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Adele groups, p-adic groups, solenoids

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[This document is
http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/12_1_adeles.pdf]

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Although we will also give the more typical descriptions, we also show that the adeles $\mathbb{A}$, p-adic integers $\mathbb{Z}_p$, p-adic rationals $\mathbb{Q}_p$, and related objects are already right under our noses, if only we can see them. These things are *not* artificial or mere stylistic choices. It is unwise to ignore them.

Several important technical issues lead to such things. One is closer examination of *Hecke operators*, for example on holomorphic modular forms:

$$T_N f \;=\; \sum_{\gamma \in X_N} f|_{2k}\gamma \qquad\qquad (f \text{ holomorphic weight } 2k \text{ for } \Gamma = SL_2(\mathbb{Z}))$$

where X_N is integer-entry 2-by-2 matrices with determinant 1, with the weight-$2k$ action

$$\big(f|_{2k}\gamma\big)(z) \;=\; f\big(\frac{az+b}{cz+d}\big)\,(cz+d)^{-2k} \qquad\qquad (\text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

An obstacle to a straightforward proof of the *self-adjointness* of these operators with respect to the Petersson inner product

$$\langle f, F\rangle \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\,\overline{F}(z)\,y^{2k}\,\frac{dx\,dy}{y^2}$$

is that, although the *sum* over X_N is a modular form of level one, the individual *summands* are typically *not* level one, but *higher* level, usually level N. That is, the most straightforward assertion we can rely upon is that for $\gamma \in X_N$ the γ^{th} summand is a modular form only for

$$\Gamma_N \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$$

That is, the straightforward assertion is that

$$\big(f|_{2k}\gamma\big)\big|_{2k}\beta \;\;=\;\; f|_{2k}\gamma \qquad\qquad (\text{for } \beta \in \Gamma_N)$$

To see how the sufficient condition $\beta \in \Gamma_N$ arises, use the associativity of the action:

$$f|_{2k}(\gamma\beta) \;=\; f|_{2k}(\gamma\beta\gamma^{-1}\cdot\gamma) \;=\; \big(f|_{2k}\gamma\beta\gamma^{-1}\big)\big|_{2k}\gamma$$

Thus, $f|_{2k}\gamma$ is *assured* invariant under β when $\gamma\beta\gamma^{-1} \in \Gamma_1$, for f of level 1. For example,

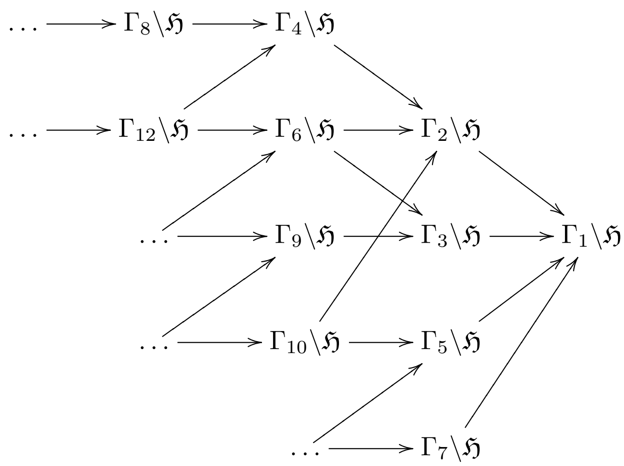
$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix}\gamma^{-1} \;=\; \begin{pmatrix} a & b/N \\ cN & d \end{pmatrix} \qquad\qquad (\text{for } \gamma = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix})$$

which is in $\Gamma_1 = SL_2(\mathbb{Z})$ only for $b = 0 \bmod N$. Similarly,

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma^{-1} \;=\; \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix} \qquad\qquad (\text{for } \gamma = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix})$$

is in Γ_1 only for $c = 0 \bmod N$. Generally, with $\gamma \in X_N$, similar computations show that $\beta \in \Gamma_N$ is the simplest sufficient condition for $(f|_{2k}\gamma)|_{2k}\beta = f|_{2k}\gamma$.

To treat the summands in $T_N f$ directly, we might enlarge the space of modular forms to include those for Γ_N for all N. The full family of quotients $\Gamma_N\backslash\mathfrak{H}$, indexed by positive integers N *ordered by divisibility*, is partly depicted in

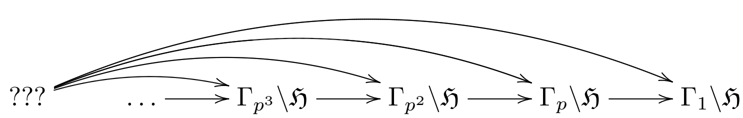


(where $\Gamma_N z \to \Gamma_M z$ for $M|N$)

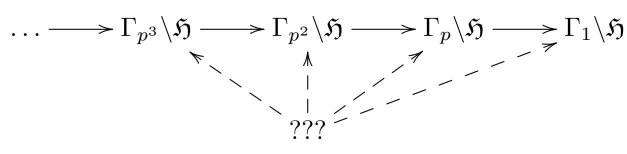
The poset [1] of positive integers ordered by divisibility is very interesting, but adds a further complication, so we might consider subfamilies corresponding to powers of a single prime, with their simpler partial order structures:

$$\ldots \longrightarrow \Gamma_{p^3}\backslash\mathfrak{H} \longrightarrow \Gamma_{p^2}\backslash\mathfrak{H} \longrightarrow \Gamma_p\backslash\mathfrak{H} \longrightarrow \Gamma_1\backslash\mathfrak{H}$$

