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Harmonic analysis on spheres

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Harmonic analysis on the *circle* $S^1 \approx \mathbb{R}/\mathbb{Z}$ uses Fourier series expansions of functions and generalized functions

$$f \sim \sum_{n\in\mathbb{Z}} \Big(\int_{\mathbb{R}/\mathbb{Z}} f(t)\, e^{-2\pi i n t}\, dt \Big)\, e^{2\pi i n x}$$

The exponential functions are *simple, special* functions, in several regards. Exponentials are *eigenfunctions* of the translation-invariant differential operator $\frac{d}{dx}$, and the maps $x \to e^{i\xi x}$ for $\xi \in \mathbb{C}$ are *continuous group homomorphisms* to $\mathbb{C}^\times$. The senses in which Fourier expansions converge or diverge play a key role.

Harmonic analysis on the *line* uses Fourier inversion expansions of functions and generalized functions

$$f \sim \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(t)\, e^{-2\pi i \xi t}\, dt \Big)\, e^{2\pi i \xi x}\, d\xi$$

This is more complicated than on the circle, because the line is not *compact*. On $\mathbb{R}$ the exponential functions, still eigenfunctions for $\frac{d}{dx}$ and still giving group homomorphisms to $\mathbb{C}^\times$, are no longer in $L^2(\mathbb{R})$. Similarly, Fourier inversion expresses functions as *integrals* of exponential functions, not as *sums*. Senses of convergence of Fourier integrals are commensurately more complicated than Fourier series.

Both the circle and the real line are *abelian groups*. The abelian-ness simplifies the harmonic analysis. Unsurprisingly, harmonic analysis related to *non-abelian* groups is more complicated. One immediate complication is that for non-abelian G many subgroups H are not *normal*, equivalently, quotients G/H are *not groups*. By contrast, the quotient $\mathbb{R}/\mathbb{Z}$ of the real line by the integers presents the circle as a *group*, not merely as a *quotient space* of a group.

A tractable non-abelian, but still *compact*, situation is spheres $S^{n-1} \subset \mathbb{R}^n$, which are quotients [1]

$$S^{n-1} \approx SO(n-1) \backslash SO(n)$$

of rotation groups $SO(n)$. Spheres themselves are rarely groups, but *are* acted-upon transitively by rotation groups. Specifying a rotation-invariant measure on S^{n-1} gives a function space

$$L^2(S^{n-1}) = \text{completion of } C^o(S^{n-1}) \text{ with respect to } |f|_{L^2} = \Big(\int_{S^{n-1}} |f|^2 \Big)^{1/2}$$

With corresponding hermitian inner-product

$$\langle f, F \rangle = \int_{S^{n-1}} f \cdot \overline{F}$$

[1] Rotation groups as *orthogonal groups* and *special orthogonal groups* are reviewed below.

the translation action $f \to g \cdot f$ of $g \in SO(n)$ on functions f by $(g \cdot f)(x) = f(xg)$ is *unitary*, meaning that it preserves the inner product: using the translation-invariance of the measure,

$$\langle g \cdot f, g \cdot F \rangle \;=\; \int_{S^{n-1}} f(xg)\,\overline{F}(xg)\;dx \;=\; \int_{S^{n-1}} f(x)\,\overline{F}(x)\;d(g^{-1}x) \;=\; \int_{S^{n-1}} f(x)\,\overline{F}(x)\;dx \;=\; \langle f, F \rangle$$

For example, *Weyl's equidistribution criterion* for point sets on S^{n-1} requires identification of a convivial orthonormal basis for $L^2(S^{n-1})$. In fact, we will prove an orthogonal decomposition

$$L^2(S^{n-1}) \;=\; \text{(completion of)} \bigoplus_{d=0}^{\infty} H_d \qquad\qquad \text{(orthogonal)}$$

into *finite-dimensional* $O(n)$-stable subspaces H_d. In particular,

$$H_d \;=\; \{\text{harmonic, degree } d \text{ homogeneous polynomials on } \mathbb{R}^n\}$$

where the *harmonic* condition is $\Delta f = 0$, with the usual Euclidean Laplacian

$$\Delta \;=\; \Delta^{\mathbb{R}^n} \;=\; \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

Letting proj_d be the orthogonal projection of $L^2(S^{n-1})$ to H_d, expansions of $f \in L^2(S^{n-1})$ as

$$f \;=\; \sum_{d=0}^{\infty} f_d \;=\; \sum_{d=0}^{\infty} \text{proj}_d(f) \qquad\qquad (\text{with } f_d \in H_d)$$

are sometimes called *Fourier-Laplace series*. On S^1, these expansions are readily compared to the usual Fourier expansions, as done below.

While L^2 convergence follows from simple generalities about Hilbert spaces, on S^{n-1} with $n-1 > 1$, *pointwise* convergence of Fourier-Laplace series is somewhat more complicated than for Fourier series, because the the worst-case sup-norms of orthonormal bases of H_d for $S^{n-1} \subset \mathbb{R}^n$ grow like d^{n-2} as d increases. We prove a Sobolev imbedding theorem: let f be in $L^2(S^{n-1})$ with Fourier-Laplace expansion $\sum_d f_d$ with $f_d \in H_d$. Suppose

$$\sum_d (1+d)^s \cdot |f_d|^2_{L^2} \;<\; +\infty \qquad\qquad (\text{for some } s > \dim S^{n-1})$$

Then f is *continuous*, and its Fourier-Laplace series converges *uniformly pointwise* to f. The left-hand side of the displayed inequality is the s^{th} *Sobolev norm* (squared).

Some ideas from the discussion of sup-norms can be used to show that the spaces H_d are *irreducible* for the action of $SO(n)$, that is, given a non-zero $f \in H_d$, the collection of finite linear combinations of translates $g \cdot f$ of f by $g \in SO(n)$ is all of H_d. That is, H_d is an *irreducible representation* of $SO(n)$.

The irreducibility of H_d enables an easy proof of *Hecke's identity*

$$\int_{\mathbb{R}^n} f(x)\, e^{-\pi\langle x,x\rangle} \cdot e^{-2\pi i\langle \xi,x\rangle}\; dx \;=\; i^{-d} \cdot f(\xi)\, e^{-\pi\langle \xi,\xi\rangle} \qquad\qquad (\text{for } f \in H_d)$$

by reducing to the case $f(x_1, \ldots, x_n) = (x_1 \pm ix_2)^d$, which allows direction computation.

Hecke's identity is essential to ascertain the behavior of *harmonic theta series*

$$\theta(z) \;=\; \theta_{n,f}(z) \;=\; \sum_{m=(m_1,\ldots,m_n)\in\mathbb{Z}^n} f(m)\, e^{-\pi i(m_1^2+\ldots+m_n^2)\,z} \qquad\qquad (\text{with } z \in \mathfrak{H})$$

under inversion $z \to -1/z$, to see that these theta series are *modular forms* of weight $\frac{m}{2} + d$. Indeed, for m odd, the weight is half-integral, a significantly more complicated case than the integral-weight case. For example, these theta series arise when carrying out a Weyl-criterion equidistribution analysis for integer lattice points projected to the unit sphere, as we see later.

1. *Calculus on spheres*

To use the rotational symmetry of spheres, we want *eigenfunctions* for rotation-invariant *differential operators* on spheres, and expect that these eigenfunctions will be the analogues of exponential functions on the circle or line. Thus, we must identify/construct rotation-invariant differential operators on spheres. Also, we need rotation-invariant *measures* or *integrals* on spheres. Rather than writing formulas in coordinates, we first describe these objects by their desired properties, then give constructions in terms of the ambient Euclidean space.

For $1 \le n \in \mathbb{Z}$, the usual unit $(n-1)$-sphere is

$$S \;=\; S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \;:\; x_1^2 + \ldots + x_n^2 = 1\}$$

Write $\Delta^S = \Delta^{S^{n-1}}$ for the desired rotation-invariant second-order [2] differential operator on functions on S, and let $f \to \int_S f$ denote the desired rotation-invariant (positive) integral. We call Δ^S the *Laplacian* on the sphere. All functions here are indefinitely differentiable. [3] Two desired properties are

$$\begin{array}{lcll} \int_{S^{n-1}} (\Delta^S f) \cdot \varphi & = & \int_{S^{n-1}} f \cdot (\Delta^S \varphi) & \textit{(self-adjointness)} \\ \int_{S^{n-1}} (\Delta^S f) \cdot \overline{f} & \le & 0 & \textit{(definiteness)} \end{array}$$

with equality only for f *constant*. We also assume that Δ^S has *real coefficients*, in the abstracted sense that

$$\overline{\Delta^S f} \;=\; \Delta^S \,\overline{f}$$

There is the natural complex hermitian inner product

$$\langle f, F \rangle \;=\; \int_{S^{n-1}} f \cdot \overline{F}$$

A typical linear algebra conclusion, via a typical argument:

[1.0.1] Corollary: Granting existence of invariant Δ^S and invariant measure on S^{n-1}, with the self-adjointness and definiteness properties, eigenvectors f, F for Δ^S with *distinct* eigenvalues are orthogonal with respect to the inner product $\langle , \rangle$. Any eigenvalues are *non-positive real* numbers.

