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# Fourier analysis on finite abelian groups

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1. Fourier analysis on finite abelian groups
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There are Fourier expansions on finite abelian groups essentially identical in form to Fourier expansions of periodic functions on the real line. This follows from the spectral theory of unitary operators on finite-dimensional complex vector spaces.

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## 1. Fourier analysis on finite abelian groups

The main conclusion, existence of *Fourier expansions* on finite abelian groups, can be given independently of most of the ideas of the proofs.

Let  $G$  be a finite abelian group, and  $L^2(G)$  the complex vectorspace of complex-valued functions<sup>[1]</sup> on  $G$ , with inner product<sup>[2]</sup>

$$\langle f, \varphi \rangle = \sum_{x \in G} f(x) \overline{\varphi(x)}$$

A **character**  $\chi$  on a group is a group homomorphism<sup>[3]</sup>

$$\chi : G \longrightarrow \mathbb{C}^\times$$

Let  $\widehat{G}$  be the collection of characters  $\chi : G \rightarrow \mathbb{C}^\times$ . For  $f$  a complex-valued function on  $G$ , the **Fourier transform**  $\widehat{f}$  of  $f$  is the function on  $\widehat{G}$  defined by

$$\widehat{f}(\chi) = \langle f, \chi \rangle \quad (\text{for } \chi \in \widehat{G})$$

The **Fourier expansion** or **Fourier series** of  $f$  is

$$f \sim \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi) \chi$$

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[1] In general, for a space  $X$  with some sort of *integral* on it, the notation  $L^2(X)$  means functions  $f$  so that  $\int_X |f|^2 < \infty$ . On *finite* sets integrals become sums, possibly weighted, and this finiteness condition becomes *vacuous*. Nevertheless, it is good to use this notation as a reminder of the larger context.

[2] The notation  $L^2(G)$  is meant to suggest the presence of the *inner product* on this space of functions. On a general space  $X$  with an integral, the inner product is  $\langle f_1, f_2 \rangle = \int_X f_1 \overline{f_2}$ .

[3] The term *character* has different meanings in different contexts. The simplest sense is a group homomorphism to  $\mathbb{C}^\times$ . However, an equally important use is for the *trace* of a group homomorphism  $\rho : G \rightarrow GL_n(k)$  from  $G$  to invertible  $n$ -by- $n$  matrices with entries in a field  $k$ . In the latter sense,

$$(\text{character of } \rho)(g) = \text{trace}(\rho(g))$$

For *infinite*-dimensional representations, further complications appear. Except from context, there is no way to know which sense is intended.

[1.0.1] **Theorem:** On a finite abelian group, the Fourier expansion of a complex-valued function  $f$  represents  $f$ , in the sense that, for every  $g \in G$ ,

$$f(g) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi) \chi(g)$$

The elements of  $\widehat{G}$  form an orthogonal basis for  $L^2(G)$ . In particular, the Fourier coefficients are *unique*.

[1.0.2] **Remark:** What are we *not* doing? The theorem asserts nothing directly about the collection  $\widehat{G}$  of characters of  $G$ . Its proof uses no information about these characters. Its proof uses nothing about the structure theorem for finite abelian groups. All that is used is a *spectral theorem*.

The proof is in the following paragraphs.

[1.1] **Translation action on functions** The distinguishing feature of functions on a *group* is that the group acts on itself by right or left multiplication (or whatever the group operation is called), thereby moving around the *functions on it*.

The group operation in  $G$  will be written *multiplicatively*, not *additively*, to fit better with other notational conventions.

