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The product expansion $\Delta(z) = e^{2\pi iz} \prod (1 - e^{2\pi inz})^{24}$

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1. Siegel's proof
2. Weil's proof

[Siegel 1954] showed a very simple, if somewhat unmotivated, argument for the product expansion of $\Delta(z)$. [Weil 1968] reproved the product expansion by a more complicated method related to the *converse theorems* in [Weil 1967], the latter arising as plausibility checks on the Taniyama-Shimura conjecture.

We reproduce both arguments. As expected, both make heavy use of various coincidences. Both use the one-dimensionality of the space of holomorphic cusp forms of weight 12 for $SL_2(\mathbb{Z})$ and generation of $SL_2(\mathbb{Z})$ by translation $z \rightarrow z + 1$ and inversion $z \rightarrow -1/z$. A function of $e^{2\pi iz}$ is clearly invariant under $z \rightarrow z + 1$. It remains to prove that the product expression must be proven to have the functional equation of a weight 12 modular form under $z \rightarrow -1/z$.

1. Siegel's proof

Siegel's argument is simple but *ad-hoc*. With η the 24^{th} root of Δ , with $q = e^{2\pi iz}$, taking a logarithm,

$$\frac{1}{12} \pi iz - \log \eta(z) = - \sum_{\ell=1}^{\infty} \log(1 - q^\ell) = \sum_{k, \ell \geq 1} \frac{1}{k} q^{k\ell} = \sum_{k \geq 1} \frac{1}{k} \cdot \frac{q^k}{1 - q^k} = \sum_{k \geq 1} \frac{1}{k} \cdot \frac{1}{q^{-k} - 1}$$

This suggests proving the functional equation in the form

$$\pi i \frac{z + z^{-1}}{12} + \frac{1}{2} \log \frac{z}{i} = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{e^{-2\pi ikz} - 1} - \frac{1}{q^{-2\pi ik/z} - 1} \right)$$

Let

$$f(w) = \cot w \cdot \cot w/z$$

and let ν run over values $(n + \frac{1}{2})\pi$ for $0 \leq n \in \mathbb{Z}$. Then $f(\nu w)/w$ has *simple* poles at $w = \pm \pi k/\nu$ and at $w = \pm \pi k z/\nu$, with respective residues

$$\frac{1}{\pi k} \cot \frac{\pi k}{z} \quad \text{and} \quad \frac{1}{\pi k} \cot \pi k z \quad (\text{for } k = 1, 2, 3, \dots)$$

and a *triple* pole at $w = 0$ with residue $-\frac{1}{3}(z + z^{-1})$. Let γ be the path tracing counter-clockwise the outline of the parallelogram with vertices $1, z, -1, -z$. By residues,

$$\pi \frac{z + z^{-1}}{12} + \int_{\gamma} f(\nu w) \frac{dw}{8w} = \frac{i}{2} \sum_{k \geq 1} \frac{1}{k} (\cot \pi k z + \cot \pi k/z) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{q^{-k} - 1} - \frac{1}{q^{-k/z} - 1} \right)$$

The parameters n or ν only appear in the contour integral on the left-hand side. To evaluate it, let as $n \rightarrow +\infty$. In this limit, $f(\nu w)$ is uniformly bounded on γ , and has limiting values on the sides (excluding the vertices, where there are discontinuities) $1, -1, 1, -1$, respectively. The limit of the contour integral is

$$\int_{\gamma} f(\nu w) \frac{dw}{8w} = \left(\int_1^z - \int_z^{-1} + \int_{-1}^{-z} - \int_{-z}^1 \right) \frac{dw}{w} = 4 \log \frac{z}{i}$$

This gives the functional equation. ///

2. Weil's proof

As in [Weil 1968], consider the Dirichlet series^[1]

$$L(s) = \zeta(s) \cdot \zeta(s+1) = \sum_{m,n} \frac{1}{m} \frac{1}{(mn)^s} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{1}{d} \right) \frac{1}{n^s}$$

The *completed* version

$$\Lambda(s) = (2\pi)^{-s} \Gamma(s) L(s)$$

has functional equation inherited from $\zeta(s)$:

$$\Lambda(-s) = \Lambda(s)$$

Noting that $\zeta(2)/2\pi = \pi/12$,

$$\Lambda(s) = \frac{\pi/12}{s-1} - \frac{1}{2s^2} - \frac{\pi/12}{s+1} + (\text{holomorphic})$$