In both cases, we want a *single object* on which all the functions on all spaces $\Gamma_{p^n}\backslash\mathfrak{H}$ live, with quotient maps to every $\Gamma_{p^n}\backslash\mathfrak{H}$, fitting together so that all triangles *commute* [2] in the diagram



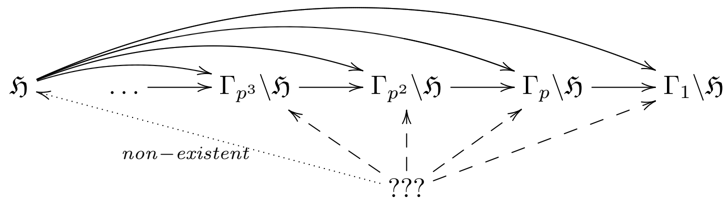
The corresponding diagram for the full family of $\Gamma_N\backslash\mathfrak{H}$ is messier, but similar. The admittedly natural candidate $\mathfrak{H}$ itself, with obvious quotient maps $q_N : \mathfrak{H} \to \Gamma_N\backslash\mathfrak{H}$ fails in several ways. First, although functions $f \in L^2(\Gamma_N\backslash\mathfrak{H})$ become functions on $\mathfrak{H}$ by composing with q_N, the function $f \circ q_N$ is *not* in $L^2(\mathfrak{H})$ with respect to the $SL_2(\mathbb{R})$-invariant measure $dx\,dy/y^2$. Second, the collection of *all* functions on $\mathfrak{H}$ is far *too large* in the sense that most functions are *not* invariant under any Γ_N. Third, certainly the *true* object ??? fits into a diagram



[1] *Poset* is the common abbreviation for *partially ordered set*, meaning a set S with an ordering $\le$ with expected properties: *transitivity* $x \le y$ and $y \le z$ implies $x \le z$, *reflexivity* $x \le x$, and $x \le y$ and $y \le x$ implies $x = y$, but possibly lacking the dichotomy requirement that *either* $x \le y$ *or* $y \le x$.

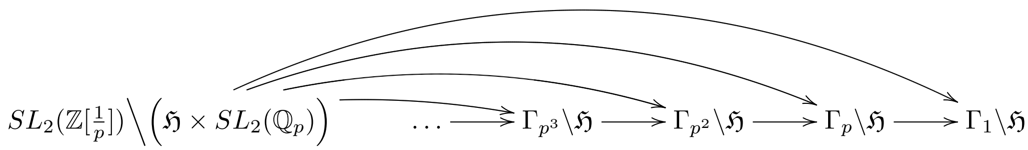
[2] For a diagram to *commute* means that the same outcomes are obtained no matter which route is followed throught the diagram.

with commuting triangles but it turns out that there is *no* map ??? $\to \mathfrak{H}$ fitting into



(fails ...)

We will see that the correct object ??? in the diagram is

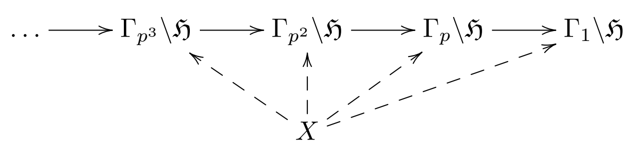


where $\mathbb{Q}_p$ is the field of p-adic numbers, the fraction field of the p-adic *integers* $\mathbb{Z}_p$, the latter often defined as the completion of $\mathbb{Z}$ with respect to the p-adic metric $d(x,y) = |x-y|_p$ with the p-adic norm

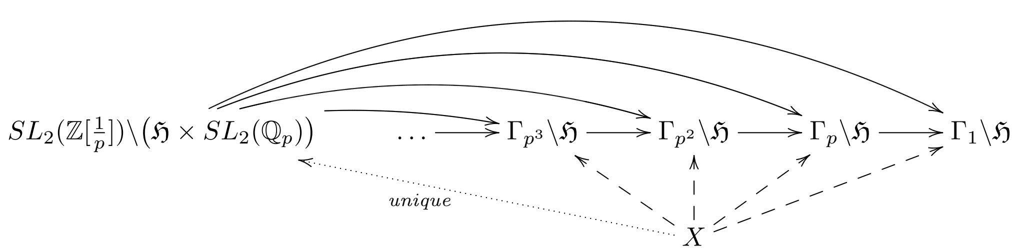
$$|p^\ell \cdot m|_p \;=\; p^{-\ell} \qquad\qquad \text{(for integers } m \text{ prime to } p\text{)}$$

and $\mathbb{Z}[\frac{1}{p}]$ is the ring of rational numbers with denominators only allowed to be powers of p. The action of $\gamma \in SL_2(\mathbb{Z}[\frac{1}{p}])$ on $\mathfrak{H} \times SL_2(\mathbb{Q}_2)$ is $g(z \times h) = gz \times gh$ with linear fractional action on $z \in \mathfrak{H}$ and left matrix multiplication on $h \in SL_2(\mathbb{Q}_p)$. We will see that the diagonally-imbedded copy of $SL_2(\mathbb{Z}[\frac{1}{p}])$ is a *discrete* subgroup of $SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$ much as $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$.

