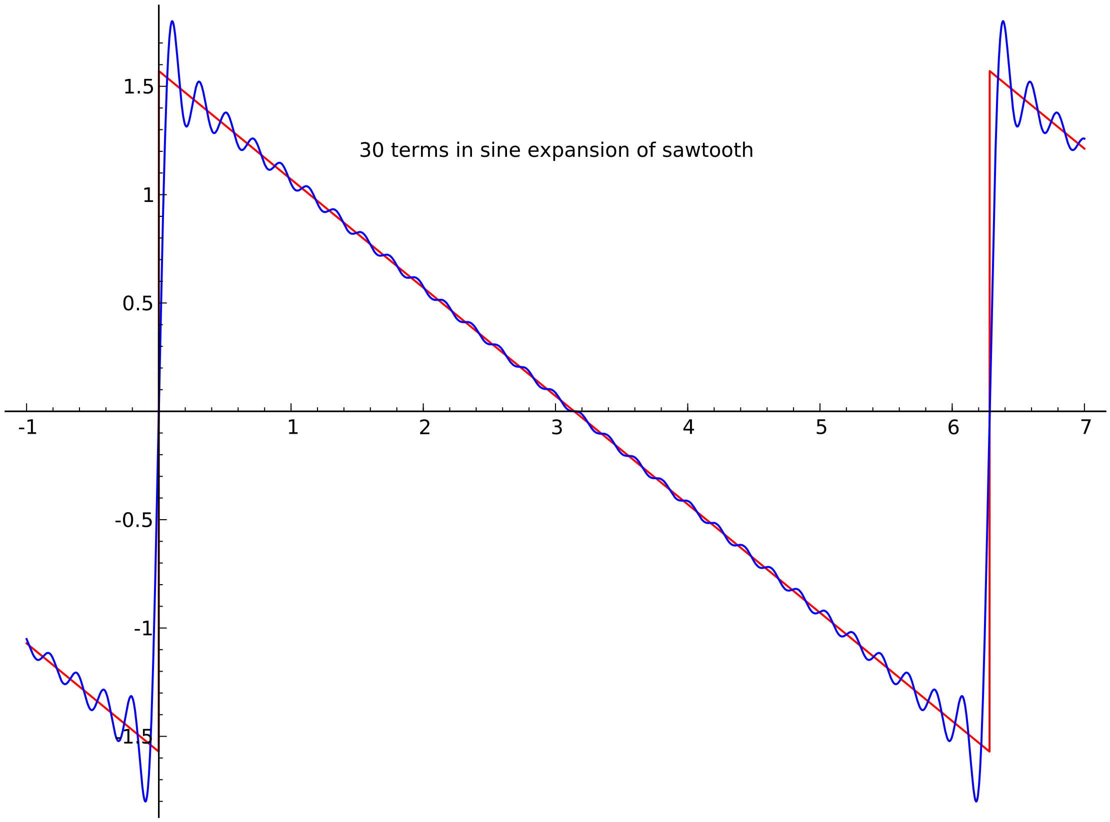

7 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7


11 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7


15 terms in sine expansion of sawtooth


20 terms in sine expansion of sawtooth


25 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7


30 terms in sine expansion of sawtooth
1.5
1
0.5
-0.5
-1
-1.5
-1
1
2
3
4
5
6
7

SAGE code to generate PDFs:

```
def sawtooth(x):
    if (0 <= x < 2*math.pi):
        return math.pi/2 - x/2
    elif (2*math.pi <= x):
        return 3*math.pi/2 - x/2
    elif (-2*math.pi < x <0):
        return - math.pi/2 - x/2
    else:
        return

def f(x,n):
    outp=0
    for i in [1..n]:
        outp += sin(i*x)/i
    return outp

for n in [1..30]: # generate the first 30 partial sums

    def g(x): return f(x,n)

    G = plot(sawtooth, xmin=-1, xmax=7, rgbcolor=(1,0,0))
    G += plot(g, xmin=-1, xmax=7)

    G += text(n.str() + " terms in sine expansion of sawtooth", (3,1.2), rgbcolor=(0,0,0))

    G.save("Wilbraham-Gibbs" + n.str() + ".pdf") # save to file
```

Shipman's device:

```
\input miniltx.tex
\def\Gin@driver{pdftex.def}
\input graphicx.sty
\resetatcatcode
```

Bibliography

[Gibbs 1898-9] J.W. Gibbs, *Fourier's series*, Nature **59** (1898), 200; (1899), 606.

[Wilbraham 1848] H. Wilbraham, *On a certain periodic function*, The Cambridge and Dublin Math. J. **3** (1848), 198-201.