The power series in $q = e^{2\pi iz}$ with the same coefficients is

$$F(z) = \sum_{m,n} \frac{1}{m} q^{mn} = \sum_n \left(\sum_m \frac{1}{m} (q^n)^m \right) = - \sum_n \log(1 - q^n)$$

Recall Dedekind's eta-function

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

Then

$$F(z) = - \left(\frac{\log q}{24} + \sum_n \log(1 - q^n) \right) + \frac{\log q}{24} = - \log \eta + \frac{\pi iz}{12}$$

From the obvious Fourier-Mellin transform relation

$$\Lambda(s) = \int_0^{\infty} y^s F(iy) \frac{dy}{y} \quad (\text{for } \operatorname{Re}(s) > 1)$$

Fourier-Mellin inversion gives

$$F(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (z/i)^{-s} \Lambda(s) ds \quad (\text{for } \sigma > 1)$$

Following Hecke and Weil, move the vertical line to $\operatorname{Re}(s) = -\sigma$, picking up residues at 1, 0, -1:

$$\begin{aligned} F(z) &= \int_{-\sigma-i\infty}^{-\sigma+i\infty} (z/i)^{-s} \Lambda(s) ds + \left(\frac{\pi}{12} \cdot (z/i)^{-1} + \frac{1}{2} \log(z/i) - \frac{\pi}{12} \cdot (z/i) \right) \\ &= \int_{-\sigma-i\infty}^{-\sigma+i\infty} (z/i)^{-s} \Lambda(s) ds + \frac{\pi i}{12z} + \frac{1}{2} \log(z/i) - \frac{\pi z}{12i} \end{aligned}$$

[1] Weil was well aware that $L(s)$ is essentially the Mellin transform of the *constant coefficient* in the Laurent expansion in s at $s = 1$ of the Eisenstein series $E_s = \sum \frac{y^s}{|cz+d|^{2s}}$. The nature of that constant coefficient is part of the *Kronecker limit formula*.

The functional equation $\Lambda(-s) = \Lambda(s)$ allows conversion of the integral on $\operatorname{Re}(s) = -\sigma$ into

$$\frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} (z/i)^{-s} \Lambda(-s) ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{-1}{z/i}\right)^{-s} \Lambda(s) ds = F(-1/z)$$

That is,

$$F(z) = F(-1/z) + \frac{\pi i}{12z} + \frac{1}{2} \log(z/i) - \frac{\pi z}{12i}$$

Using $F(z) = \pi iz/12 - \log \eta(z)$, this is

$$\frac{\pi iz}{12} - \log \eta(z) = \frac{\pi i(-1/z)}{12} - \log \eta(-1/z) + \frac{\pi i}{12z} + \frac{1}{2} \log(z/i) - \frac{\pi z}{12i}$$

which simplifies to

$$\log \eta(z) = \log \eta(-1/z) - \frac{1}{2} \log(z/i)$$

Exponentiating and taking the 24th power:

$$\eta^{24}(z) = \eta^{24}(-1/z) \cdot (z/i)^{-12}$$

or

$$\eta^{24}(-1/z) = z^{12} \cdot \eta^{24}(z)$$

That is, η^{24} has the two functional equations

$$\eta^{24}(z+1) = \eta^{24}(z) \quad \eta^{24}(-1/z) = z^{12} \cdot \eta^{24}(z)$$

and goes to 0 as $\operatorname{Im}(z) \rightarrow +\infty$. Since $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

giving the maps $z \rightarrow -1/z$ and $z \rightarrow z+1$, evidently η is a *holomorphic cuspform* of weight 12, with leading Fourier coefficient 1. Thus, it is $\Delta(z)$, and we have the product expansion

$$\Delta(z) = \eta^{24}(z) = e^{2\pi iz} \prod_{n \geq 1} (1 - e^{2\pi inz})$$

[2.0.1] Remark: In fact, Weil's connection between a simple converse theorem and a product formula is anomalous.

[Hurwitz 1881] A. Hurwitz, *[thesis]*, Math. Ann. **18** (1881), 528-592.

[Siegel 1954] C.L. Siegel, *A simple proof of $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/i}$* , Mathematika **1** (1954), p. 4 (Ges. Abh. Bd. III, Springer 1966, p. 188).

[Weil 1967] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149-156.

[Weil 1968] A. Weil, *Sur une formule classique*, J. Math. Soc. Japan **20** (1968), 400-402. (*Oeuvres Sci.* Vol. III, Springer, New York, 1979, 198-200.)
