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Unwinding and integration on quotients

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The simplest case of *unwinding* is for $f \in C_c^o(\mathbb{R})$:

$$\int_{\mathbb{R}/\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} f(x+n) \Big)\, dx \;\; = \;\; \int_{\mathbb{R}} f(x)\, dx$$

In fact, the integral on the quotient $\mathbb{R}/\mathbb{Z}$ is unequivocally *characterized*[1] by this relation, once we know that the averaged functions $\sum_n f(x+n)$ are at least *dense* in $C^o(\mathbb{R}/\mathbb{Z})$. As corollary, for $F \in C^o(\mathbb{R}/\mathbb{Z})$, since $F \cdot f \in C_c^o(\mathbb{R})$,

$$\int_{\mathbb{R}/\mathbb{Z}} F(x) \Big(\sum_{n\in\mathbb{Z}} f(x+n) \Big)\, dx \; = \; \int_{\mathbb{R}/\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} F(x)\, f(x+n) \Big)\, dx$$

$$= \; \int_{\mathbb{R}/\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} F(x+n)\, f(x+n) \Big)\, dx \; = \; \int_{\mathbb{R}} F(x)\, f(x)\, dx$$

We need analogous assertions with less elementary group actions and less transparent representatives for the quotients. For example, with $\Gamma = SL_2(\mathbb{Z})$ and $\mathfrak{H}$ the upper half-plane, integration on $\Gamma\backslash\mathfrak{H}$ is characterized by requiring, for all $f \in C_c^o(\mathfrak{H})$,

$$\int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma} f(\gamma z) \Big)\, \frac{dx\, dy}{y^2} \;\; = \;\; \int_{\mathfrak{H}} f(z)\, \frac{dx\, dy}{y^2}$$

once we know that the averages $\sum_{\gamma\in\Gamma} f(\gamma z)$ are at least *dense* in $C_c^o(\Gamma\backslash\mathfrak{H})$. In fact, such averaging maps are universally *surjective* on compactly-supported continuous functions, as demonstrated just below.

An important variant[2] uses $f \in C_c^o(\Gamma_\infty\backslash\mathfrak{H})$ for a *subgroup* Γ_∞ of Γ. By the surjectivity of averaging maps, take $\varphi \in C_c^o(\mathfrak{H})$ such that

$$\sum_{\beta\in\Gamma_\infty} \varphi \circ \beta \;\; = \;\; f$$

[1] The Riesz-Markov-Kakutani theorem asserts that every (continuous) functional on compactly-supported continuous functions on a reasonable topological space X is $f \to \int_X f(x)\, d\mu(x)$ for some measure μ. Relying on this, specification of a functional (integration) on $C_c^o(X)$ specifies a measure. In fact, we care more about the integral than about the measure.

[2] This variant of *unwinding* arose most prominently in the Rankin-Selberg method, where $\int_{\Gamma\backslash\mathfrak{H}} |f|^2 \cdot E_s$ for cuspform f and Eisenstein series E_s is unwound using the definition of E_s as wound *up* from y^s. This theme is pervasive in the theory of automorphic forms.

so then

$$\int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f\circ\gamma \Big) \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Big(\sum_{\beta\in\Gamma_\infty} (\varphi\circ\beta)\circ\gamma \Big) \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma} \varphi\circ\gamma \Big)$$

$$= \int_{\mathfrak{H}} \varphi \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} \Big(\sum_{\beta\in\Gamma_\infty} \varphi\circ\beta \Big) \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} f$$

The corollary with $F \in C^o(\Gamma\backslash\mathfrak{H})$ and $f \in C^o_c(\Gamma_\infty\backslash\mathfrak{H})$ is

$$\int_{\Gamma\backslash\mathfrak{H}} F\cdot \Big(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f\circ\gamma \Big) \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (F\cdot f)\circ\gamma \Big) \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} F\cdot f$$

Letting $L^1(G)$, $L^1(\Gamma\backslash\mathfrak{H})$, and $L^1(\Gamma_\infty\backslash\mathfrak{H})$ be the completions of $C^o_c(\mathfrak{H})$, $C^o_c(\Gamma\backslash\mathfrak{H})$, and $C^o_c(\Gamma_\infty\backslash\mathfrak{H})$ with respect to the corresponding L^1 norms

$$|f|_{L^1(\mathfrak{H})} \;=\; \int_{\mathfrak{H}} |f| \qquad |f|_{L^1(\Gamma\backslash\mathfrak{H})} \;=\; \int_{\Gamma\backslash\mathfrak{H}} |f| \qquad |f|_{L^1(\Gamma_\infty\backslash\mathfrak{H})} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} |f|$$

Extension by continuity gives the same *unwinding* property of integrals on L^1 spaces.