We will see that this object *does* have the desired universal property, that for every other object X with a compatible family of maps (making triangles commute)

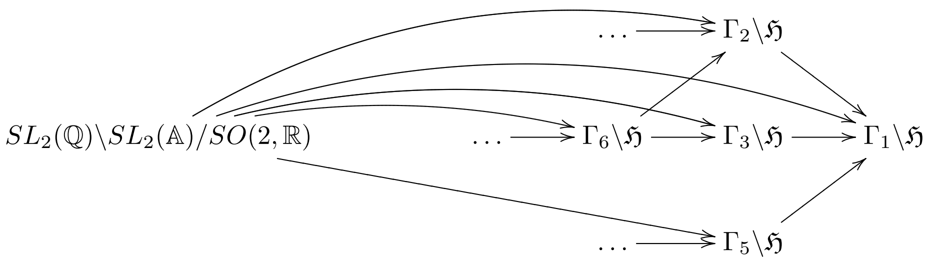


there is a *unique* map $X \to SL_2(\mathbb{Z}[\frac{1}{p}])\backslash\big(\mathfrak{H} \times SL_2(\mathbb{Q}_2)\big)$ giving a commutative diagram

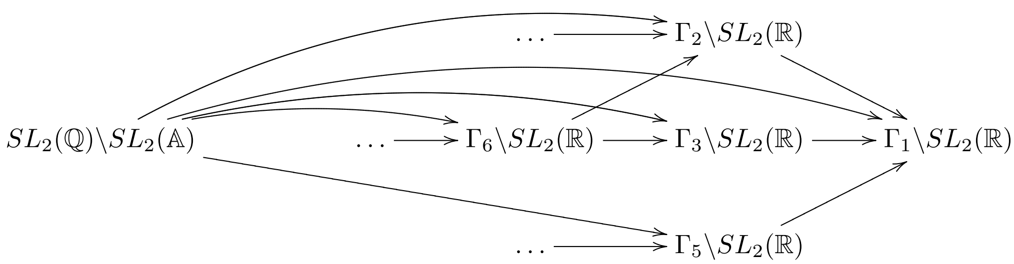


This situation also reveals a natural action of the group $SL_2(\mathbb{Q}_p)$ on the *collection* of quotients $\Gamma_{p^n}\backslash\mathfrak{H}$. This is very far from obvious!

Keeping in mind that $\mathfrak{H} \approx SL_2(\mathbb{R})/SO(2,\mathbb{R})$, the full diagram using all Γ_N's has a similar *universal object*, with $\mathbb{A}$ the ring of rational *adeles*[3]

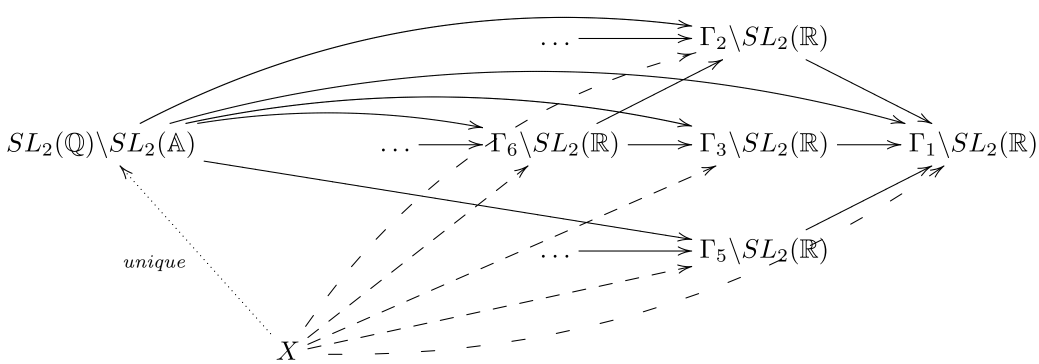


For many reasons we might want to get the group $SO(2,\mathbb{R})$ out of the way, so that $SL_2(\mathbb{R})$ can *act on the right* as do the groups $SL_2(\mathbb{Q}_p)$, and we have a diagram



This reveals the natural action of $SL_2(\mathbb{A})$ on the right on the *family* of quotients $\Gamma_N\backslash SL_2(\mathbb{R})$, incorporating an action of *every* group $SL_2(\mathbb{Q}_p)$ as well as $SL_2(\mathbb{R})$.

We will prove the *universal mapping property* that, for every space X and compatible family of maps $X \to \Gamma_N\backslash SL_2(\mathbb{R})$, there is a unique map $X \to SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})$ fitting into a commutative diagram

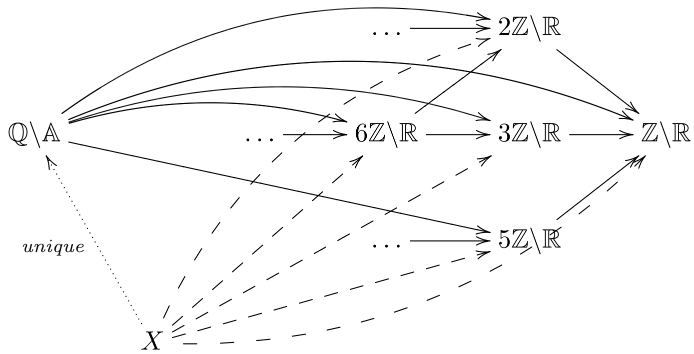


Standard terminology for this property is that $SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})$ is the *(projective) limit* of the quotients $\Gamma_N\backslash SL_2(\mathbb{R})$.