Proof: Let $\Delta^S f = \lambda \cdot f$ and $\Delta^S F = \mu \cdot F$. Assume $\lambda \neq 0$ (or else interchange the roles of λ and μ). Then

$$\langle f, f \rangle \;=\; \frac{1}{\lambda} \int_S (\Delta^S f) \cdot \overline{f} \;=\; \frac{1}{\lambda} \int_S f \, \overline{\Delta^S f} \;=\; \frac{\overline{\lambda}}{\lambda} \int_S f \, \overline{f}$$

Since $\lambda \neq 0$, f is not identically 0, the integral of $f \cdot \overline{f}$ is not 0, and $\lambda = \overline{\lambda}$, so λ is *real*. The negative definiteness of Δ^S and positive-ness of the invariant measure on S give

$$\lambda \cdot \langle f, f \rangle \;=\; \int_S (\Delta^S f) \cdot \overline{f} \;<\; 0$$

[2] Unlike the circle $\mathbb{R}/\mathbb{Z}$ and line $\mathbb{R}$ with d/dx, there are no rotation-invariant *first-order* scalar-valued differential operators on higher-dimensional spheres. Fortunately, invariant *second-order* operators are ubiquitous, and can be derived from the Euclidean Laplacian.

[3] The notion of *differentiability* for functions on a sphere can be given in several ways, all equivalent. At one extreme, the most pedestrian is to declare a function f on S^{n-1} differentiable if the function $F(x) = f(x/|x|)$ on $\mathbb{R}^n - 0$ is differentiable on $\mathbb{R}^n - 0$. At the other extreme one gives S^{n-1} its usual structure of *smooth manifold*, which incorporates a notion of differentiable function. Happily, the choice of definition doesn't matter much, since we won't be attempting to directly *compute* derivatives, but only use *properties* of differentiation.

so $\lambda < 0$. Next,

$$\langle f, F\rangle \;=\; \frac{1}{\lambda}\int_S (\Delta^S f)\cdot \overline{F} \;=\; \frac{1}{\lambda}\int_S f\cdot \overline{\Delta^S F} \;=\; \frac{\overline{\mu}}{\lambda}\int_S f\cdot \overline{F}$$

The eigenvalues λ, μ are real, so for $\mu/\lambda \neq 1$ necessarily the integral is 0. ///

The standard *special orthogonal group* [4] is

$$SO(n) \;=\; \{g \in GL_n(\mathbb{R}) \,:\, g^\top g = 1_n \quad \text{and} \quad \det g = 1\}$$

and acts on S by *right* [5] matrix multiplication,

$$k \,\times\, x \;\longrightarrow\; xk \qquad\qquad (\text{for } x \in S^{n-1} \text{ and } k \in SO(n))$$

considering elements of $\mathbb{R}^n$ as *row* vectors. Refer to the action of elements of $SO(n)$ on $S = S^{n-1}$ as *rotations* or *translations.* [6] It is useful that for $g \in SO(n)$, inverting both sides of $g^\top g = 1$ gives $g^{-1}(g^\top)^{-1} = 1$, and then $1 = gg^\top$.

The usual inner product $\langle , \rangle$ on $\mathbb{R}^n$

$$\langle x, y\rangle \;=\; xy^\top \qquad\qquad (\text{for } \textit{row} \text{ vectors } x, y)$$

is preserved by $SO(n)$, so $SO(n)$ preserves lengths and angles:

$$\langle xk, yk\rangle \;=\; (xk)(yk)^\top \;=\; x\,(kk^\top)\,y^\top \;=\; x\,y^\top \;=\; \langle x, y\rangle$$

For any k such that $k^\top k = 1$

$$(\det k)^2 \;=\; \det k^\top \cdot \det k \;=\; \det(k^\top k) \;=\; \det 1_n \;=\; 1$$

Thus, $\det k = \pm 1$. Recall:

[1.0.2] Claim: The action of $SO(n)$ on S^{n-1} is *transitive.* [7]

Proof: We show that, given $x \in S$ there is $k \in SO(n)$ such that $e_1 k = x$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. That is, we construct $k \in SO(n)$ such that the top row of k is x. Indeed, complete x to an $\mathbb{R}$-basis $x, x_2, x_3, \ldots, x_n$ for $\mathbb{R}^n$. Then apply the *Gram-Schmidt* process [8] to find an orthonormal (with respect to the standard inner product) basis $x, v_2, \ldots, v_n$ for $\mathbb{R}^n$. The condition $kk^\top = 1$, when expanded, is

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[4] The *full* orthogonal group is defined by $g^\top g = 1_n$ *without* the determinant condition. The determinant condition preserves *orientation*, however orientation is defined. The modifier *special* refers to the determinant condition.

[5] The choice of *right* matrix multiplication on *row* vectors is not essential, but *right* action on vectors has minor notational advantages when discussing functions on spaces of vectors.

[6] *Proof* that $SO(n)$ is exactly all the rotations would require a precise definition of *rotation*, which we avoid.

[7] Recall that *transitivity* means that for all $x, y \in S$ there is $g \in O(n)$ such that $gx = y$. If $SO(n)$ is all rotations of the sphere, then physical intuition makes the transitivity plausible. But this requires *two* assumptions: that all rotations are given by $SO(n)$, and that intuition about rotations is accurate in higher dimensions. We can do better.

[8] Recall that, given a basis $v_1, \ldots, v_n$ for a (real or complex) vector space with an inner product (real-symmetric or complex hermitian), the Gram-Schmidt process produces an *orthogonal* or *orthonormal* basis, as follows. Replace v_1 by $v_1/|v_1$ to give it length 1. Then replace v_2 first by $v_2 - \langle v_2, v_1\rangle v_1$ to make it orthogonal to v_1 and then by $v_2/|v_2|$ to give it length 1. Then replace v_3 first by $v_3 - \langle v_3, v_1\rangle v_1$ to make it orthogonal to v_1, then by $v_3 - \langle v_3, v_2\rangle v_2$ to make it orthogonal to v_2, and then by $v_3/|v_3|$ to give it length 1. And so on.

the assertion that the rows of k form an orthonormal basis, so taking $x, v_2, \ldots, v_n$ as the rows of k, we have k such that $e_1 k = x$. As noted just above, the determinant of this k is ± 1. To ensure that it is 1, replace v_n by $-v_n$ if necessary. This still gives $e_1 k = x$, giving the transitivity. ///

The *isotropy group* $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\text{(isotropy group)} \;=\; SO(n)_{e_n} \;=\; \{\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \;:\; A \in SO(n-1)\} \;\approx\; SO(n-1)$$

Thus, by transitivity, as $SO(n)$-spaces

$$S^{n-1} \;\approx\; SO(n-1)\backslash SO(n)$$

The action of $g \in SO(n)$ on *functions* f on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) is [9]

$$(g \cdot f)(x) \;=\; f(xg)$$

The rotation invariance conditions are

$$\begin{cases} \displaystyle\int_{S^{n-1}} g \cdot f \;=\; \int_{S^{n-1}} f & (\text{for } g \in SO(n)) \\ \Delta^S(g \cdot f) \;=\; g \cdot (\Delta^S f) & (\text{for } g \in SO(n)) \end{cases}$$

[1.0.3] Remark: To prove *existence* of an invariant integral and invariant Laplacian, there are several approaches, of varying expense and quality. For immediate purposes, we choose a pedestrian *ad hoc* approach, despite its ugliness. A better argument will be given later.

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2. *The spherical Laplacian*

One direct proof of *existence* of the invariant Laplacian on $S = S^{n-1}$ uses the imbedding of the sphere $S = S^{n-1}$ in $\mathbb{R}^n$. The usual Euclidean Laplacian

$$\Delta \;=\; \Delta^{\mathbb{R}^n} \;=\; \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_n}\right)^2 \qquad (\text{on } \mathbb{R}^n)$$

is $SO(n)$-invariant [10] on differentiable functions on $\mathbb{R}^n$. For a function f on S^{n-1}, create a function F on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f \;=\; (\text{restriction to } S \text{ of }) \; \Delta F$$

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[9] If we let the group act by *left* multiplication, and then try to define $(g \cdot f)(x) = f(gx)$, we have a problem, as follows. For $g, h \in SO(n)$, *associativity* fails, since

$$((gh) \cdot f)(x) \;=\; f((gh)\,x) \;=\; f(g(hx)) \;=\; (g \cdot f)(hx) \;=\; (h \cdot (g \cdot f))(x)$$

instead of $g \cdot (h \cdot f)$. A left action on the set requires defining $(g \cdot f)(x) = f(g^{-1}x)$. Avoiding the inverse is a pleasant minor economy.