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1. *Surjectivity of averaging maps*

By convention, a *topological group* is a *locally compact, Hausdorff* topological space G with a *continuous* group operation $G\times G\to G$, and continuous *inversion* map $g\to g^{-1}$. To avoid pathologies with regard to measures on products, we require that topological groups have a *countable basis.*

Let dg be a right G-invariant measure [3] on G, meaning that for $f\in C^o_c(G)$

$$\int_G f(g)\,dg \;=\; \int_G f(gh)\,dg \;=\; \int_G f(g)\,d(gh^{-1}) \;=\; \int_G f(g)\,dg \qquad \text{(for all } h\in G)$$

Let $\delta_G : G\to(0,+\infty)$ be the *modular function* of G, gauging the discrepancy between left and right invariant measures, in the sense that $\mathrm{meas}\,(gE) = \delta_G(g)\cdot\mathrm{meas}\,(E)$ for a measurable set $E\subset G$. Then $\delta_G^{-1}(g)\,dg$ is a *left* invariant measure.

Let H be a closed subgroup of G, with right invariant measure dh, and modular function δ_H.

[1.0.1] Lemma: The *averaging map* $\alpha : C^o_c(G)\to C^o_c(H\backslash G)$ by

$$(\alpha f)(g) \;=\; \int_H f(hg)\,dh$$

is *surjective.*

Proof: Let $q: G\to H\backslash G$ be the quotient map. Let U be a neighborhood of $1\in G$ having compact closure $\overline{U}$. For each $g\in G$, gU is a neighborhood of g. The images $q(gU)$ are open, by the characterization of the quotient topology. Given $F\in C^o_c(H\backslash G)$, the support $\mathrm{spt}(F)$ of F is covered by the opens $q(gU)$, and admits a finite subcover $q(g_1U),\ldots,q(g_nU)$. The set

$$C \;=\; q^{-1}(\mathrm{spt}(F)) \;\cap\; \Big(g_1\overline{U}\cup\ldots\cup g_n\overline{U}\Big) \;\subset\; G$$

[3] Right or left G-invariant positive regular Borel measures on G are (right or left) *Haar* measures on G.

is compact, and $q(C) = \mathrm{spt}(F) \subset H\backslash G$. Let $f \in C_c^o(G)$ be identically 1 on a compact neighborhood of C, and non-negative real-valued everywhere. Then $\alpha f \in C_c^o(H\backslash G)$ is strictly positive on the compact set $\mathrm{spt}(F)$, so has a strictly positive lower bound μ there.

By continuity, $V = \{x \in H\backslash G : \alpha f(x) > \mu/2\}$ is an open subset of $H\backslash G$ containing $q(C) = \mathrm{spt}(F)$, and $\alpha f(x) \geq \mu/2$ on the closure $\overline{V}$. Then $F/\alpha f$ is continuous on V. Let

$$\Phi(x) \;=\; \begin{cases} f(x) \cdot \frac{F(qx)}{\alpha f(x)} & (\text{for } x \in V) \\ 0 & (\text{for } x \notin V) \end{cases}$$

By design, $\alpha(\Phi) = F$, and Φ is continuous in V. This is the main argument.

Continuity of Φ in the interior of the complement of $\overline{V}$ is clear, but one might worry about continuity of Φ at points on the boundary ∂V of V:

Since $\overline{V}$ is compact, ∂V is compact. Thus, there is a neighborhood N of ∂V on which $\alpha f > \mu/4$. Every neighborhood of every point of ∂V contains a point *not* in V, so not in $\mathrm{spt}(F)$, so by continuity f is 0 on ∂V. Given $\varepsilon > 0$, there is an open neighborhood N' of the compact set ∂V on which $|F| < \varepsilon$. On the neighborhood $N'' = N \cap N'$ of ∂V the continuous quotient $F/\alpha f$ is bounded by $4\varepsilon/\mu$. Thus, assigning values 0 to Φ on ∂V is compatible both with values 0 off $\overline{V}$ and values $F/\alpha f$ in V. ///

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2. *Invariant measures and integrals on quotients* $H\backslash G$

[2.0.1] **Theorem:** The quotient $H\backslash G$ has a right G-invariant measure if and only if $\delta_G\big|_H = \delta_H$. In that case, the integral is unique up to scalars, and is characterized as follows. For given right Haar measure dh on H and for given right Haar measure dg on G there is a unique invariant measure $d\dot{g}$ on $H\backslash G$ such that for $f \in C_c^o(G)$

$$\int_{H\backslash G} \Big(\int_H f(hg)\, dh \Big)\, dg \;=\; \int_G f(g)\, dg \qquad (\text{for } f \in C_c^o(G))$$