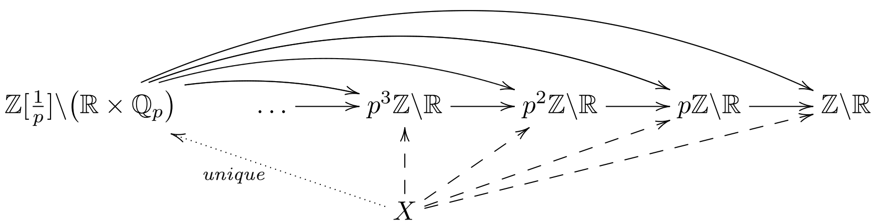
To warm up to this situation, replace Γ_N and $SL_2(\mathbb{R})$ by simpler objects, namely, replace Γ_N by $N\mathbb{Z}$ and

[3] As discussed below, $\mathbb{A}$ contains a copy of every $\mathbb{Q}_p$ as well as $\mathbb{R}$, but is smaller than the product.

$SL_2(\mathbb{R})$ by $\mathbb{R}$. In that case, we will find that $\mathbb{Q}\backslash\mathbb{A}$ is the universal object in analogous diagrams

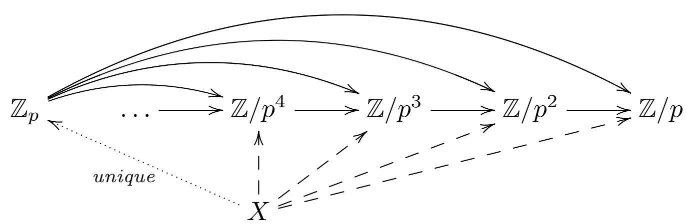


Again, first with the simpler poset of powers p^n of a fixed prime p, rather than the whole poset of positive integers ordered by divisibility: for every compatible family of maps $X \to p^n\mathbb{Z}\backslash\mathbb{R}$, there is a unique map $X \to \mathbb{Z}[\frac{1}{p}]\backslash(\mathbb{R} \times \mathbb{Q}_p)$ fitting into



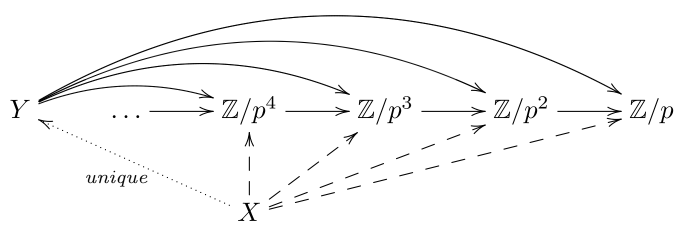
All the quotients $N\mathbb{Z}\backslash\mathbb{R}$ are *circles*, and the *(projective) limit* $\mathbb{Z}[\frac{1}{p}]\backslash\mathbb{R}$ is a *solenoid*.

As an even simpler examples of *limits*: the ring of p-adic integers $\mathbb{Z}_p$ is the *limit* of the finite rings $\mathbb{Z}/p^n$: for every compatible family of maps $X \to \mathbb{Z}/p^n$, there is a unique $X \to \mathbb{Z}_p$ giving a commutative diagram



This characterization of $\mathbb{Z}_p$ is more useful than the more-elementary description as metric completion, below.

It is striking and profoundly important that these diagrammatic universal-mapping-property characterizations determine the universal objects *uniquely* (up to unique isomorphism), as we will show. That is, for example, any *other* object Y fitting into all the diagrams



is *isomorphic* to $\mathbb{Z}_p$, and there is a *unique* isomorphism $Y \to \mathbb{Z}_p$ compatible with all the maps to the $\mathbb{Z}/p^n$'s. Further, the uniqueness proof does *not* use internal details of either the objects or the maps, but only the shape of the diagrams! Thus, the proofs themselves are *universal*.

1. *Hensel's lemma*

Kurt Hensel's 1897 conception of p-adic numbers can be illustrated in systematic solution of problems such as $x^2 = -1 \bmod 5^n$ for *all* powers 5^n of 5. [4]

Starting with either $x_1 = 2$ or $x_1 = 3$ (whose squares are 4 and 9, which are $-1 \bmod 5$), one hopes to *adjust* the solution mod 5 to be a solution mod 5^2. Namely, one hopes that for some y the modified value $x_2 = x_1 + 5y$ will satisfy $x^2 = -1 \bmod 25$. This condition simplifies usefully

$$\begin{aligned} (x_1+5y)^2 &= -1 \quad \bmod 5^2 \\ x_1^2 + 10x_1y + 25y^2 &= -1 \quad \bmod 5^2 \\ (x_1^2+1) + 10x_1y &= 0 \quad \bmod 5^2 \\ \frac{x_1^2+1}{5} + 2x_1y &= 0 \quad \bmod 5 \end{aligned}$$

since $x_1^2 + 1$ is divisible by 5. A critical point is that the y^2 term *disappears* mod 5^2, leaving in any case a *linear* problem in y. Since $-2x_1 = 1 \bmod 5$ is invertible mod 5, we can solve for

$$y = (-2x_1)^{-1} \cdot \frac{x_1^2+1}{5} = (-2\cdot 2)^{-1} \cdot \frac{2^2+1}{5} = 1 \bmod 5$$

and then

$$x_2 = x_1 + 5y = 2 + 1\cdot 5 = 7$$

satisfies

$$x_2^2 = (x_1+5y)^2 = -1 \bmod 5^2$$

This process of *improvement* can be continued indefinitely, imitating the example just done, as follows. With

$$x_n^2 = -1 \bmod 5^n$$

try to find y mod 5 such that

$$(x_n + 5^n y)^2 = -1 \bmod 5^{n+1}$$

The same rearrangement gives

$$\begin{aligned} (x_n+5^ny)^2 &= -1 \quad \bmod 5^{n+1} \\ x_n^2 + 2\cdot 5^n x_n y + 5^{2n}y^2 &= -1 \quad \bmod 5^{n+1} \\ (x_n^2+1) + 2\cdot 5^n x_n y &= 0 \quad \bmod 5^{n+1} \\ \tfrac{x_n^2+1}{5^n} + 2x_ny &= 0 \quad \bmod 5 \end{aligned}$$