[10] To verify that the usual Euclidean Laplacian is $SO(n)$-invariant is a direct computation. In fact, the Laplacian is invariant under the *full* orthogonal group $O(n)$. Let $k \in O(n)$ with ij^{th} entry k_{ij}. For F on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\Delta(k \cdot F)(x) \;=\; \sum_\ell (\frac{\partial}{\partial x_\ell})^2 F(\ldots, \sum_i x_i k_{ij}, \ldots) \;=\; \sum_\ell \frac{\partial}{\partial x_\ell} \sum_s k_{\ell s} F_s(\ldots, \sum_j x_i k_{ij}, \ldots)$$

where F_s is the partial derivative of F with respect to its s^{th} argument. Taking the next derivative gives

The map $f \to F$ that creates from f on S the degree-zero positive-homogeneous [11] function F on $\mathbb{R}^n - 0$ *commutes with* the action of $SO(n)$. [12] From the definition,

$$\Delta^S \overline{f} \;=\; \overline{\Delta^S f}$$

The $SO(n)$-invariance of the spherical Laplacian follows from the $SO(n)$-invariance of the usual Laplacian: for $k \in SO(n)$

$$\Delta^S(k \cdot f) \;=\; (\Delta(k \cdot F))|_S \;=\; (k \cdot (\Delta F))|_S \;=\; k \cdot (\Delta F)|_S$$

since restriction to the sphere commutes with $SO(n)$, as does $f \to F$. Thus, Δ^S is $SO(n)$-invariant.

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3. *Eigenvectors for the spherical Laplacian*

As usual, a function on $\mathbb{R}^n$ is *harmonic* when it is annihilated by the Euclidean Laplacian $\Delta = \Delta^{R^n}$.

[3.0.1] Proposition: For f positive-homogeneous of degree s and *harmonic*, the restriction $f|_S$ of f to the sphere S^{n-1} is an *eigenfunction* for Δ^S,

$$\Delta^S(f|_S) \;=\; -s(s+n-2) \cdot (f|_S)$$

with eigenvalue $-s(s+n-2)$.

Proof: First, the effect of Δ^S on a positive-homogeneous function f of degree s is directly computed:

Let $r = |x|$ and let f_i be the partial derivative with respect to the i^{th} argument:

$$\Delta^S(f|_S) \;=\; \Delta f(x/|x|) \;=\; \Delta\left(|x|^{-s} \cdot f\right) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2}\left((r^2)^{-\frac{s}{2}} \cdot f\right)$$

$$= \sum_i \frac{\partial}{\partial x_i}\Big(-\frac{s}{2}\,(2x_i)\,(r^2)^{-(\frac{s}{2}+1)}\,f + (r^2)^{-s/2}\,f_i\Big) \;=\; \sum_i \frac{\partial}{\partial x_i}\Big(-sx_i\,(r^2)^{-(\frac{s}{2}+1)}\,f + (r^2)^{-s/2}\,f_i\Big)$$

$$= \sum_i \Big(-s\,(r^2)^{-(\frac{s}{2}+1)}\,f + sx_i\,(\frac{s}{2}+1)(2x_i)\,(r^2)^{-(\frac{s}{2}+2)}\,f - sx_i(r^2)^{-(\frac{s}{2}+1)}\,f_i$$

$$-\,\frac{s}{2}\,(2x_i)(r^2)^{-(\frac{s}{2}+1)}\,f_i \;+\; (r^2)^{-s/2}\,f_{ii}\Big)$$

$$= \; -ns\,(r^2)^{-(\frac{s}{2}+1)}\,f + sr^2\,(s+2)\,(r^2)^{-(\frac{s}{2}+2)}\,f - s(r^2)^{-(\frac{s}{2}+1)}\,sf + (r^2)^{-s/2}\,\Delta f$$

$\sum_\ell \sum_{s,t} k_{\ell s} k_{\ell t} F_{st}(\ldots, \sum_i x_i k_{ij}, \ldots)$. Interchange the order of the sums and observe that $\sum_\ell k_{\ell s} k_{\ell t}$ is the $(s,t)^{th}$ entry of $k^\top \cdot k$, which is the $(s,t)^{th}$ entry of 1_n. Thus, the whole is

$$\sum_s F_{ss}(\ldots, \sum_i x_i k_{ij}, \ldots) \;=\; (\Delta F)(xk) \;=\; (k \cdot (\Delta F))(x)$$

[11] A function F on $\mathbb{R}^m$ is *positive-homogeneous* of degree $s \in \mathbb{C}$ if, for all $t > 0$ and for all $x \in \mathbb{R}^m$, $F(tx) = t^s \cdot F(x)$. We make $F(x) = f(x/|x|)$ positive-homogeneous of degree 0, rather than $|x|^s f(x/|x|)$ positive-homogeneous of degree s, so that *constant* functions on the sphere become constant functions on $\mathbb{R}^n - 0$, and are *annihilated* by any differential operator.

[12] The map $f \to F$ commutes with the action $(k \cdot F)(x) = F(xk)$ of $SO(n)$ on functions because $F(xk) = f(xk/|xk|) = f((x/|x|) \cdot k) = (kf)(x/|x|)$.

by using Euler's identity [13] that for positive-homogeneous f of degree s,

$$\sum_i x_i \; f_i(x) \;=\; s \cdot f$$

as well as the obvious $\sum_i x_i^2 = r^2$. Simplifying,

$$\Delta(|x|^{-s} \cdot f) \;=\; -ns\, r^{-(s+2)}\, f + s(s+2)\, r^{-(s+2)}\, f - 2s\, r^{-(s+2)}\, sf + r^{-s}\, \Delta f$$

$$=\; -s(n-(s+2)+2s)\, r^{-(s+2)}\, f + r^{-s}\, \Delta f \;=\; -s(n+s-2)\, r^{-(s+2)}\, f \;+\; r^{-s}\, \Delta f$$

Then for $\Delta f = 0$, restriction to the sphere makes $r = 1$, giving the eigenfunction property. ///

The most tractable positive-homogeneous functions are homogeneous *polynomials*. Let

$$H_d \;=\; \{\text{homogeneous (total) degree } d \text{ harmonic polynomials in } \mathbb{C}[x_1, \ldots, x_n]\}$$

Let $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ be the *homogeneous* polynomials of degree d. Introduce a temporary complex-hermitian form [14]

$$(,) \;:\; \mathbb{C}[x_1, \ldots, x_n] \times \mathbb{C}[x_1, \ldots, x_n] \;\longrightarrow\; \mathbb{C}$$

by

$$(P, Q) \;=\; \Big(\overline{Q}(\partial) P(x)\Big)\Big|_{x=0}$$

where $Q(\partial)$ means to replace x_i by $\partial/\partial x_i$ in a polynomial, and $R|_{x=0}$ means to evaluate R at $x = 0$. The main point of this construction is the immediate identity

$$(\Delta f, g) \;=\; (f, r^2 g)$$

where $r^2 = x_1^2 + \ldots + x_n^2$. That is, multiplication by r^2 is *adjoint* to application of Δ with respect to $(,)$. This will be useful after proof that $(,)$ is *non-degenerate*:

[3.0.2] Claim: The form $(,)$ is positive-definite hermitian.

Proof: Observe that $(P, Q) = 0$ for *homogeneous* polynomials P, Q *unless* P, Q are of the same degree. When restricted to the *homogeneous* polynomials $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ of degree d, the form $(,)$ has an *orthogonal basis* of *distinct monomials*, since

$$(\partial {x_1}^{m_1} \ldots \partial {x_n}^{m_n})\, (x_1^{e_1} \ldots x_n^{e_n})|_{x=0} \;=\; \begin{cases} 0 & \text{(if } any\ m_i \neq e_i) \\ m_1! \ldots m_n! & \text{(if } every\ m_i = e_i) \end{cases}$$

[13] Euler's identity is readily proven by considering the function $g(t) = G(tx) = t^s G(x)$ for $t > 0$, differentiating with respect to t, and evaluating at $t = 1$:

$$st^{s-1} G(x) \;=\; g'(t) \;=\; \sum_i x_i \frac{\partial G(tx)}{\partial x_i}$$

gives $s \cdot G(x) = \sum_i x_i \frac{\partial G(x)}{\partial x_i}$.

[14] This hermitian form is not as *ad hoc* as it may look. From a slightly more sophisticated viewpoint: it pairs polynomials against *Fourier transforms* of polynomials, which are derivatives of Dirac delta at 0, which are *compactly-supported* distributions, so can be evaluated on polynomials, which are smooth. One need not think in these terms to use the properties here, which also admit elementary proofs.

Looking at the orthogonal basis of monomials, $(,)$ is *hermitian* and *positive definite* on $\mathbb{C}[x_1,\ldots,x_n]^{(d)}$. ///

[3.0.3] Claim: The map

$$\Delta \;:\; \mathbb{C}[x_1,\ldots,x_n]^{(d)} \longrightarrow \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$$

is *surjective*. Harmonic polynomials f in $\mathbb{C}[x_1,\ldots,x_n]^{(d)}$ are *orthogonal* to polynomials r^2h (with $h \in \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$) with respect to $(,)$.