Proof: First, prove the *necessity* of the condition on the modular functions. Suppose that there is a right G-invariant measure on $H\backslash G$. Let α be the averaging map $f \to \int_H f(hg)\, dh$. For $f \in C_c^o(G)$ the map

$$f \to \int_{H\backslash G} \alpha f(\dot{g})\, d\dot{g}$$

momentarily emphasizing the coordinate $\dot{g}$ on the quotient, is a right G-invariant functional (with the continuity property as above), so by uniqueness of right invariant measure on G must be a constant multiple of the Haar integral

$$f \longrightarrow \int_G f(g)\, dg$$

The averaging map behaves in a straightforward manner under left translation $L_h f(g) = f(h^{-1}g)$ for $h \in H$: for $f \in C_c^o(G)$ and for $h \in H$

$$\alpha(L_h f)(g) \;=\; \int_H f(h^{-1}xg)\, dx \;=\; \delta_H(h) \int_H f(xg)\, dx$$

by replacing x by hx. Then

$$\int_G f(g)\, dg \;=\; \int_{H\backslash G} \alpha(f)(g)\, d\dot{g} \;=\; \delta(h)^{-1} \int_{H\backslash G} \alpha(L_h f)(g)\, d\dot{g} \;=\; \delta(h)^{-1} \int_G f(h^{-1}g)\, dg$$

by comparing the iterated integral to the single integral. Replacing g by hg in the integral gives

$$\int_G f(g)\,dg \;=\; \delta(h)^{-1}\delta_G(h)\int_G f(g)\,dg$$

Choosing f such that the integral is not 0 implies the stated condition on the modular functions.

Proof of *sufficiency* starts from *existence* of Haar measures on G and on H. For simplicity, first suppose that both groups are *unimodular*. As expected, attempt to define an integral on $C_c^o(H\backslash G)$ by

$$\int_{H\backslash G} \alpha f(\dot g)\,d\dot g \;=\; \int_G f(g)\,dg$$

invoking the fact that the averaging map α from $C_c^o(G)$ to $C_c^o(H\backslash G)$ is surjective. The potential problem is *well-definedness*. It suffices to prove that $\int_G f(g)\,dg = 0$ for $\alpha f = 0$. Indeed, for $\alpha f = 0$, for all $F \in C_c^o(G)$, the integral of F against αf is certainly 0. Rearrange

$$0 \;=\; \int_G F(g)\,\alpha f(g)\,dg \;=\; \int_G\int_H F(g)\,f(hg)\,dh\,dg \;=\; \int_H\int_G F(h^{-1}g)\,f(g)\,dg\,dh$$

by replacing g by $h^{-1}g$. Replace h by h^{-1}, so

$$0 \;=\; \int_G \alpha F(g)\,f(g)\,dg$$

Surjectivity of α shows that F can be chosen so that αF is identically 1 on the support of f. Then the integral of f is 0, as claimed, proving the well-definedness for unimodular H and G.

For not-necessarily-unimodular H and G, in the previous argument the left translation by h^{-1} produces a factor of $\delta_G(h^{-1})$. Then replacing h by h^{-1} converts right Haar measure to left Haar measure, so produces a factor of $\delta_H(h)^{-1}$, and the other factor becomes $\delta_G(h)$. If $\delta_G(h)\cdot\delta_H(h)^{-1} = 1$, then the product of these two factors is 1, and the same argument goes through, proving well-definedness. ///

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3. *Uniqueness of invariant integrals*

The *uniqueness* of invariant measure and integrals on a topological group G is a special case of a more general uniqueness result for invariant functionals. The general *existence* argument is of a different nature, but in tangible circumstances existence is often clear for other reasons.

The argument here illustrates the usefulness of the *Gelfand-Pettis* or *weak* integral, itself discussed further below.

A translation-invariant function f on the real line, that is, a function with $f(x+y) = f(x)$ for all $x, y \in \mathbb{R}$, is *constant*, by a point-wise argument:

$$f(x) \;=\; (T_x f)(0) \;=\; f(0) \qquad \text{(with translation action } T_x f(y) = f(x+y))$$

The same conclusion holds for translation-invariant *distributions*, but we cannot argue in terms of point-wise values.

[3.0.1] **Theorem:** *(Uniqueness of Haar measure)* On a topological group G, [4] there is a unique *right* G-invariant element of the dual space $C_c^o(G)^*$ (up to constant multiples), namely the right-invariant integral

$$f \;\longrightarrow\; \int_G f(g)\,dg \qquad \text{(with right translation-invariant measure)}$$

The same proof gives a much broader result:

[4] A *topological group* is usually understood to be locally compact and Hausdorff. To avoid measure-theoretic pathologies, a *countable basis* is often assumed. Perhaps oddly, the local compactness excludes most topological vector spaces.

[3.0.2] Theorem: Let $V \subset C_c^o(G)$ be a *quasi-complete locally convex*[5] topological vector space of complex-valued functions on G stable under left and right translations, and containing an *approximate identity* $\{\varphi_i\}$. In this context, an *approximate identity* on a topological group G is a sequence of non-negative, compactly-supported continuous functions φ_i with $\int_G \varphi_i = 1$, whose supports shrink to $\{1_G\}$, meaning that, for every neighborhood U of 1, there is i_o such that for all $i \geq i_o$ the support of φ_i is inside U. Urysohn's lemma implies that $C_c^o(G)$ contains an approximate identity. On topological groups G that are not *unimodular*, that is, on which a right Haar measure is not a left Haar measure, nevertheless $d(h^{-1})$ for right Haar measure is a left Haar measure, so for an approximate identity $\{\varphi_i\}$ for a fixed *right* Haar measure, $\{g \to \varphi_i(g^{-1})\}$ is an approximate identity for *left* Haar measure.