[4] The abstracted notion of *metric space* did not exist until [Fréchet 1906], so Hensel could not use such an idea in 1897. The notion of *projective limit* was not developed until after 1940, for example by S. Eilenberg and S. MacLane, motivated by algebraic topology. Instead, Hensel attempted to make analogies to power series, but these analogies did not quite work right.

using the fact that x_n^2+1 is already divisible by 5^n, and that the y^2 term goes away. The last equation has a unique solution

$$y \;=\; (-2x_n)^{-1}\cdot \frac{x_n^2+1}{5^n} \bmod 5$$

where the inverse need be taken only mod 5, not modulo any higher power of 5. The new solution is

$$x_{n+1} \;=\; x_n + 5^n y$$

and satisfies

$$x_{n+1} \;=\; x_n \bmod 5^n$$

Thus, $x_n^{-1} = x_1^{-1}$ mod 5, so only a single multiplicative inverse need be computed, and the induction step is

$$y \;=\; (-2x_n)^{-1}\cdot \frac{x_n^2+1}{5^n} \bmod 5$$

This produces the sequence

$$2,\ 7,\ 57,\ 182,\ 1482,\ 13057,\ 25182,\ \ldots \quad\sim\quad \sqrt{-1}$$

This procedure is an example of *Hensel's lemma*. A little more generally:

[1.0.1] Claim: Let $f(x)\in\mathbb{Z}[x]$, p a prime, and x_1 such that

$$f(x_1)=0 \bmod p \qquad\text{and}\qquad f'(x_1)\neq 0 \bmod p$$

Then the recursion [5]

$$x_{n+1} \;=\; x_n - f(x_n)\cdot f'(x_1)^{-1} \bmod p^{n+1}$$

(where $f'(x_1)^{-1}$ is an inverse modulo p) determines a sequence of integers x_n such that

$$f(x_n) \;=\; 0 \bmod p^n$$

with the *compatibility*

$$x_{n+1} = x_n \bmod p^n$$

Proof: Amusingly, we need a *Taylor series expansion*

$$f(x+h) \;=\; f(x) + h\cdot f'(x) + (\text{error term})$$

legitimate for purely algebraic reasons, for polynomials, of the form

$$f(x+h) \;=\; f(x) + f'(x)\cdot h + E\cdot h^2$$

where E is a polynomial in x and h with coefficients in $\mathbb{Z}$. [6] Granting such an expression, let $\delta = -f'(x_n)^{-1}\cdot f(x_n)$, with inverse modulo p^n, and evaluate

$$f(x_{n+1}) \;=\; f(x_n+\delta) \;=\; f(x_n) + f'(x_n)\cdot\delta + E\cdot\delta^2$$

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[5] This recursive formula is the Newton-Raphson formula, easily derived geometrically in the real-number case, by *sliding down the tangent*, that is, by finding the intersection of the horizontal axis with the tangent line to the curve $y=f(x)$ at the point $(x_n, f(x_n))$.

[6] The *derivative* of a polynomial can be defined without taking any limits, via the usual formula $\frac{d}{dx}(x^n) = nx^{n-1}$, and requiring that this map be linear over whatever commutative ring the polynomials' coefficient lie in.

$$= f(x_n) - f'(x_n) \cdot f'(x_n)^{-1} \cdot f(x_n) + E \cdot \delta^2 \;=\; f(x_n) - f(x_n) + E(x_n) \cdot \delta^2 \;=\; E \cdot \delta^2$$

Since $f(x_n) = 0 \bmod p^n$, certainly $x_{n+1} = x_n \bmod p^n$, and $f'(x_n) \neq 0 \bmod p$, so has an inverse mod p^n, since f and f' have coefficients in $\mathbb{Z}$. And then $\delta = 0 \bmod p^n$, so $\delta^2 = 0 \bmod p^{2n}$. Since E is a polynomial with coefficients in $\mathbb{Z}$, $E \cdot \delta^2 = 0 \bmod p^{2n}$. That is,

$$f(x_{n+1}) \;=\; 0 \bmod p^{2n}$$

For $n \geq 1$, we have $2n \geq n + 1$, meeting the requirement on the recursion.

In the expression

$$x_{n+1} \;=\; x_n - f(x_n) \cdot f'(x_n)^{-1} \bmod p^{n+1}$$

since $f(x_n) = 0 \bmod p^n$, we only need $f'(x_n)^{-1}$ modulo p in order to know $x_{n+1} \bmod p^{n+1}$. Thus, it suffices to check that

$$f'(x_1)^{-1} \;=\; f'(x_n)^{-1} \bmod p$$

Indeed, since $x_{n+1} = x_n \bmod p^n$, $x_n = x_1 \bmod p$ for all n. Since f' has coefficients in $\mathbb{Z}$, we have $f'(x_n) = f'(x_1)$ for all n. Since $f'(x_1) \neq 0 \bmod p$, the inverses mod p are all the same.