Proof: For $h \in \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$, if $(\Delta f, h) = 0$ for all $f \in \mathbb{C}[x_1,\ldots,x_n]^{(d)}$, then

$$0 \;=\; (\Delta f, h) \;=\; (f, r^2 h) \qquad \text{(for all } f\text{)}$$

so $r^2h = 0$, so $h = 0$, by the positive-definiteness of $(,)$. This also proves the second assertion. ///

Iterating the claim just proven,

$$\mathbb{C}[x_1,\ldots,x_n]^{(d)} \;=\; H_d \oplus r^2 H^{d-2} \oplus r^4 H^{d-4} + \ldots$$

[3.0.4] Claim: Polynomials restricted to the n-sphere are equal to linear combinations of *harmonic* polynomials restricted to the sphere.

Proof: Use

$$\mathbb{C}[x_1,\ldots,x_n]^{(d)} \;=\; H_d \oplus r^2 H^{d-2} \oplus r^4 H^{d-4} + \ldots$$

to write a homogeneous polynomial as

$$f \;=\; f_0 + r^2 f_2 + r^4 f_4 + \ldots \qquad \text{(with each } f_i \text{ harmonic)}$$

Restricting to the sphere,

$$f|_S \;=\; (f_0 + r^2 f_2 + r^4 f_4 + \ldots)|_S \;=\; (f_0 + f_2 + f_4 + \ldots)|_S$$

since $r^2 = 1$ on the sphere. ///

[3.0.5] Remark: Again, from computations above,

$$\Delta^S f \;=\; -d(d+n-2)\cdot f \qquad \text{(for } f \in H_d\text{)}$$

Since the degree d is non-negative,

$$\lambda_d \;=\; -d(d+n-2) = -(d+\frac{n-2}{2})^2 + (\frac{n-2}{2})^2$$

the eigenvalues $\lambda_d = -d(d+n-2)$ are non-positive, and 0 only for degree $d = 0$. As $d \to +\infty$, the eigenvalues go to $-\infty$. Indeed, $\lambda_{d'} < \lambda_d \le 0$ for $d' > d$, so the spaces H_d are *distinguished* by their eigenvalues for the spherical Laplacian.

[3.0.6] Remark: For the circle S^1, the 0-eigenspace is 1-dimensional and for $d > 0$ the $(-d^2)$-eigenspace is 2-dimensional, with basis $(x \pm iy)^d$. By contrast, for $n > 1$ the dimensions of eigenspaces are *unbounded* as the degree d goes to $+\infty$. Specifically,

[3.0.7] Claim: The dimension of H_d is

$$\dim_{\mathbb{C}} H_d \;=\; \dim \mathbb{C}[x_1,\ldots,x_n]^{(d)} - \dim \mathbb{C}[x_1,\ldots,x_n]^{(d-2)} \;=\; \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}$$

Proof: From above,

$$\Delta \;:\; \mathbb{C}[x_1,\ldots,x_n]^{(d)} \longrightarrow \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$$

is surjective, so the dimension of H_d is the indicated difference of dimensions.

The dimension of the space of degree d polynomials in n variables can be determined by counting the number of monomials $x_1^{e_1}\ldots x_n^{e_n}$ with $\sum_i e_i = d$, as follows. Imagine each exponent as consisting of the corresponding number of marks, lined up, with $n-1$ additional marks to separate the marks corresponding to the n distinct variables x_i, for a total of $n+d-1$. The choice of location of the separating marks is the binomial coefficient. ///

[3.0.8] Corollary: The dimension of H_d grows like d^{n-2} as $d \to +\infty$. ///

[3.0.9] Corollary: The dimensions of the polynomial λ-eigenspaces for the spherical Laplacian Δ grow like $|\lambda|^{\frac{n}{2}-1}$. ///

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4. *Invariant integrals and measures on spheres*

We used the assumed *existence* of an $SO(n)$-invariant integral on S^{n-1} to be sure that eigenvalues for the spherical Laplacian Δ^S are non-positive, in determining all eigenvectors.

To prove existence of an invariant integral we can write a *formula* [15] as follows, using the fact that the usual measure on $\mathbb{R}^n$ is $SO(n)$-invariant, since the absolute value of the determinant of an element of $SO(n)$ is 1. For γ a fixed smooth non-negative function on $[0,\infty)$ with

$$\int_{\mathbb{R}^n} \gamma(|x|^2)\,dx \;=\; 1$$

for a continuous function f on S, define

$$\int_{S^{n-1}} f \;=\; \int_{\mathbb{R}^n} \gamma(|x|^2)\, f\Big(\frac{x}{|x|}\Big)\,dx$$

For convenience, we may at some moments suppose that γ has compact support and vanishes identically on a neighborhood of 0. [16] For $k \in SO(n)$ we have the $SO(n)$-invariance of the integral:

$$\int_S k\cdot f \;=\; \int_{R^n} \gamma(|x|^2)\, f(\frac{xk}{|xk|})\,dx \;=\; \int_{R^n} \gamma(|xk^{-1}|^2)\, f(\frac{x}{|x|})\,dx \;=\; \int_{R^n} \gamma(|x|^2)\, f(\frac{x}{|x|})\,dx \;=\; \int_S f$$

by changing variables to replace x by xk^{-1}, and using $|xk^{-1}| = |x|$. Proof of the desired integration-by-parts-twice result from this clunky viewpoint is a little more trouble:

[4.0.1] Proposition: For differentiable functions f, φ on $S = S^{n-1}$,

$$\int_{S^{n-1}} (\Delta^S f)\cdot\varphi \;=\; \int_{S^{n-1}} f\cdot\Delta^S\varphi$$

---

[15] If we were already happy with *Haar measure* on $SO(n)$ and invariant measures on quotient spaces such as S^{n-1}, then we would not need another construction. We rarely need more than the *existence* (and essential uniqueness) of such an integral.

[16] Compact support and the vanishing condition may be convenient, since then there are no boundary terms in integrating by parts on $\mathbb{R}^n$.

Further, Δ^S is *negative-definite* in the sense that

$$\int_S (\Delta^S f) \cdot \overline{f} \le 0$$

with equality only for f *constant*.

Proof: Let $F(x) = f(x/r)$ and $\Phi(x) = \varphi(x/r)$, where $r = |x|$. By definition,

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_{\mathbb{R}^n} \gamma(r^2)\; r^2 \cdot (\Delta F)(x)\; \Phi(x)\; dx$$

where the r^2 is inserted so that $r^2 \Delta F$ is positive-homogeneous of degree 0 as required by the integration formula. [17] Integrating by parts, this becomes

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i}\, \frac{\partial}{\partial x_i} \left(r^2 \cdot \gamma(r^2)\; \Phi(x) \right)\; dx$$

Let $\delta(r^2) = r^2 \gamma(r^2)$. Then

$$\frac{\partial}{\partial x_i} \left[r^2 \cdot \gamma(r^2)\, \Phi(x) \right] \;=\; \frac{\partial}{\partial x_i} \left[\delta(r^2)\; \Phi(x) \right] \;=\; 2x_i \delta'(r^2) \Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i}$$

The whole is

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i} \left[2x_i \delta'(r^2)\Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i} \right] dx \;=\; -\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i}\, \delta(r^2) \frac{\partial \Phi}{\partial x_i}\; dx$$

since Euler's identity asserts that

$$\sum_i x_i \frac{\partial F}{\partial x_i} \;=\; 0$$

because F is positive-homogeneous of degree 0. That last expression for the integral is symmetric in F and Φ, giving the desired integration-by-parts result. Finally, with $\Phi = \overline{F}$, the last expression for the integral is visibly non-positive, and is 0 only if $\partial F / \partial x_i = 0$ for all i, only if F is constant, only if f is constant. ///

[4.0.2] Remark: This argument is unsatisfying, since it does not extend to more general situations. A more universal existence argument for G-invariant measures on quotient spaces $H\backslash G$ will be given later.

[4.0.3] Corollary: For distinct degrees d, d', the spaces H_d and $H_{d'}$ are mutually orthogonal.

Proof: Earlier, we saw that the eigenvalues of Δ^S on distinct H_d's are distinct. Distinct eigenspaces of Δ^S are orthogonal. ///

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5. L^2 spectral decompositions on spheres

The idea of *spectral decomposition* on the sphere is that functions on the sphere should be *sums of eigenfunctions* for the spherical Laplacian. For L^2 functions the convergence should be in L^2. For *smooth* functions, the sum should converge well in a *uniform pointwise* sense. As usual, L^2 convergence does not imply pointwise convergence.

[17] Since $F(x) = f(x/r)$ is positive-homogeneous of degree 0, ΔF is positive-homogeneous of degree -2, so we need to adjust it by r^2.

[5.0.1] Remark: The proof here is relatively elementary, but sadly fails to suggest a causal mechanism that might apply in other situations. A better, somewhat more costly, proof will be given later.

[5.0.2] Theorem: The collection of finite linear combinations of homogeneous harmonic polynomials restricted to S^{n-1} is *dense* in $L^2(S^{n-1})$.