Then there is a unique *right* G-invariant element of the dual space V^* (up to constant multiples), namely

$$f \longrightarrow \int_G f(g)\, dg \qquad \text{(with right translation-invariant measure)}$$

Proof: Let $R_g f(y) = f(yg)$ be the right translation action of G on functions on G, and $L_g f(y) = f(g^{-1}y)$ left translation. Let φ_i be an approximate identity for a fixed right Haar measure dg. Since $\int_G \varphi_i(g)\, dg = 1$ and φ_i is non-negative, $\varphi_i(g)\, dg$ is a probability measure[6] on the (compact) support of φ_i. Thus, for any $f \in V$, we have a V-valued Gelfand-Pettis integral

$$R_{\varphi_i} f \;=\; \int_G \varphi_i(g)\, R_g f\; dg \;\in\; \text{closure of convex hull of } \{\varphi_i(g) f : g \in G\} \;\subset\; V$$

The assumption $V \subset C_c^o(G)$ is understood to entail that right and left translation action of G on V are *continuous*, meaning that $G \times V \to V$ by $g \times f \to R_g f$ and $g \to L_g f$ are (jointly) continuous. By continuity, given a neighborhood N of 0 in V, $R_{\varphi_i} f \in f + N$ for all sufficiently large i. Letting

$$\check{\varphi}_i(g) \;=\; \varphi_i(g^{-1})$$

so that $\{\check{\varphi}_i\}$ is an approximate identity for *left* Haar measure $d(g^{-1})$, also $L_{\check{ph}_i} f \to f$.

For a right-invariant (continuous) functional u on the space of functions V,

$$u(f) \;=\; \lim_i u\Big(g \to \int_G \check{\varphi}_i(h)\, f(h^{-1}g)\, d(h^{-1})\Big) \;=\; \lim_i u\Big(g \to \int_G \check{ph}_i(h^{-1})\, f(hg)\, dh\Big)$$

replacing h by h^{-1}. After further replacing h by hg^{-1}, then move the functional u inside the integral via the Gelfand-Pettis integral property, and invoke the invariance:

$$u(f) \;=\; \lim_i u\Big(g \to \int_G \check{ph}_i(gh^{-1})\, f(h)\, dh\Big) \;=\; \lim_i \int_G u\Big(g \to, \check{\varphi}_i(gh^{-1})\Big)\, f(h)\, dh$$

$$= \lim_i \int_G u\Big(g \to, \check{\varphi}_i(g)\Big)\, f(h)\, dh \;=\; \Big(\lim_i u\check{\varphi}_i\Big) \cdot \int_G f(h)\, dh$$

Apparently the limit exists, and gives the constant. ///

[5] The class of *quasi-complete, locally convex* topological vectorspaces includes essentially all reasonable examples: Hilbert, Banach, and Fréchet spaces, as well as *LF spaces*, that is, strict colimits of Fréchet, such as $C_c^o(\mathbb{R})$ and $C_o^\infty(\mathbb{R})$. Also included are these spaces' weak-star *duals*, and other spaces of mappings such as the *strong operator topology* on mappings between Hilbert spaces, in addition to the *uniform* operator topology. The general definition of quasi-completeness requires a bit more background on topological vector spaces.

[6] As usual, a *probability measure* is simply a non-negative, regular Borel measure with total measure 1.

4. *Preview of vector-valued integrals*

Rather than *constructing* integrals as limits following [Bochner 1935], [Birkhoff 1935], *et alia*, we use the [Gelfand 1936]-[Pettis 1938] *characterization* of integrals. Existence of Gelfand-Pettis integrals is proven separately, for vector spaces with adequate *completeness* properties. [7]

A useful sample application is broad justification of *differentiation of an integral* with respect to a parameter.

Let V be a topological vectorspace. For a continuous V-valued function f on a measure space X a *Gelfand-Pettis integral* of f is $I_f \in V$ such that

$$\lambda(I_f) \;=\; \int_X \lambda \circ f \qquad\qquad (\text{for all } \lambda \in V^*)$$

When it exists and is unique, this vector I_f would be denoted by

$$I_f \;=\; \int_X f \;=\; \int_X f(x)\,dx$$

In contrast to *construction* of integrals as limits of Riemann sums, the Gelfand-Pettis *characterization* is a property no reasonable notion of integral would lack. Since this property is an irreducible minimum, this definition of integral is called a *weak integral*.