The factorials occurring in the usual form of the Taylor expansion appear to be a problem. However, any polynomial $P(x) = \sum_i b_i x^i$ with coefficients in $\mathbb{Z}$ can be written in the form

$$P(x+h) \;=\; c_0 + c_1 \cdot h + c_2 \cdot h^2 + \ldots \qquad \text{(a finite expansion)}$$

with c_i polynomials in x by substituting $x + h$ in P and expanding in powers of h. Thus, the issue is to see that, in this expansion

$$c_1(x) \;=\; f'(x)$$

The requisite expansion is *linear*[7] in the polynomial P, it suffices to consider $P(x) = x^n$. By the Binomial Theorem

$$(x+h)^n \;=\; x^n + nx^{n-1} \cdot h + E \cdot h^2$$

where E is a polynomial in x and h, with coefficients in $\mathbb{Z}$. Since nx^{n-1} is the derivative of x^n, we have the desired sort of Taylor expansion, and Hensel's procedure will succeed. ///

[1.0.2] Remark: No special properties of the ring $\mathbb{Z}$ were used above, so the same argument succeeds, and this simple case of Hensel's lemma applies, to prime ideals in arbitrary commutative rings with identity.

[1.0.3] Remark: The *compatibility* $x_{n+1} = x_n \bmod p^n$ on a sequence of integers x_n exactly says that

$$\ldots \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1$$

fits into the diagram

$$\ldots \longrightarrow \mathbb{Z}/p^3 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p$$

[7] This *linearity* is that the expansion for the sum of two polynomials is the sum of the corresponding expansions.

2. *Metric definition of p-adic integers $\mathbb{Z}_p$ and p-adic rationals $\mathbb{Q}_p$*

The p-adic *norm* or *absolute value* $|n|_p$ of an integer $p^e m$ with m prime to p is

$$p\text{-adic norm of } m \cdot p^e \;=\; |m \cdot p^e|_p \;=\; p^{-e}$$

Additionally declare that $|0|_p = 0$. The p-adic *metric* on $\mathbb{Z}$ is

$$p\text{-adic distance } m \text{ to } n \;=\; |m - n|_p$$

The p-adic norm on $\mathbb{Q}$ extends that on $\mathbb{Z}$, namely

$$|p^n \cdot \frac{c}{d}|_p \;=\; p^{-n} \qquad\qquad (\text{integers } c, d \text{ prime to } p \text{ and } n \in \mathbb{Z})$$

Perhaps counter-intuitively, p is small and $1/p$ is large:

$$|p|_p \;=\; \frac{1}{p} \qquad\qquad |\frac{1}{p}|_p \;=\; p$$

In summary, *high divisibility by p* means p-adically *small*.

[2.0.1] Remark: In a context where the p-adic norm is the only norm used, we may suppress the subscript.

[2.0.2] Example: The sequences $\{x_n\}$ produced via Hensel's lemma to achieve $f(x_n) = 0 \bmod p^n$ (for given $f(x) \in \mathbb{Z}[x]$) are *Cauchy sequences*[8] in the p-adic metric, since $x_{m+1} = x_m \bmod p^m$ implies that for all pairs of indices $m \le n$

$$|x_m - x_n|_p \le |p^m|_p = p^{-m}$$

And, the fact that $f(x_n) = 0 \bmod p^n$ gives

$$\lim_n f(x_n) = 0 \qquad (\text{in the } p\text{-adic metric})$$

We define p-adic things as *completions*[9]

$$\begin{cases} p\text{-adic integers } \mathbb{Z}_p & = \quad p\text{-adic metric completion of } \mathbb{Z} \\ p\text{-adic numbers } \mathbb{Q}_p & = \quad p\text{-adic metric completion of } \mathbb{Q} \end{cases}$$

but we should check that the p-adic metric really is a metric. The positivity and symmetry of the associated p-adic metric are immediate, but the triangle inequality is not so immediate.

[2.0.3] Example: Partly for the visual effect, we note that

$$1 + 2 + 4 + 8 + 16 + \ldots \;=\; -1 \qquad (\text{in } \mathbb{Z}_2)$$

[8] The p-adic sense of *Cauchy sequence* is completely analogous to that in $\mathbb{R}$, namely, that a sequence $\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$ there is N such that for all $m, n \ge N$ we have $|x_m - x_n|_p < \varepsilon$.

[9] A metric space is *complete* if every *Cauchy sequence* converges. A *completion* of a metric space X is often defined by a construction (given in the appendix below), but, as discussed shortly, the *idea* is that the completion is the smallest complete metric space containing the given one.

This is genuinely valid in $\mathbb{Z}_2$. Superficially, thinking of the real numbers, this might be perceived as a *corruption* of the familiar identity

$$1 + r + r^2 + r^3 + \ldots \;=\; \frac{1}{1-r} \qquad\qquad (\text{for } |r| < 1)$$

for *real* or *complex* r, and with the usual real or complex absolute value $|r|$. But it is *not* a flawed version at all, since it is literally correct 2-adically.

Regarding the triangle inequality, in fact, we have a strange stronger property: [10]

[2.0.4] Proposition: *(ultrametric inequality)* For x, y in $\mathbb{Q}$,

$$|x + y| \;\le\; \max\{|x|,\ |y|\}$$

In fact, *equality* occurs in this last inequality, except possibly when $|x| = |y|$. Thus, in terms of the p-adic metric $d(x, y)$,

$$d(x, y) \;\le\; \max\{d(x, z),\ d(z, y)\}$$

with *equality* except possibly when $d(x, z) = d(z, y)$.