Proof: By Weierstraß' approximation theorem, as in the appendix, polynomials are dense in $C^o(\mathbb{R}^n)$, meaning in the topology given by sups on compacts. Even though it has no interior, the sphere is compact, so restrictions of polynomials to S^{n-1} approximate continuous functions on the sphere in sup norm. Our description of $L^2(S^{n-1})$ is as the L^2 completion of $C^o(S^{n-1})$, so $C^o(S^{n-1})$ is dense in $L^2(S^{n-1})$ in the L^2 topology. Since the sphere has finite total measure, density in sup norm implies density in L^2 norm, so restrictions of polynomials to S^{n-1} are dense in $L^2(S^{n-1})$.

We showed that every homogeneous degree d polynomial f is expressible as

$$f \;=\; f_d + r^2 \cdot f_{d-2} + r^4 \cdot f_{d-4} + \ldots \qquad\qquad (\text{where } f_j \in H_j)$$

Restricting to the sphere, $r = 1$, so

$$f \;=\; f_d + f_{d-2} + f_{d-4} + \ldots \qquad\qquad (\text{restricted to the sphere})$$

Thus, harmonic polynomials are dense in L^2. ///

[5.0.3] Corollary:

$$L^2(S^{n-1}) \;=\; \text{completion of} \bigoplus_{d \geq 0} H_d$$

Proof: Given f in the orthogonal complement to that completion, let φ be a harmonic polynomial approximating f to within $\varepsilon > 0$ in L^2 norm. Write $\varphi = \sum_d \varphi_d$ where φ_d is homogeneous of degree d. The various spaces H_d are mutually orthogonal, since they have distinct eigenvalues for Δ^S. Thus, orthogonality of f to every H_d assures that $\varphi_d = 0$ for all d. Thus, $|f|_{L^2} < \varepsilon$. This holds for every $\varepsilon > 0$, so $f = 0$ in L^2. ///

[5.0.4] Corollary: Thus, every L^2 function f on S^n has at least an L^2 Fourier-Laplace expansion

$$f \;=\; \sum_{d=0}^{\infty} f_d \qquad\qquad (\text{in } L^2 \text{ sense})$$

where f_d is the orthogonal projection in $L^2(S^{n-1})$ of f to the space H_d of homogeneous degree d harmonic polynomials restricted to the sphere.

[5.0.5] Remark: All this discussion applies to the circle $S^1 \subset \mathbb{R}^2$, presented as $\{(x,y) : x^2 + y^2 = 1\}$. For the circle, there are also parametrizations $\theta \to e^{2\pi i\theta}$ or $\theta \to e^{i\theta}$, corresponding to the presentations $S^1 \approx \mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/2\pi\mathbb{Z}$. The comparison is that the exponentials $\theta \to e^{2\pi in\theta}$ with $n \in \mathbb{Z}$ are exactly the restrictions to $\{x^2 + y^2 = 1\}$ of the harmonic polynomials $(x+iy)^n$ for $n \geq 0$, and $(x-iy)^{|n|}$ for $n \leq 0$.

Thus, the Weierstraß approximation theorem and the above discussion yield yet-another proof that exponentials form an orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$.

6. *Sup-norms of spherical harmonics on* S^{n-1}

To eventually assess the *pointwise* convergence of Fourier series on the sphere, we must be aware that, unlike the *bounded* functions $e^{in\theta}$ on the circle S^1, for $f \in H_d$ on S^n with $n > 1$ there is *no* trivial comparison [18] of the two norms

$$|f|_{C^o} = \sup_{x\in S^{n-1}} |f(x)| \qquad |f|_{L^2} = \left(\int_S |f(x)|^2\, dx\right)^{1/2}$$

Nevertheless, a little work gives a useful comparison: [19]

[6.0.1] Proposition: Let $f \in H_d$. Then

$$|f|_{C^o} \;\le\; \sqrt{\frac{\dim H_d}{\mathrm{vol}\,(S^{n-1})}} \;\cdot\; |f|_{L^2}$$

And the estimate is *sharp*, in the sense that there is a function in H_d for which equality holds.

[6.0.2] Remark: The occurrence of the square root of the total measure of S^{n-1} compensates for the fact that the square root of the total measure enters in the L^2-norm, and does *not* enter in the sup norm.

[6.0.3] Remark: Using the dimension computation from above, as $d \to +\infty$

$$\sqrt{\dim H_d} \;\sim\; \sqrt{\binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}} \;\sim\; d^{\frac{n}{2}-1}$$

Proof: For $x \in S$, from the Riesz-Fischer theorem, the functional $f \to f(x)$ is necessarily given by

$$f(x) \;=\; \langle f, F_x\rangle \qquad\qquad (\text{for some } F_x \in H_d)$$

The Cauchy-Schwarz-Bunyakowsky inequality gives

$$|f(x)| \;=\; |\langle f, F_x\rangle| \;\le\; |f|_{L^2}\cdot |F_x|_{L^2}$$

which bounds the value $f(x)$ in terms of its L^2-norm with constant being the L^2-norm of F_x.

We show that $F_x(x)$ is independent of $x \in S^{n-1}$. Let $g \in SO(n)$, with action $(g\cdot f)(x) = f(xg)$. Again, by design, the action of $g \in SO(n)$ on functions is *unitary* in the sense that

$$\langle g\cdot f, g\cdot F\rangle \;=\; \int_{S^{n-1}} f(xg)\,\overline{F(xg)}\,dx \;=\; \int_{S^{n-1}} f(x)\,\overline{F(x)}\,dx \;=\; \langle f, F\rangle$$

[18] Since the spaces H_d are finite-dimensional, each has a unique topology compatible with the vector space structure. (The latter fact is not trivial to prove, but is not too hard.) Thus, the fact that the sup-norm and L^2-norm are comparable on these finite-dimensional spaces is clear *a priori*. But this qualitative fact is irrelevant to the issue at hand. What is *not* clear is growth of the comparison constants in the parameter d, and this greater precision is needed for Sobolev-type estimates.

[19] In fact, the proof of the following proposition uses few of the specifics of this situation, and, indeed, the same argument proves that for a compact group K acting transitively on a set X, with a finite-dimensional K-stable space V of functions on X, for $f \in V$ we have the same comparison of sup-norm and L^2-norm, namely $|f|_{C^o} \le \sqrt{(\dim V)/\mathrm{vol}\,(X)}\cdot |f|_{L^2}$. This inequality is interesting even for *finite* groups.

by replacing x by xg^{-1} in the integral. The functions F_x have natural relations among themselves, namely,

$$\langle f, F_{xg}\rangle \;=\; f(xg) \;=\; (g\cdot f)(x) \;=\; \langle g\cdot f, F_x\rangle \;=\; \langle f, g^{-1}\cdot F_x\rangle$$

Thus,

$$F_{xg} \;=\; g^{-1}\cdot F_x$$

Since the action of g is unitary, the L^2-norm of the function F_x is the same for every $x\in S$. In particular,

$$F_x(x) \;=\; \langle F_x, F_x\rangle \;=\; \langle g^{-1}\cdot F_x, g^{-1}F_x\rangle \;=\; F_{xg}(xg)$$

Since $SO(n)$ is transitive on the sphere, $F_x(x)$ is independent of x.

Further,

$$|F_x(y)| \;=\; |\langle F_x, F_y\rangle| \;\le\; |F_x|_{L^2}\cdot|F_y|_{L^2} \;=\; |F_x|^2_{L^2}$$

because all the L^2 norms are the same. This determines the sup-norm

$$|F_x|_{C^o} \;=\; F_x(x) = |F_x|^2_{L^2}$$

The L^2 norm is evaluated as follows. Express F_x in terms of an orthonormal basis $\{f_i\}$ for H_d

$$F_x \;=\; \sum_i \langle F_x, f_i\rangle\cdot f_i$$

Evaluating both sides at x gives

$$F_x(x) \;=\; \sum_i \langle F_x, f_i\rangle\cdot f_i(x) \;=\; \sum_i \overline{f_i(x)}\cdot f_i(x) \;=\; \sum_i |f_i(x)|^2$$

Again, the value $F_x(x) = |F_x|^2_{L^2}$ is independent of x, since for $g\in SO(n)$ and $x\in S$

$$F_{xg}(xg) = \langle F_{xg}, F_{xg}\rangle = \langle g\cdot F_x, g\cdot F_x\rangle = \langle F_x, F_x\rangle = F_x(x)$$

Integrating $F_x(x) = \sum_i |f_i(x)|^2$ over S, using the fact that $F_x(x)$ is independent of $x\in S^{n-1}$,

$$\mathrm{vol}\,(S^{n-1})\cdot F_x(x) \;=\; \dim_{\mathbb{C}}\, H_d$$

Then

$$F_x(x) \;=\; |F_x|_{C^o} \;=\; |F_x|^2_{L^2}$$

gives

$$|F_x|_{L^2} \;=\; \sqrt{F_x(x)} \;=\; \sqrt{\frac{\dim H_d}{\mathrm{vol}\,(S^{n-1})}}$$

Combining this with $|f(x)| \;\le |F_x|_{L^2}\cdot|f|_{L^2}$ from above,

$$|f|_{C^o} \;\le\; |F_x|_{L^2}\cdot|f|_{L^2}\sqrt{\frac{\dim H_d}{\mathrm{vol}\,(S^{n-1})}}\cdot|f|_{L^2}$$

as claimed. Further, we saw that

$$|F_x|_{C^o} \;=\; |F_x|_{L^2}\cdot|F_x|_{L^2} \;=\; \sqrt{\frac{\dim H_d}{\mathrm{vol}\,(S^{n-1})}}\cdot|F_x|_{L^2}$$

so the estimate is sharp. ///

7. *Pointwise convergence of Fourier-Laplace series*

The comparison of sup-norms and L^2 norms on H_d yields useful criteria for pointwise convergence. The following L^2 Sobolev criterion proves the most useful in practice.