Uniqueness of the integral is immediate when the dual V^* *separates points*, meaning that for $v \neq v'$ in V there is $\lambda \in V^*$ with $\lambda v \neq \lambda v'$. This separation property certainly holds for Hilbert spaces: the map $\lambda w = \langle w, v - v' \rangle$ is a continuous linear functional and $\lambda(v - v') \neq 0$ gives $\lambda v \neq \lambda v'$. The separation property for Banach spaces and more generally is part of the *Hahn-Banach theorem*. [8] [9] For the rest of this discussion, all topological vector spaces are assumed locally convex.

Similarly, *linearity* of $f \to I_f$ follows when V^* separates points. The remaining issue is *existence*. [10]

We integrate nice functions: compactly-supported and continuous, on measure spaces with *finite, positive, Borel* measures. In this situation, all the $\mathbb{C}$-valued integrals

$$\int_X \lambda \circ f \;=\; \int_X \lambda\big(f(x)\big)\,dx$$

exist for elementary reasons, being integrals of compactly-supported $\mathbb{C}$-valued continuous functions on a compact set with respect to a finite Borel measure.

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[7] Precisely, things work out fine for quasi-complete, locally convex topological vectorspaces. Again, this class includes Hilbert, Banach, Fréchet spaces, *LF spaces* such as $C^o_c(\mathbb{R})$ and $C^\infty_o(\mathbb{R})$), these spaces' weak-star *duals*, and spaces of mappings such as the *strong operator topology* on mappings between Hilbert spaces, in addition to the *uniform* operator topology.

[8] Hahn-Banach holds for all *locally convex* topological vector spaces, that is, topological vector space with a local basis at 0 consisting of *convex* sets. This includes Fréchet spaces, strict colimits of Fréchet spaces such as $C^o_c(\mathbb{R})$ or $C^\infty_c(\mathbb{R})$, dual spaces of these, and essentially every reasonable space.

[9] Although every reasonable topological vector space is locally convex, we can construct topological vector spaces *without* this property, whose main utility is illustrating the possibility of failure of local convexity.

[10] We require that the integral of a V-valued function be in the space V itself, rather than in a larger space containing V, such as a double dual V^{**}, for example. Some discussions of integration do allow integrals to exist in larger spaces.

The crucial requirement on V for *existence* turns out to be that *the convex hull of a compact set has compact closure*. It is not too hard to show that Hilbert, Banach, or Fréchet spaces have this property, because of their *metric completeness*.

Non-metrizable spaces need a subtler notion of completeness, *quasi-completeness*, meaning that *bounded* Cauchy *nets* converge. [11]

[4.0.1] Theorem: Let X be a compact Hausdorff topological space with a *finite*, positive, Borel measure. Let V be a locally convex topological vectorspace in which the *closure of the convex hull of a compact set is compact*. Then *continuous compactly-supported* V-valued functions f on X have Gelfand-Pettis integrals. Further,

$$\int_X f \;\in\; \text{meas}\,(X)\cdot\Big(\text{closure of convex hull of } f(X)\Big) \qquad\qquad (\textit{Proof later.})$$

[4.0.2] Remark: The conclusion that the integral of f lies in the closure of a convex hull, is the substitute for the estimate of a $\mathbb{C}$-valued integral by the integral of its absolute value.

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5. *Mapping property of Gelfand-Pettis integrals*

The characterization of Gelfand-Pettis integrals immediately and easily yields useful applications, many of them fitting under the following umbrella.

Let X be a compact, Hausdorff topological space with a positive, regular Borel measure. Let $T : V \to W$ be a continuous linear map of locally convex, quasi-complete topological vector spaces.

[5.0.1] Corollary: For a continuous V-valued function f on X,

$$T\Big(\int_X f\Big) \;=\; \int_X T\circ f$$

Proof: The right-hand side is the Gelfand-Pettis integral of the continuous, compactly-supported W-valued function $T\circ f$, while the left-hand side is the image under T of the Gelfand-Pettis integral of f.

Since W^* separates points, the equality will follow from proving that

$$\mu\Big(T\big(\int_X f\big)\Big) \;=\; \mu\Big(\int_X T\circ f\Big) \qquad\qquad (\text{for all } \mu\in W^*)$$

Noting that $\mu\circ T\in V^*$, from the characterization of the Gelfand-Pettis integrals,

$$\mu\Big(T\big(\int_X f\big)\Big) \;=\; (\mu\circ T)\Big(\int_X f\Big) \;=\; \int_X (\mu\circ T)f \;=\; \int_X \mu(T\circ f) \;=\; \mu\Big(\int_X T\circ f\Big)$$

as desired. ///

[5.1] Example: differentiation under the integral

For $F\in C^k\big([a,b]\times[c,d]\big)$, we claim that the $C^k[c,d]$-valued function f on $[a,b]$ given by

$$f(x)(y) \;=\; F(x,y)$$

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[11] In topological vectorspaces lacking countable local bases, quasi-completeness is more relevant than completeness. For example, the weak $*$-dual of an infinite-dimensional Hilbert space is *never* complete, but is always quasi-complete. This example is non-trivial.