Proof: Let $x = p^m \cdot a/b$ and $y = p^n \cdot c/d$ with a, b, c, d prime to p, and positive or negative integers m, n. Without loss of generality, we can suppose that $m \le n$. Certainly

$$x + y = p^m \cdot \frac{a}{b} + p^n \cdot \frac{c}{d} \;=\; \frac{p^m ad + p^n cb}{bd} \;=\; p^m \cdot \frac{ad + p^{n-m} cb}{bd}$$

Note that, by unique factorization, bd is still prime to p. For $m < n$, the numerator in the fraction in

$$x + y \;=\; p^m \cdot \frac{ad + p^{n-m} cb}{bd}$$

is prime to p. Thus, for $m < n$, that is, for $|x| > |y|$,

$$|x + y| = p^{-m} = |x| = \max\{|x|,\ |y|\}$$

When $m = n$, that is, when $|x| = |y|$, the numerator may be *further* divisible by p in some cases. Thus, for $|x| = |y|$,

$$|x + y| \le p^{-m} = \max\{|x|,\ |y|\}$$

Then

$$d(x, y) \;=\; |x - y| \;=\; |(x - z) + (z - y)| \;\le\; \max\{|x - z|,\ |z - y|\} \;=\; \max\{d(x, z),\ d(z, y)\}$$

with equality unless possibly when $d(x, z) = d(z, y)$. ///

An *isometry* $f : X \to Y$ of metric spaces X, Y is a set map from X to Y such that distances are preserved, namely,

$$d_Y(f(x),\ f(x')) = d_X(x, x')$$

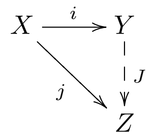
for all x, x' in X. [11] A metric space is *complete* if every Cauchy sequence converges.

---

[10] It is traditional at this point to say the the *ultrametric property* proven in the proposition can be construed as asserting that all p-adic triangles are *isosceles*.

[11] Beware that in some contexts an *isometry* is presumed to be a *bijection* to the target, in addition to preserving distance. In our context we specifically do *not* assume that isometries are surjections to the target spaces.

[2.0.5] Definition: A *completion* of a metric space X is a *complete* metric space Y and an isometry $i : X \to Y$ such that, for every isometry $j : X \to Z$ to a *complete* metric space Z, there is a unique isometry $J : Y \to Z$ giving a commutative diagram



As usual with mapping-property characterizations, there is at most one completion, up to unique isometric isomorphism. In the appendix below we give the usual construction, which proves *existence*.

As expected, define

$$\begin{cases} p\text{-adic integers } \mathbb{Z}_p & = \quad p\text{-adic metric completion of } \mathbb{Z} \\ \\ p\text{-adic numbers } \mathbb{Q}_p & = \quad p\text{-adic metric completion of } \mathbb{Q} \end{cases}$$

As usual, operations on the completions are defined as limits, and well-definedness must be proven. That is, for Cauchy sequences of rational numbers x_n and y_n with $x_n \to a$ and $y_n \to b$ p-adically, define

$$a + b \;=\; \lim_n (x_n + y_n) \qquad a \cdot b \;=\; \lim_n (x_n \cdot y_n) \qquad |a| \;=\; \lim_n |x_n|$$

[2.0.6] Proposition: The p-adic norm is a continuous function on the completion. The p-adic norm is *multiplicative* on the completions, that is,

$$|ab| \;=\; |a| \cdot |b|$$

Addition, multiplication, and multiplicative inverse (away from 0) are *continuous* maps in the p-adic metric. Also,

$$\mathbb{Z}_p \;=\; \{x \in \mathbb{Q}_p : |x| \le 1\} \;=\; \{x \in \mathbb{Q}_p : |x| < p\}$$

In particular, $\mathbb{Z}_p$ is both closed and open in $\mathbb{Q}_p$. The p-adic integers $\mathbb{Z}_p$ form an *integral domain*, and $\mathbb{Q}_p$ is its *field of fractions*. On $\mathbb{Q}_p$ it is still true that the *ultrametric inequality* holds:

$$|x + y| \;\le\; \max\{|x|, |y|\} \quad (\text{with } \textit{equality} \text{ except possibly when } |x| = |y|)$$

[2.0.7] Corollary: Polynomials with p-adic coefficients give continuous functions on $\mathbb{Q}_p$. ///

[2.0.8] Remark: The ultrametric property makes the set of x with $|x| \le 1$ a *subring*, since otherwise this set would not be closed under addition. There is no analogous subring of $\mathbb{R}$.

Proof: From the *general* theory of metric spaces the metric $d(,)$ on the completion is defined by taking limits

$$d(a, b) \;=\; \lim_n d(x_n, y_n)$$

where x_n, y_n are rational and $x_n \to a$ and $y_n \to b$. Part of the general assertion is that this is *well-defined*, that is, is independent of the Cauchy sequences approaching a and b. In the present situation, the extension of the p-adic norms to the completion is a special case, taking $b = 0$, so

$$|a| \;=\; d(a, 0) \;=\; \lim_n |x_n - 0| \;=\; \lim_n |x_n|$$

We have the expected

$$|a + b| \;=\; |a - (-b)| \;=\; d(a, b) \;\le\; d(a, 0) + d(0, b) \;=\; |a| + |b|$$

The p-adic continuity of the p-adic norm on $\mathbb{Q}$ is immediate from the continuity of the metric on the completion, which is a general fact about completions.