[7.0.1] Corollary: *(Sobolev imbedding)* A function $f \in L^2(S^{n-1})$ with Fourier-Laplace expansion $f = \sum_d f_d$ with $f_d \in H_d$ satisfying

$$\sum_d (1+d)^s \cdot |f_d|^2_{L^2} \;<\; \infty \qquad\qquad (\text{for any } s > \dim S^{n-1} = n-1)$$

is *continuous*, and its Fourier-Laplace expansion converges *uniformly pointwise* to f.

Proof: For any $s \in \mathbb{R}$,

$$\sup_{x\in S^{n-1}} \Big|\sum_d f_d(x)\Big| \;\ll\; \sum_d \sqrt{\dim H_d} \cdot |f_d|_{L^2} \;\ll\; \sum_d (1+d)^{\frac{n-2}{2}} \cdot |f_d|_{L^2}$$

$$= \sum_d (1+d)^{\frac{s}{2}} \cdot |f_d|_{L^2} \cdot \frac{1}{(1+d)^{\frac{s-(n-2)}{2}}} \;=\; \Big(\sum_d (1+d)^s \cdot |f_d|^2_{L^2}\Big)^{1/2} \cdot \Big(\sum_d \frac{1}{(1+d)^{s-(n-2)}}\Big)^{1/2}$$

by Cauchy-Schwarz-Bunyakowsky. For $s > n-1$, the latter sum is finite. Thus, for any $s > n-1$, if

$$\sum_d (1+d)^s \cdot |f_d|^2_{L^2} \;<\; +\infty$$

then the Fourier-Laplace series $\sum_d f_d$ converges uniformly pointwise. That is, the finite partial sums converge uniformly. Since the partial sums are continuous, and a uniform limit of continuous functions is continuous, the limit is continuous.

Since f is the L^2 limit of the finite partial sums, and the C^o limit is also in L^2, it must be that f is that C^o limit, so is continuous, and the partial sums converge uniformly to f itself. ///

[7.0.2] Remark: One might think of the superficially simpler analogous assertion and proof wherein the L^2 norms of the Fourier components f_d are *not* squared: a similar chain of inequalities

$$\sup_{x\in S^{n-1}} \Big|\sum_d f_d(x)\Big| \;\ll\; \sum_d \sqrt{\dim H_d} \cdot |f_d|_{L^2} \;\ll\; \sum_d (1+d)^{\frac{n-2}{2}} \cdot |f_d|_{L^2}$$

proves uniform convergence when the last sum is finite. However, Hilbert spaces are far more congenial than Banach spaces, in many technical regards.

8. *Irreducibility of representation spaces for $O(n)$*

Certainly $O(n)$ stabilizes each H_d, because the action of $O(d)$ on functions on $\mathbb{R}^n$ commutes with $\Delta = \Delta^{R^n}$, and the linearity of the action preserves the degree of polynomials.

[8.0.1] Claim: The vector spaces H_d are *irreducible* for $O(n)$, in the sense that their only subspaces stable under the action of $O(n)$ are $\{0\}$ and the whole H_d.

Proof: The proof will show that every finite-dimensional $O(n)$-stable subspace of any H_d contains a non-zero $O(n-1)$-fixed vector, where $O(n-1)$ is the isotropy subgroup of the last standard basis element e_n.

Then we show that each H_d contains a *unique* $O(n-1)$-fixed vector. The $O(n)$-invariance of $\langle,\rangle$ assures that the orthogonal complement to any $O(n)$-stable subspace of H_d is itself $O(n)$-stable, so if H_d were *not* irreducible, there would be at least a two-dimensional space of $O(n-1)$-fixed vectors in H_d, contradiction.

We will need to *average* over rotations by $O(n-1)$, so we need an invariant measure/integral on $O(n-1)$ itself. That is, with the action $(h\cdot\varphi)(y) = \varphi(yh)$ for $h, y \in O(n-1)$ and continuous φ on $O(n-1)$, require

$$\int_{O(n-1)} h\cdot\varphi \;=\; \int_{O(n-1)} \varphi$$

Discussion of the invariant integral is postponed to the end.

First, we show that a non-zero $O(n)$-stable finite-dimensional subspace V of H_d contains a non-zero $O(n-1)$-fixed vector. Take not-identically-zero $f \in V$. For $f(e_n) \neq 0$, the averaged version

$$F \;=\; \int_{O(n-1)} g\cdot f$$

is designed to be $O(n-1)$-invariant: for $h \in O(n-1)$,

$$h\cdot F \;=\; h\cdot \int_{O(n-1)} g\cdot f\; dg \;=\; \int_{O(n-1)} hg\cdot f\; dg \;=\; \int_{O(n-1)} g\cdot f\; dg$$

by replacing g by $h^{-1}g$. Note that we need confidence that the translation action can be moved inside the integral, when the integral is viewed as an integral of an H_d-valued function on $O(n-1)$. And F is not accidentally identically 0, because its value at e_n is a non-zero constant multiple of that of f:

$$F(e_n) \;=\; \int_{O(n-1)} (g\cdot f)(e_n)\; dg \;=\; \int_{O(n-1)} f(e_n g)\; dg \;=\; \int_{O(n-1)} f(e_n)\; dg \;=\; f(e_n)\int_{O(n-1)} 1\; dg$$

If by mischance $f(e_n) = 0$, then use the $O(n)$-stability to rotate f by $h \in O(n)$ so that $f(e_n h) \neq 0$. This is possible, since $O(n)$ is transitive on S^{n-1} and f is not identically 0. Thus, modulo facts about integration on $O(n-1)$, any $O(n)$-stable subspace of H_d contains a non-zero $O(n-1)$-fixed vector.

Next, let V be a non-zero $O(n)$-stable subspace of H_d. We claim that the orthogonal complement W of V inside H_d is still $O(n)$-stable. Indeed, for $w \in W$, $v \in V$, and $g \in O(n)$, by unitariness of the action of $O(n)$,

$$\langle g\cdot w, v\rangle \;=\; \langle g\cdot w, gg^{-1}\cdot v\rangle \;=\; \langle w, g^{-1}\cdot v\rangle \;=\; 0$$

since $g^{-1}\cdot v \in V$. Thus, if $W \neq \{0\}$, then it has a non-zero $O(n-1)$-fixed vector.

Now prove that H_d has at most one $O(n-1)$-fixed vector, up to scalar multiples. An $O(n-1)$ invariant function f can be written

$$f(x_1, \ldots, x_{n-1}, x_n) \;=\; F(\rho, x_n) \qquad\qquad (\text{where } \rho = \sqrt{x_1^2 + \ldots + x_{n-1}^2})$$

Using subscripts to denote partial derivatives with respect to arguments, the harmonic condition on such functions is

$$0 \;=\; \sum_{1\le i\le n-1} \frac{\partial^2}{\partial x_i^2} F(\rho, x_n) \;+\; \frac{\partial^2 F}{\partial x_n^2} \;=\; \sum_{1\le i\le n-1} \frac{\partial}{\partial x_i}\Big(\frac{\frac{1}{2}\cdot 2x_i}{\rho} F_1\Big) \;+\; F_{22}$$

$$=\; \sum_{1\le i\le n-1} \Big(\frac{1}{\rho}F_1 - \frac{x_i^2}{\rho^3}F_1 + \frac{x_i^2}{\rho^2}F_{11}\Big) \;+\; F_{22} \;=\; \frac{n-2}{\rho}F_1 + F_{22} + F_{11}$$