is *continuous* as a $C^k[c,d]$-valued function, when the latter is given the standard Banach-space structure

$$|\varphi|_{C^k} \;=\; \sup_{0\le i\le k}\,\sup_{y\in[c,d]} |\varphi^{(i)}(y)|$$

Indeed, the partial derivatives of F (up to order k) with respect to its second argument are continuous on the compact set $[a,b]\times[c,d]$, so are *uniformly* continuous. That is, given $\varepsilon>0$, there is $\delta>0$ such that for *all* $x\in[a,b]$ and all $0\le i\le k$,

$$\sup_{0\le i\le k}\,\sup_{y\in[c,d]} \Big|\frac{\partial^i}{\partial y^i}F(x,y)-\frac{\partial^i}{\partial y^i}F(x',y)\Big| \;<\; \varepsilon \qquad\qquad (\text{for } |x-x'|<\delta)$$

This is exactly what is meant by saying that f is a continuous $C^k[c,d]$-valued function. By design, $T=\frac{\partial}{\partial y}$ is a continuous linear map $C^k[c,d]\to C^{k-1}[c,d]$. Then the Gelfand-Pettis mapping property gives

$$\frac{\partial}{\partial y}\int_a^b F(x,y)\,dx \;=\; T\int_a^b f(x)\,dx \;=\; \int_a^b Tf(x)\,dx \;=\; \int_a^b \frac{\partial}{\partial y}F(x,y)\,dx$$

That is, the interchange of these two limit processes is legitimate for *general* reasons.

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6. *Appendix: apocryphal lemma $X\approx G/G_x$ and other background*

Along with other background, the main point of this appendix is to prove that, with mild hypotheses, a topological space X acted upon transitively by a *topological* group G is homeomorphic to the quotient G/G_x, where G_x is the isotropy group of a chosen point x in X.

Not assuming a mastery of point-set topology, nor a mastery of ideas about topological *groups*, several basic ideas need development in the course of the proof. Everything here is completely standard and widely useful. The discussion includes a variant of the *Baire Category Theorem*[12] for locally compact Hausdorff spaces.

[6.0.1] Remark: Ignoring the topology, that is, as *sets*, the bijection $G/G_x\approx X$ by $g\cdot G_x\leftrightarrow gx$ is easy to see. In contrast, the *topological* aspects are not trivial. Surprisingly, the topology of the group G completely determines the topology of the set X on which it acts.

[6.0.2] Proposition: Let G be a locally compact, Hausdorff topological group[13] and X a locally compact Hausdorff topological space with a continuous transitive action of G upon X. [14] Suppose that G has a *countable basis.* [15] Let x be any fixed element of X, and G_x the isotropy group[16] The natural map

$$G/G_x\to X \text{ by } gG_x\to gx$$

[12] The more common form of the Baire Category Theorem asserts that a *complete metric space* is *not* a countable union of closed sets each containing no non-empty open set.

[13] As expected, this means that G is a group and is a topological space, the group multiplication is a continuous map $G\times G\to G$, and inversion is continuous. The *local compactness* is the requirement that every point has an open neighborhood with compact closure. The Hausdorff requirement is that any two distinct points $x\neq y$ have open neighborhoods $U\ni x$ and $V\ni y$ that are disjoint, that is, $U\cap V=\phi$.

[14] As expected, continuity of the action means that $G\times X\to X$ by $g\times x\to gx$ is continuous. The transitivity means that for any $x\in X$ the set of images of x by elements of G is the whole set X, that is, $\{gx: g\in G\}=X$.

[15] That is, there is a countable collection B (the basis) of open sets in G such that *any* open set is a union of sets from the basis B.

[16] As usual, the isotropy (sub-) group of x in G is the subgroup of group elements fixing x, namely, $G_x:=\{g\in G: gx=x\}$.

is a homeomorphism.

Proof: We must do a little systematic development of the topology of topological groups in order to give a coherent argument.

[6.0.3] Claim: In a locally compact Hausdorff space X, given an open neighborhood U of a point x, there is a neighborhood V of x with compact closure $\overline{V}$ and $\overline{V} \subset U$.

Proof: By local compactness, x has a neighborhood W with compact closure. Intersect U with W if necessary so that U has compact closure $\overline{U}$. Note that the compactness of $\overline{U}$ implies that the *boundary* [17] ∂U of U is compact. Using the Hausdorff-ness, for each $y \in \partial U$ let W_y be an open neighborhood of y and V_y an open neighborhood of x such that $W_y \cap V_y = \phi$. By compactness of ∂U, there is a finite list $y_1, \ldots, y_n$ of points on ∂U such that the sets U_{y_i} cover ∂U. Then $V = \bigcap_i V_{y_i}$ is open and contains x. Its closure is contained in $\overline{U}$ and in the complement of the open set $\bigcup_i W_{y_i}$, the latter containing ∂U. Thus, the closure $\overline{V}$ of V is contained in U. ///

[6.0.4] Claim: The map $gG_x \to gx$ is a continuous bijection of G/G_x to X.