The group  $G$  acts on the vector space  $L^2(G)$  of functions on itself by *translation*: for  $g \in G$ , the *translate*  $T_g f$  of a function  $f$  by  $g$  is the function on  $G$  defined by<sup>[4]</sup>

$$(T_g f)(x) = f(xg) \quad (\text{for function } f, \text{ and } x, g \in G)$$

The maps-on-function  $T_g$  are vectorspace endomorphisms of the vectorspace of functions on  $G$ :

$$\begin{cases} T_g(f_1 + f_2)(x) = (f_1 + f_2)(xg) = f_1(xg) + f_2(xg) = T_g f_1(x) + T_g f_2(x) & (\text{additivity}) \\ T_g(c \cdot f)(x) = (c \cdot f)(xg) = c(f(xg)) = (c \cdot (T_g f))(x) & (\text{scalar } c) \end{cases}$$

To reduce clutter, the action of  $g \in G$  on functions  $f$  may be written simply  $gf$  or  $g \cdot f$ . The **associativity** property

$$(gh)f = g(hf) \quad (\text{for } g, h \in G, \text{ function } f)$$

comes from the associativity of the group operation itself:

$$((gh)f)(x) = f(x(gh)) = f((xg)h) = (hf)(xg) = (g(hf))(x)$$

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[4] For *non-abelian* groups  $G$ , there are two translation actions, namely, *left* and *right*

$$\begin{cases} T_g^{\text{right}} f(x) = f(xg) \\ T_g^{\text{left}} f(x) = f(g^{-1}x) \end{cases}$$

The inverse in the left translation is for *associativity*

$$T_{gh}^{\text{left}} f = T_g(T_h f)$$

For abelian groups, the two translation actions become essentially the same thing, insofar as either alone gives all the information that the two together could give. Also, for abelian groups, the inverse in the definition of the left translation action loses some of its significance, since for *abelian* groups  $g \rightarrow g^{-1}$  is a group automorphism.

The associativity property is equivalent to the assertion that the map  $g \rightarrow T_g$  is a *group homomorphism* from  $G$  to  $\mathbb{C}$ -linear automorphisms of  $L^2(G)$  (and that the identity element of  $g$  acts trivially).

Since  $g \rightarrow T_g$  is a group homomorphism, the abelian-ness of  $G$  implies that the linear maps  $T_g, T_h$  commute: since  $gh = hg$ ,

$$T_g \circ T_h = T_{gh} = T_{hg} = T_h \circ T_g \quad (\text{for all } g, h \in G)$$

Since  $G$  is finite, there is a positive integer  $N$  such that, for all  $g \in G$ ,  $g^N = e \in G$ . Thus,

$$\chi(g)^N = \chi(g^N) = \chi(e) = 1 \quad (\text{for any } \chi \in \widehat{G})$$

That is, the values of  $\chi$  lie on the unit circle in  $\mathbb{C}^\times$ , so  $|\chi(g)| = 1$ . In particular,  $\chi$  is **unitary** in the sense that

$$\chi(g)^{-1} = \overline{\chi(g)}$$

We claim that the linear operators  $T_g$  are also *unitary*, in the sense that

$$\langle T_g f, T_g F \rangle = \langle f, F \rangle \quad (\text{for } g \in G, \text{ functions } f, F)$$

To prove this, compute directly:

$$\langle T_g f, T_g F \rangle = \sum_{h \in G} (T_g f)(h) \overline{(T_g F)(h)} = \sum_{h \in G} f(hg) \overline{F(hg)}$$

Change variables in the sum, by replacing  $h$  by  $hg^{-1}$ . Here the fact that  $G$  is a *group* is used:  $g^{-1}$  exists, and is closed under the group law:

$$\sum_{h \in G} f(hg) \overline{F(hg)} = \sum_{h \in G} f(h) \overline{F(h)} = \langle f, F \rangle$$

proving the unitarity.