Note that the $O(n-1)$-invariance entails further that $F(\rho, y)$ is *even* as a function of ρ. Using the homogeneity, $F(0,1) = f(e_n) = 1$, and the even-ness, write

$$F(\rho, y) \;=\; y^d + c_2 y^{d-2}\rho^2 + c_4 y^{d-4}\rho^4 + \ldots$$

and use the harmonic condition to solve recursively for the coefficients:

$$\begin{aligned} 0 \;&=\; \Big(\frac{n-2}{\rho}F_1 + F_{22} + F_{11}\Big)\Big(y^d + c_2 y^{d-2}\rho^2 + c_4 y^{d-4}\rho^4 + \ldots\Big) \\ &=\; d(d-1)y^{d-2} + c_2\Big((d-2)(d-3)y^{d-4}\rho^2 + \frac{n-2}{\rho}2y^{d-2}\rho^1 + 2y^{d-2}\cdot 1\Big) + \ldots \\ &=\; d(d-1)y^{d-2} + c_2\Big((d-2)(d-3)y^{d-4}\rho^2 + (n-2)2y^{d-2}\cdot 1 + 2y^{d-2}\cdot 1\Big) + \ldots \end{aligned}$$

For example, equating the y^{d-2} coefficients gives

$$0 \;=\; d(d-1) + c_2\Big((d-2)(d-3) + 2(n-2) + 2\Big)$$

The coefficient of c_2 is strictly positive, so is non-zero, and c_2 is completely determined. Similarly, all coefficients c_{2j} are determined recursively:

$$0 \;=\; c_{2j}(2j)(2j-1) + c_{2j+2}\Big((2j-2)(2j-3) + (d-2j+2)(n-2) + (d-2j+2)(d-2j+1)\Big)$$

The coefficient of c_{2j+2} is a sum of three products of non-negative integers, and at least one of the three products is strictly positive. Thus, up to normalizing constant, there is at most one $O(n-1)$-invariant element in H_d. Existence was already proven, by averaging over $O(n-1)$. ///

[8.0.2] Remark: Note that we did not use any feature of the invariant integral on $O(n)$ except its *existence*, and the possibility of moving the translation action from outside an integral to inside, when the integrand is vector-valued, with values in the finite vector space H_d. The sanest argument for existence would be to invoke general existence results for Haar (invariant) measures on topological groups. Nevertheless, a more elaborate version of our contriving a rotation-invariant measure on S^{n-1} by using the standard (translation and rotation-invariant) measure on the ambient Euclidean space, can be made to give an invariant measure on any orthogonal group $O(n)$ from the standard measure on the n^2-dimensional Euclidean space M_n of n-by-n real matrices, as follows.

First, we would note that $G = GL_n(\mathbb{R})$ is of full measure inside M_n. Second, from the spectral theorem for self-adjoint operators on real vector spaces, one shows that every $g \in G$ has a unique expression $g = s \cdot k$ with $k \in O(n)$ and s positive-definite symmetric. Letting $s^{\frac{1}{2}}$ of a symmetric positive-definite matrix s refer to its *unique* positive-definite square root, observe that

$$\Big(gg^\top\Big)^{-\frac{1}{2}} \cdot g \;=\; \Big((sk)(sk)^\top\Big)^{-\frac{1}{2}} \cdot sk \;=\; \Big(skk^\top s\Big)^{-\frac{1}{2}} \cdot sk \;=\; (s^2)^{-\frac{1}{2}} \cdot sk \;=\; k$$

With dx denoting the usual measure on $M_n \approx \mathbb{R}^{n^2}$,

$$f \;\longrightarrow\; \int_{GL_n(\mathbb{R})} e^{-\mathrm{tr}(xx^\top)} \cdot f\Big(\big(xx^\top\big)^{-\frac{1}{2}} \cdot x\Big)\, dx$$

gives a right $O(n)$-invariant integral on functions f on $O(n)$.

[8.0.3] Remark: For a continuous *vector-valued* integrand f on a compact (finite-measure) space X, and for a *continuous* linear map $T : V \to W$ from the topological vector space V where f takes its values to another topological vector space W, it is natural to hope that the obvious relation

$$T\Big(\int_X f\Big) \;=\; \int_X T\circ f$$

Indeed, under very mild assumptions on V, W, for continuous, compactly-supported V-valued f, this relation holds, as shown by Gelfand and Pettis independently in the 1930s. In the present situation, we only needed the result for $V = W$ finite-dimensional, and various more primitive arguments give the result, *depending* on one's characterization of integrals in the first place. With considerable hindsight, for purposes similar to those here, and even many far more sophisticated, it may be best to *characterize* a V-valued integral $\int_X f$ as being the element of V such that

$$\lambda\Big(\int_X f\Big) \;=\; \int_X \lambda \circ f \qquad\qquad \text{(for all continuous linear } \lambda : V \to \mathbb{C})$$

This is a *Gelfand-Pettis* integral. [20] It reduces properties of vector-valued integrals to properties of scalar-valued ones. For finite-dimensional vector-valued integrands, elementary considerations suffice, and oughtn't be entangled with more serious general considerations, but it is reassuring that even in considerable generality (integrands with values in locally convex, quasi-complete topological vector spaces), Gelfand-Pettis integrals of continuous, compactly supported functions do exist and are unique.

[8.0.4] **Remark:** For non-abelian groups of linear operators on a vector space such as $L^2(S^{n-1})$, and for non-commutative rings of such operators, it is unreasonable to hope for simultaneous *eigenvectors*. The correct generalization, or substitute, is *irreducible representations*, just as H_d is irreducible for $O(n)$. That is, instead of hoping to express general vectors as linear combinations of *eigenvectors*, we should only hope to express general vectors as linear combinations of vectors from *irreducible subspaces*, such as H_d. Typically, for non-abelian groups, these irreducibles are *not* one-dimensional, as we see for H_d.

Despite the loss of the relevance of the simpler notion of *simultaneous eigenvector*, irreducible subspaces are a very useful substitute, as illustrated below in proof of Hecke's identity about Fourier transforms.

[8.0.5] **Remark:** We proved that each of the irreducibles H_d of $O(n)$ occurring in $L^2(S^{n-1})$ has a unique (up to scalars) $O(n-1)$-fixed vector. If we could prove that *every* irreducible of $O(n)$ contains at most one $O(n-1)$-fixed vector, then the two compact groups $O(n-1), O(n)$ would be said to be a *Gelfand pair*.

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9. *Hecke's identity*

[9.0.1] **Theorem:** *(Hecke)* For homogeneous degree d *harmonic* polynomial f on $\mathbb{R}^n$,

$$\text{Fourier transform } f(x)\, e^{-\pi|x|^2} \;=\; i^{-d} \cdot f(x)\, e^{-\pi|x|^2}$$

That is,

$$\int_{\mathbb{R}^n} f(x)\, e^{-\pi|x|^2} \cdot e^{-2\pi i\langle \xi, x\rangle}\, dx \;=\; i^{-d} \cdot f(\xi)\, e^{-\pi|\xi|^2}$$

Proof: We reduce to the case $f(x_1, x_2, x_3, \ldots, x_n) = (x_1 + ix_2)^d$ by using the irreducibility of H_d as $O(n)$-representation space, as follows. Certainly $\varphi(x) = (x_1 + ix_2)^d$ is in H_d. Suppose for the moment that it satisfies Hecke's identity. By irreducibility, the collection of finite linear combinations of translates of φ by $O(n)$ is the whole space H_d. Recall that Fourier transform *commutes* with the action of $g \in O(n)$: with $\langle \xi, x\rangle = x\xi^\top$ for row vectors x, ξ,

$$\int_{\mathbb{R}^n} F(xg) \cdot e^{-2\pi i\langle \xi, x\rangle}\, dx \;=\; \int_{\mathbb{R}^n} F(x) \cdot e^{-2\pi i\langle \xi, xg^{-1}\rangle}\, dx \;=\; \int_{\mathbb{R}^n} F(x) \cdot e^{-2\pi i\langle \xi g, x\rangle}\, dx \;=\; \widehat{F}(\xi g)$$

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[20] Sometimes Gelfand-Pettis integrals are called *weak* integrals, but in this use *weak* refers to the hypothesis, not the conclusion. A different approach to integration of vector-valued functions, imitating the Riemann or Lebesgue constructions, is the Bochner integral, which we do not need here.

by changing variables, since

$$\langle \xi, xg^{-1}\rangle \;=\; xg^{-1}\xi^\top \;=\; xg^\top \xi^\top \;=\; x(\xi g)^\top \;=\; \langle \xi g, x\rangle$$

Thus, it suffices to prove Hecke's identity for $(x_1 + ix_2)^d$.