Proof: First, $G \times X \to X$ by $g \times y \to gy$ is continuous by definition of the continuity of the action. Thus, with fixed $x \in X$, the restriction to $G \times \{x\} \to X$ is still continuous, so $G \to X$ by $g \to gx$ is continuous. The quotient topology on G/G_x is the unique topology on the *set* (of cosets) G/G_x such that any continuous $G \to Z$ constant on G_x cosets factors through the quotient map $G \to G/G_x$. That is, we have a commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & Z \\ \downarrow & \nearrow & \\ G/G_x & & \end{array}$$

Thus, the induced map $G/G_x \to X$ by $gG_x \to gx$ is continuous. ///

[6.0.5] Remark: We need to show that $gG_x \to gx$ is *open* to prove that it is a homeomorphism.

[6.0.6] Claim: For a given point $g \in G$, every neighborhood of g is of the form gV for some neighborhood V of 1.

Proof: First, again, $G \times G \to G$ by $g \times g \to gh$ is continuous, by assumption. Then, for fixed $g \in G$, the map $h \to gh$ is continuous on G, by restriction. And this map has a continuous inverse $h \to g^{-1}h$. Thus, $h \to gh$ is a homeomorphism of G to itself. In particular, since $1 \to g \cdot 1 = g$, neighborhoods of 1 are carried to neighborhoods of g, as claimed. ///

[6.0.7] Claim: Given an open neighborhood U of 1 in G, there is an open neighborhood V of 1 such that $V^2 \subset U$, where

$$V^2 = \{gh : g, h \in V\}$$

Proof: The continuity of $G \times G \to G$ assures that, given the neighborhood U of 1, the inverse image W of U under the multiplication $G \times G \to G$ is open. Since $G \times G$ has the product topology, W contains an open of the form $V_1 \times V_2$ for opens V_i containing 1. With $V = V_1 \cap V_2$, we have $V^2 \subset V_1 \cdot V_2 \subset U$ as desired.
///

[17] As usual, the *boundary* of a set E in a topological space is the intersection $\overline{E} \cap \overline{E^c}$ of the closure $\overline{E}$ of E and the closure $\overline{E^c}$ of the complement E^c of E.

[6.0.8] Remark: Similarly, but more simply, since inversion $g \to g^{-1}$ is continuous and is its own (continuous) inverse, for an open set V the image $V^{-1} = \{g^{-1} : g \in V\}$ is open. Thus, for example, given a neighborhood V of 1, replacing V by $V \cap V^{-1}$ replaces V by a smaller *symmetric* neighborhood, meaning that the new V satisfies $V^{-1} = V$.

The following result is not strictly necessary, but sheds some light on the nature of topological groups.

[6.0.9] Claim: Given a set E in G,

$$\text{closure } E = \bigcap_U E \cdot U$$

where U runs over open neighborhoods of 1. [18]

Proof: A point $g \in G$ is in the closure of E if and only if every neighborhood of g meets E. That is, from just above, every set gU meets E, for U an open neighborhood of 1. That is, $g \in E \cdot U^{-1}$ for every neighborhood U of 1. We have noted that inversion is a homeomorphism of G to itself (and sends 1 to 1), so the map $U \to U^{-1}$ is a bijection of the collection of neighborhoods of 1 to itself. Thus, g is in the closure of E if and only if $g \in E \cdot U$ for every open neighborhood U of 1, as claimed. ///

[6.0.10] Remark: This allows us to give another proof, for topological groups, of the fact that, given a neighborhood U of 1 in G, there is a neighborhood V of 1 such that $\overline{V} \subset U$. (We did prove this above for locally compact Hausdorff spaces generally.)

Proof: First, from the continuity of $G \times G \to G$, there is V such that $V \cdot V \subset U$. From the previous claim, $\overline{V} \subset V \cdot V$, so $\overline{V} \subset V \cdot V \subset U$, as claimed. ///

[6.0.11] Remark: We can improve the conclusion of the previous remark using the local compactness of G, as follows. Given a neighborhood U of 1 in G, there is a neighborhood V of 1 such that $\overline{V} \subset U$ and $\overline{V}$ is *compact*. Indeed, local compactness means exactly that there is a local basis at 1 consisting of opens with compact closures. Thus, given V as in the previous remark, shrink V if necessary to have the compact closure property, and still $\overline{V} \subset V \cdot V \subset U$, as claimed.

[6.0.12] Corollary: For an open subset U of G, given $g \in U$, there is a compact neighborhood V of $1 \in G$ such that $gV^2 \subset U$.

Proof: The set $g^{-1}U$ is an open containing 1, so there is an open $W \ni 1$ such that $W^2 \subset g^{-1}U$. Using the previous claim and remark, there is a compact neighborhood V of 1 such that $V \subset W$. Then $V^2 \subset W^2 \subset g^{-1}U$, so $gV^2 \subset U$ as desired. ///

[6.0.13] Claim: Given an open neighborhood V of 1, there is a countable list $g_1, g_2, \ldots$ of elements of G such that $G = \bigcup_i g_iV$.