For a *single* linear operator  $T$  on a complex vector space  $V$ , and for a complex number  $\lambda$ , the  $\lambda$ -eigenspace  $V_\lambda$  of  $T$  on  $V$  is

$$V_\lambda = \{v \in V : Tv = \lambda \cdot v\}$$

The **Spectral Theorem** for a *single* unitary operator  $T$  on a finite-dimensional complex vector space with inner product  $\langle, \rangle$  asserts that  $V$  decomposes as an orthogonal direct sum of eigenspaces of  $T$ :

$$V = \bigoplus_{\lambda} V_\lambda \quad (\text{orthogonal direct sum})$$

We claim that another unitary operator  $S$  *commuting with  $T$*  stabilizes the  $T$ -eigenspaces  $V_\lambda$ . To see this, take  $v \in V_\lambda$ :

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot Sv$$

since the linearity of  $S$  implies that  $S$  commutes with scalar multiplication.

This sets up an induction, as follows. We want to prove that a group  $H$  of *commuting* unitary operators on a finite-dimensional complex vectorspace  $V$  with hermitian inner product  $\langle, \rangle$  has *simultaneous* eigenspaces  $V_\lambda$  on  $V$  giving an orthogonal direct sum decomposition  $V = \bigoplus_{\lambda} V_\lambda$ .

In this situation, the notion of *eigenvalue* must be a little more complicated than individual numbers: for *each*  $T \in H$ , there must be a number  $\lambda_T \in \mathbb{C}$ . That is, an *eigenvalue* is really a *map*  $T \rightarrow \lambda_T$  from  $H$  to  $\mathbb{C}$ .

In this context, for two eigenvalues  $\lambda, \mu$  to be *distinct* means that  $\lambda_T \neq \mu_T$  for *some*  $T \in H$  (not necessarily for *all*  $T \in H$ ).

Now we do the induction. Suppose we have the conclusion for vector spaces of dimension  $< n$ . Let  $V$  be of dimension  $n$ . First, a silly case: if all operators  $T \in H$  are *scalar*, then *every* vector is a simultaneous eigenvector for all the operators in  $H$ , and we are done. So now consider the (serious) case that *not* all operators in  $H$  are scalar. Let  $T \in H$  be a non-scalar operator. By the spectral theorem for unitary operators,  $V$  has an orthogonal decomposition into eigenspaces for  $T$ , implicitly with different eigenvalues. Since  $T$  is non-scalar, every one of these eigenspaces has dimension  $< n$ . By induction, and by the fact that the operators all commute, each such eigenspace decomposes as an orthogonal direct sum of *simultaneous* eigenspaces for  $H$ . Thus, the whole space  $V$  is an orthogonal direct sum of simultaneous eigenspaces. This completes the induction.

Thus, in the case that  $H$  is the group of automorphisms  $T_g$  with  $g \in G$  and  $V = L^2(G)$ ,

$$L^2(G) = \bigoplus_{\lambda} L^2(G)_{\lambda} \quad (\text{with simultaneous eigenvalues } \lambda : G \rightarrow \mathbb{C}^{\times})$$

In fact, for  $H$  an abelian group of unitary automorphisms, we claim that the eigenvalues  $T \rightarrow \lambda_T$  for  $T \in H$  are *group homomorphisms*  $H \rightarrow \mathbb{C}^{\times}$ : for  $S, T \in H$ , and for  $v$  in the  $\lambda$ -eigenspace,

$$\lambda_{ST} \cdot v = (ST)(v) = S(T(v)) = S(\lambda_T \cdot v) = \lambda_T \cdot S(v) = \lambda_T \cdot \lambda_S \cdot v$$

Thus,

$$\lambda_{ST} = \lambda_T \cdot \lambda_S \quad (\text{for all } S, T \in H, \text{ simultaneous eigenvalue } \lambda : H \rightarrow \mathbb{C}^{\times})$$

That is, for *groups* of automorphisms, *eigenvalues are characters*. We'll write  $V_{\chi}$  instead of  $V_{\lambda}$  to emphasize this information, and for  $L^2(G)$  write

$$L^2(G) = \bigoplus_{\chi \in \widehat{G}} L^2(G)_{\chi}$$

We will see that each  $L^2(G)_{\chi}$  is one-dimensional, spanned by  $\chi$  itself.