Indeed, since the Fourier transform factors over the coordinates, and

$$e^{-\pi|x|^2} \;=\; e^{-\pi(x_1^2+\ldots+x_n^2)} \;=\; e^{-\pi(x_1^2+x_2^2)} \cdot e^{-\pi(x_3^2+\ldots+x_n^2)}$$

it suffices to treat the two-dimensional case. Let $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, and also $w = \xi_1 + i\xi_2$ and $\overline{w} = \xi_1 - i\xi_2$. Since $x_1\xi_1 + x_2\xi_2 = \frac{1}{2}(z\overline{w} + \overline{z}w)$,

$$\Big(\text{Fourier transform } z^d\, e^{-\pi z\overline{z}}\Big)(w) \;=\; \int_{\mathbb{C}} z^d\, e^{-\pi z\overline{z}} \cdot e^{-\pi i(z\overline{w}+\overline{z}w)}\; dx_1\, dx_2$$

Since

$$(-\pi i z)^d \;=\; \Big(\frac{\partial}{\partial \overline{w}}\Big)^d e^{-\pi i(z\overline{w}+\overline{z}w)}$$

we have

$$(-\pi i z)^d \cdot \Big(\text{Fourier transform } z^d\, e^{-\pi z\overline{z}}\Big)(w) \;=\; \Big(\frac{\partial}{\partial \overline{w}}\Big)^d \int_{\mathbb{C}} e^{-\pi z\overline{z}} \cdot e^{-\pi i(z\overline{w}+\overline{z}w)}\; dx_1\, dx_2$$

Recall that the Gaussian $e^{-\pi|x|^2}$ is its own Fourier transform, so this is

$$(-\pi i z)^d \cdot \Big(\text{Fourier transform } z^d\, e^{-\pi z\overline{z}}\Big)(w) \;=\; \Big(\frac{\partial}{\partial \overline{w}}\Big)^d e^{-\pi w\overline{w}} \;=\; (-\pi w)^d \cdot e^{-\pi w\overline{w}}$$

Cancelling the factors of π gives Hecke's identity. ///

[9.0.2] Remark: Thus, harmonic-polynomial multiples of Gaussians are *eigenfunctions* for Fourier transform. There are many other distinguishable eigenfunctions, but the present examples will have applications in construction of theta series relevant to equidistribution problems.

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10. *Appendix: Bernstein's proof of Weierstraß approximation*

Weierstraß proved that polynomials are dense in $C^o(C)$ for compact subsets C of $\mathbb{R}^n$, with respect to sup-norm. Decades later, Stone greatly abstracted this. Prior to Stone, S. Bernstein gave a memorable argument for Weierstraß' concrete case, with the additional virtue of suggesting similarly intuitive arguments for various function spaces on topological spaces with transitive group actions.

[10.0.1] Theorem: Given $f \in C^o(U)$ for U open in $\mathbb{R}^n$, for compact $C \subset U$ and $\varepsilon > 0$, there is a polynomial P such that

$$\sup_{x\in C} |f(x) - P(x)| \;<\; \varepsilon$$

Proof: The idea is to create a sequence P_ℓ of polynomials on $\mathbb{R}^n$ whose restrictions φ_ℓ to a fixed compact, such as the cube

$$C \;=\; \{x = (x_1, \ldots, x_n) \;:\; |x_i| \le 1, \text{ for all } i\}$$

form an *approximate identity*, in the sense that their masses bunch up at 0.

More precisely, we want the restrictions φ_ℓ to C to be non-negative, to have integrals 1 on the smaller cube $\frac{1}{2}C$, and to satisfy

$$\lim_{\ell\to\infty} \frac{\int_{\delta C} \varphi_\ell}{\int_{\frac{1}{2}C} \varphi_\ell} \;=\; 1 \qquad\qquad \text{(for each fixed positive } \delta \le \tfrac{1}{2}\text{)}$$

Granting all that, we can show that the *mollifications* of f by the φ_ℓ approach f in the sup norm:

$$\lim_{\ell\to\infty} \sup_{x\in\frac{1}{2}C} \Big| f(x) - \int_{\frac{1}{2}C} f(x+y)\,\varphi_\ell(y)\,dy \Big| \;=\; 0 \qquad\qquad (\text{as } \ell\to\infty)$$

Indeed,

$$\int_{\frac{1}{2}C} f(x+y)\,\varphi_\ell(y)\,dy \;=\; \int_{\delta C} f(x+y)\,\varphi_\ell(y)\,dy + \int_{\frac{1}{2}C-\delta C} f(x+y)\,\varphi_\ell(y)\,dy$$

$$=\; f(x)\int_{\delta C} \varphi_\ell(y)\,dy + \int_{\delta C} \Big(f(x+y) - f(x)\Big)\,\varphi_\ell(y)\,dy + \int_{\frac{1}{2}C-\delta C} f(x+y)\,\varphi_\ell(y)\,dy$$

The first integral goes to 1 as $\ell\to\infty$, for fixed $\delta>0$, by the bunching-up property. The second integral goes to 0 uniformly in x as $\delta\to 0$, by the uniform continuity of f on C. The third integral goes to 0 as $\ell\to\infty$, since the masses of the φ_ℓ bunch up inside δC. Thus, assuming we have such polynomials,

$$\lim_{\ell\to\infty} \int_{\frac{1}{2}C} f(x+y)\,\varphi_\ell(y)\,dy \;=\; f(x) \qquad\qquad (\text{uniformly in } x\in\tfrac{1}{2}C)$$

At the same time,

$$\int_{\frac{1}{2}C} f(x+y)\,\varphi_\ell(y)\,dy \;=\; \int_{x+\frac{1}{2}C} f(y)\,\varphi_\ell(-x+y)\,dy$$

is a superposition of polynomials of degrees at most that of φ_ℓ. The space V of such polynomials is finite-dimensional. Thus, this integral of a compactly-supported continuous V-valued function lies in V. That is, this integral is equal to a *polynomial*, as a function. This would prove the theorem.

To make suitable polynomials P_ℓ, it suffices to treat the single-variable case. Let

$$P_\ell(x) \;=\; (1-x^2)^\ell \qquad\qquad (\text{for } x\in\mathbb{R})$$

First, determine where the second derivative vanishes: solve

$$0 \;=\; \frac{d}{dx}\Big(-2\ell x(1-x^2)^{\ell-1}\Big) \;=\; 4\ell(\ell-1)x^2(1-x^2)^{\ell-2} - 2\ell(1-x^2)^{\ell-1}$$

$$=\; 2\ell\cdot\Big((\ell-1)x^2 - (1-x^2)\Big)\cdot(1-x^2)^{\ell-2}$$

Thus, in the interior of $[-1,1]$, the second derivative vanishes at $\pm 1/\sqrt{\ell}$, so the curve bends downward in $[-1/\sqrt{\ell}, 1/\sqrt{\ell}]$, and bends upward outside that interval. In particular, the line segments from the points $(\pm 1/\sqrt{\ell}, 0)$ to $(0,1)$ are below the graph of P_ℓ, so

$$\int_{|x|\le 1/\sqrt{\ell}} P_\ell(x)\,dx \;\ge\; \frac{2}{\sqrt{\ell}}$$

On the other hand, the standard fact that

$$\lim_{\ell\to\infty}(1-\frac{x}{\ell})^\ell \;=\; e^{-x}$$

suggests a certain approach. For example,

$$P_\ell\Big(\frac{\sqrt{\log \ell}}{\sqrt{\ell}}\Big) \;=\; \Big(1 - \frac{\log \ell}{\ell}\Big)^\ell$$

Since $\log(1-x) \le -x$ for $x \ge 0$,

$$\log P_\ell\Big(\frac{\sqrt{\log \ell}}{\sqrt{\ell}}\Big) \;\le\; -\log \ell$$

Thus,

$$P_\ell\Big(\frac{\sqrt{\log \ell}}{\sqrt{\ell}}\Big) \;\le\; \frac{1}{\ell}$$

Thus we have a sufficient bunching-up result: obviously $\frac{1}{\sqrt{\ell}} \;<\; \frac{\log \ell}{\sqrt{\ell}}$, so

$$\int_{|x|<\frac{\log \ell}{\sqrt{\ell}}} P_\ell(x)\,dx \;\;\ge\;\; \frac{2}{\sqrt{\ell}}$$

while

$$\int_{\frac{\log \ell}{\sqrt{\ell}}<|x|<1} P_\ell(x)\,dx \;\;\le\;\; \frac{2}{\ell}$$

That is, letting

$$\varphi_\ell(x) \;=\; \frac{P_\ell(x)}{\displaystyle\int_{|x|\le\frac{1}{2}} P_\ell(x)\,dx}$$

gives the single-variable approximate identity desired. The product

$$\varphi_\ell(x_1)\ldots\varphi_\ell(x_n)$$

is the desired collection for $\mathbb{R}^n$. ///

[10.0.2] Remark: Although it is unnecessary for the above, it is interesting to determine the integral of the single-variable P_ℓ over $[-1,1]$. Integrating by parts repeatedly, it is

$$\int_{-1}^1 (1-x)^\ell\cdot(1+x)^\ell\,dx \;=\; \frac{\ell}{\ell+1}\int_{-1}^1 (1-x)^{\ell-1}\cdot(1+x)^{\ell+1}\,dx \;=\; \frac{\ell(\ell-1)}{(\ell+1)(\ell+2)}\int_{-1}^1 (1-x)^{\ell-2}\cdot(1+x)^{\ell+2}\,dx$$

$$=\;\ldots\;=\;\frac{\ell!\,\ell!}{(2\ell)!}\int_{-1}^1 (1+x)^{2\ell}\,dx \;=\; \frac{\ell!\,\ell!}{(2\ell)!}\,\frac{2^{2\ell+1}}{2\ell+1}$$

[Stein-Weiss 1978] E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Press, 1978.