Proof: To see this, first let $U_1, U_2, \ldots$ be a countable basis. For $g \in G$, by definition of a basis,

$$gV = \bigcup_{U_i \subset gV} U_i$$

Thus, for each $g \in G$, there is an index $j(g)$ such that $g \in U_{j(g)} \subset gV$. Do note that there are only countably many such indices. For each index i appearing as $j(g)$, let g_i be an element of G such that $j(g_i) = i$, that is,

$$g_i \in U_{j(g_i)} \subset g_i \cdot V$$

[18] This characterization of the closure of a subset of a topological group is very different from anything that happens in general topological spaces. To find a related result we must look at more restricted classes of spaces, such as *metric* spaces. In a metric space X, the closure of a set E is the collection of all points $x \in X$ such that, for every $\varepsilon > 0$, the point x is within ε of some point of E.

Then, for every $g \in G$ there is an index i such that

$$g \in U_{j(g)} = U_{j(g_i)} \subset g_i \cdot V$$

This shows that the union of these $g_i \cdot V$ is all of G. ///

A subset E of a topological space is **nowhere dense** if its closure contains no (non-empty) open set. [19]

[6.0.14] Claim: *(Variant of Baire Category theorem)* A locally compact Hausdorff topological space is *not* a countable union of nowhere dense sets. [20]

Proof: Let W_n be closed sets containing no non-empty open subsets. Thus, any non-empty open U meets the complement of W_n, and $U - W_n$ is a *non-empty* open. Let U_1 be a non-empty open with compact closure, so $U_1 - W_1$ is *non-empty* open. From the discussion above, there is a non-empty open U_2 whose *closure* is contained in $U_1 - W_1$. Continuing inductively, there are non-empty open sets U_n with compact closures such that

$$U_{n-1} - W_{n-1} \supset \bar{U}_n$$

Certainly

$$\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \ldots$$

Then $\bigcap \bar{U}_i \neq \phi$, by compactness. [21] [22] Yet this intersection fails to meet any W_n. In particular, it *cannot* be that the union of the W_n's is the whole space. ///

Now we can prove that $G/G_x \approx X$, using the viewpoint we've set up.

Given an open set U in G and $g \in U$, let V be a compact neighborhood of 1 such that $gV^2 \subset U$. Let $g_1, g_2, \ldots$ be a countable set of points such that $G = \bigcup_i g_i V$. Let $W_n = g_n V x \subset X$. By the transitivity, $X = \bigcup_i W_i$.

We observed at the beginning of this discussion that $G \to X$ by $g \to gx$ is continuous, so W_n is compact, being a continuous image of the compact set $g_n V$. So W_n is closed since it is a compact subset of the Hausdorff space X.

By the (variant) Baire category theorem, some $W_m = g_m V x$ contains a non-empty open set S of X. For $h \in V$ so that $g_m h x \in S$,

$$gx = g(g_m h)^{-1}(g_m h)x \in gh^{-1}g_m^{-1}S$$

[19] The union of all open subsets of a given set is its *interior*. Thus, a set is nowhere dense if its closure has empty interior.

[20] The more common verison of the Baire category theorem asserts the same conclusion for *complete metric* spaces. The argument is structurally identical.

[21] In Hausdorff topological spaces X compact sets C are closed, proven as follows. Fixing x not in C, for each $y \in C$, there are opens $U_y \ni x$ and $V_y \ni y$ with $U \cap V = \phi$, by the Hausdorff-ness. The U_y's cover C, so there is a finite subcover, $U_{y_1}, \ldots, U_{y_n}$, by compactness. The finite *intersection* $W_x = \bigcap_i V_{y_i}$ is open, contains x, and is disjoint from C. The union of all W_x's for $x \notin C$ is open, and is exactly the complement of C, so C is closed.

[22] The intersection of a nested sequence $C_1 \supset C_2 \supset \ldots$ of non-empty compact sets C_n in a Hausdorff space X is non-empty. Indeed, the complements $C_n^c = X - C_n$ are open (since compact sets are closed in Hausdorff spaces), and if the intersection were empty, then the union of the opens C_n^c would cover C_1. By compactness of C_1, there is a finite subcollection $C_1^c, \ldots, C_n^c$ covering C_1. But $C_1^c \subset \ldots \subset C_n^c$, and C_n^c omits points in C_n, which is non-empty, contradiction.

Every group element $y \in G$ acts by homeomorphisms of X to itself, since the continuous inverse is given by y^{-1}. Thus, the image $gh^{-1}g_m^{-1}S$ of the open set S is open in X. Continuing,

$$gh^{-1}g_m^{-1}S \subset gh^{-1}g_m^{-1}g_mVx \subset gh^{-1}Vx \subset gV^{-1} \cdot Vx \subset Ux$$

Therefore, gx is an interior point of Ux, for all $g \in U$. ///

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