On one hand, every  $\chi \in \widehat{G}$  is a complex-valued function on  $G$ , so is in  $L^2(G)$ . In fact, we claim that  $\chi \in V_{\chi}$ :

$$(T_g \chi)(h) = \chi(hg) = \chi(h) \chi(g) = \chi(g) \chi(h) \quad (\text{since } \mathbb{C}^{\times} \text{ is abelian})$$

On the other hand, we claim that  $V_{\chi}$  is *exactly* scalar multiples  $\mathbb{C} \cdot \chi$  of  $\chi$ . To see this, let  $f \in V_{\chi}$ . With  $e$  the identity element of  $G$ ,

$$f(g) = f(e \cdot g) = (T_g f)(e) = \chi(g) \cdot f(e) = f(e) \cdot \chi(g)$$

That is,

$$f = f(e) \cdot \chi \quad (\text{for } f \in V_{\chi})$$

By the orthogonality of  $V_{\chi}$  and  $V_{\tau}$  for distinct  $\chi, \tau$ , the characters are an *orthogonal basis* for  $L^2(G)$ . Their lengths are readily determined, using the earlier-noted *unitariness*  $\bar{\chi} = \chi^{-1}$ :

$$\langle \chi, \chi \rangle = \sum_{g \in G} \chi(g) \cdot \bar{\chi}(g) = \sum_{g \in G} \chi(g) \cdot \chi(g)^{-1} = \sum_{g \in G} 1 = |G|$$

Any  $f \in L^2(G)$  can be written as a linear combination of orthogonal basis elements  $e_i$

$$f = \sum_i \frac{\langle f, e_i \rangle \cdot e_i}{\langle e_i, e_i \rangle}$$

Using the orthogonal basis  $\chi \in \widehat{G}$ ,

$$f = \sum_{\chi \in \widehat{G}} \frac{\langle f, \chi \rangle \cdot \chi}{\langle \chi, \chi \rangle} = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \langle f, \chi \rangle \cdot \chi$$

This is an equality of functions on the finite set  $G$ , and  $\widehat{f}(\chi)$  is defined to be  $\langle f, \chi \rangle$ , so

$$f(g) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \langle f, \chi \rangle \cdot \chi(g) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi) \cdot \chi(g) \quad (\text{for all } g \in G)$$

This proves the representability of functions on finite abelian groups by their Fourier series. ///

## 2. Appendix: spectral theorem for unitary operators

Let  $V$  be a finite-dimensional complex vector space with a hermitian inner product  $\langle \cdot, \cdot \rangle$ . A linear map  $T : V \rightarrow V$  is *unitary* if it preserves the inner product, in the sense that

$$\langle Tv, Tw \rangle = \langle v, w \rangle \quad (\text{for all } v, w \in V)$$

Thus, the *adjoint*  $T^*$  of a unitary operator  $T$  has the property

$$\langle v, w \rangle = \langle Tv, Tw \rangle = \langle v, T^*Tw \rangle$$

Subtracting,  $\langle v, T^*Tw - w \rangle = 0$  for all  $v$ , so  $T^*Tw = w$  for all  $w \in V$ . That is, unitary  $T$  is *invertible*, and  $T^* = T^{-1}$ . This also shows that  $T^*T = TT^*$ .

The *inverse* of a unitary operator is unitary, since

$$\langle T^{-1}v, T^{-1}w \rangle = \langle T^*v, T^{-1}w \rangle = \langle v, TT^{-1}w \rangle = \langle v, w \rangle$$

Eigenvalues  $\lambda$  of a unitary operator  $T$  are of absolute value 1, since for a  $\lambda$ -eigenvector  $v$

$$\lambda \bar{\lambda} \langle v, v \rangle = \langle \lambda v, \lambda v \rangle = \langle Tv, Tv \rangle = \langle v, v \rangle$$

In particular, eigenvalues  $\lambda$  are non-zero, and  $\lambda^{-1} = \bar{\lambda}$ .

Given  $\lambda \in \mathbb{C}$ , let

$$\lambda\text{-eigenspace of } T = V_\lambda = \{v \in V : Tv = \lambda \cdot v\}$$

[2.0.1] **Theorem:** The vectorspace is an *orthogonal* direct sum

$$V = \bigoplus_{\lambda} V_\lambda \quad (\text{eigenspaces of unitary } T)$$

*Proof:* We grant ourselves the more elementary fact that, because  $V$  is finite-dimensional and  $\mathbb{C}$  is algebraically closed, there is at least *one* one eigenvalue  $\lambda$  and non-zero eigenvector  $v$  for  $T$ . Thus, the  $\lambda$ -eigenspace  $V_\lambda$  is not  $\{0\}$ .

Now the unitariness is used, to set up an induction on dimension. We claim that  $T$  stabilizes the orthogonal complement

$$V_\lambda^\perp = \{w \in V : \langle w, v \rangle = 0 \text{ for all } v \in V_\lambda\}$$

Indeed, for  $w$  in that orthogonal complement and  $v \in V_\lambda$ ,

$$\langle Tw, v \rangle = \langle w, T^*v \rangle = \langle w, T^{-1}v \rangle = \langle w, \lambda^{-1}v \rangle = \lambda \langle w, v \rangle = 0 \quad (\text{for all } v \in V_\lambda)$$

The restriction of a unitary operator  $T$  to a  $T$ -stable subspace is obviously still unitary. By induction on the dimension of the vectorspace,  $V_\lambda^\perp$  is an orthogonal direct sum of  $T$ -eigenspaces:  $V_\lambda^\perp = \bigoplus_\mu V'_\mu$ . Then

$$V = V_\lambda \oplus \bigoplus_\mu V'_\mu$$

is the orthogonal direct sum decomposition of the whole space. ///

### 3. Appendix: cancellation lemma

The orthogonality of distinct characters can be proven directly.

Let  $G$  be a finite group, not necessarily abelian. First, we have the **cancellation lemma**:

[3.0.1] **Lemma**: For a non-trivial group homomorphism  $\sigma : G \rightarrow \mathbb{C}^\times$ ,

$$\sum_{g \in G} \sigma(g) = 0$$

*Proof*: Let  $g_o \in G$  be such that  $\sigma(g_o) \neq 1$ . Then

$$\sum_{g \in G} \sigma(g) = \sum_{g \in G} \sigma(g_o g) = \sum_{g \in G} \sigma(g_o) \sigma(g) = \sigma(g_o) \sum_{g \in G} \sigma(g)$$

by replacing  $g$  by  $g_o g$  in the sum, using the fact that left multiplication by  $g_o$  is a bijection of  $G$  to itself. Subtracting,

$$(1 - \sigma(g_o)) \cdot \sum_{g \in G} \sigma(g) = 0$$

Since  $\sigma(g_o) \neq 1$ , necessarily the sum is 0. ///

[3.0.2] **Corollary**: Let  $\sigma \neq \tau$  be group homomorphisms  $G \rightarrow \mathbb{C}^\times$ . Then

$$\sum_{g \in G} \sigma(g) \bar{\tau}(g) = 0$$

*Proof*: Since  $G$  is finite, there is  $N$  such that  $g^N = e$  for every  $g \in G$ . Thus,

$$\tau(g)^N = \tau(g^N) = \tau(e) = 1$$

Thus,  $\tau(g)$  is a root of unity, and  $|\tau(g)| = 1$ . In particular,  $\bar{\tau}(g) = \tau(g)^{-1}$ . Then  $\sigma \bar{\tau} = \sigma \tau^{-1}$  is a character of  $G$ , and is not the trivial character. The previous lemma gives the vanishing. ///