The multiplicativity $|xy| = |x| \cdot |y|$ follows for $x, y \in \mathbb{Q}$ from the fact that the ideal $p\mathbb{Z}$ is prime in $\mathbb{Z}$. That is, writing $x = p^m \cdot a/b$ and $y = p^n \cdot c/d$ with a, b, c, d relatively prime to p,

$$xy = p^{m+n} \cdot (ac)/(bd)$$

and by the primality of p the products ac and bd are still prime to p. Then for $x_n \to a$ and $y_n \to b$ with x_n, y_n in $\mathbb{Q}$,

$$|ab| \;=\; \lim_n |x_n y_n| \;=\; \lim_n (|x_n| \cdot |y_n|) \;=\; \lim_n |x_n| \cdot \lim_n |y_n| \;=\; |a| \cdot |b|$$

by continuity of multiplication of real numbers, since $|x_n| \to |a|$ and thus $\{|x_n|\}$ is Cauchy in $\mathbb{R}$ (as is $\{|y_n|\}$).

Continuity of addition is easy, from

$$|(x+y) - (x'+y')| \;\le\; |x - x'| + |y - y'|$$

For multiplication,

$$|(xy) - (x'y')| \;\le\; |x(y-y')| + |(x-x')y'| \;\le\; |x(y-y')| + |(x-x')(y'-y)| + |(x-x')y|$$

$$= \; |x||y-y'| + |x-x'||y'-y| + |x-x'||y|$$

Thus, given x, y in $\mathbb{Q}_p$ and x', y' sufficiently close to them, the products are close.

For multiplicative inverses, let $x \neq 0$. From

$$1 \;=\; |1| \;=\; |x \cdot x^{-1}| \;=\; |x| \cdot |x^{-1}|$$

we have

$$|x^{-1}| \;=\; \frac{1}{|x|}$$

and the continuity of inversion in $\mathbb{R}^\times$ gives the result.

Unsurprisingly, since $\mathbb{Z}$ is a subset of $\mathbb{Q}$, there is the same containment relation between their completions. Since the p-adic absolute value of $x \in \mathbb{Z}$ is at most 1, we immediately have containment in one direction, namely

$$\mathbb{Z}_p \;\subset\; \{x \in \mathbb{Q}_p : |x| \le 1\}$$

On the other hand, for $y \in \mathbb{Q}_p$ with $|y| \le 1$, since $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$, there is $r \in \mathbb{Q}$ arbitrarily close to y. For $|y - r| \le 1$,

$$|r| \;\le\; \max\{|r-y|, |y|\} \;\le\; 1$$

so $|r| \le 1$ itself. Thus, it suffices to show that r itself can be approximated arbitrarily well by elements of $\mathbb{Z}$.

Since $|r| \le 1$, $r = p^n \cdot \frac{a}{b}$ with $a, b \in \mathbb{Z}$ relatively prime to p and $n \ge 0$. For $b \neq 0 \bmod p$, Hensel's lemma gives a sequence of integers x_i such that

$$b \cdot x_i \;=\; 1 \bmod p^i$$

That is, x_i is a Cauchy sequence of integers approaching a multiplicative inverse b^{-1} of b in $\mathbb{Q}_p$. By continuity of the norm,

$$|b^{-1}| \;=\; \lim_i |x_i| \;=\; \lim_n 1 \;=\; 1$$

since p does not divide any of the integers x_i. Thus,

$$\lim_i p^n \cdot a \cdot x_i \;=\; p^n \cdot a/b \;=\; r$$

That is, as i varies the integers $p^n \cdot a \cdot x_i$ get close to r. This proves that $\mathbb{Z}$ is dense in $\{y \in \mathbb{Q}_p : |y| \le 1\}$, so $\mathbb{Z}_p$ is exactly the latter set, as claimed.

Since the possible values of the p-adic norm are only powers of p, the condition $|x| < p$ implies $|x| \le 1$.

If $ab = 0$, then $|ab| = 0$, and by multiplicativity $|a| \cdot |b| = 0$, and a or b is 0. Thus, $\mathbb{Z}_p$ is an integral domain.

Since $\mathbb{Q}$ is a field, it follows fairly easily that its completion $\mathbb{Q}_p$ is a field. To see that $\mathbb{Q}_p$ has no proper subfield that contains $\mathbb{Z}_p$, recall that any element $x \in \mathbb{Q}_p$ with $|x| \le 1$ lies in $\mathbb{Z}_p$. For $|x| > 1$, x cannot be 0, so has an inverse (since $\mathbb{Q}_p$ is a field), and $|x^{-1}| < 1$, so lies in $\mathbb{Z}_p$. This proves that $\mathbb{Q}_p$ is the fraction field of $\mathbb{Z}_p$.

Finally, the ultrametric inequality persists, seen as follows. Given $a, b \in \mathbb{Q}_p$, let x_n, y_n be rational numbers with $x_n \to a$ and $y_n \to b$. Given $\varepsilon > 0$, let n be large enough such that $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$. The *triangle inequality* gives

$$|x_n + y_n - (a+b)| \;<\; |x_n - a| + |y_n - b| \;<\; 2\varepsilon$$

and

$$|a+b| \le |(a-x_n)+(b-y_n)|+|x_n+y_n| \le 2\varepsilon+\max\{|x_n|,|y_n|\} \le 3\varepsilon+\max\{|x_n|-\varepsilon,|y_n|-\varepsilon\} < 3\varepsilon+\max\{|a|,|b|\}$$

This is true for every $\varepsilon > 0$, giving the ultrametric inequality. ///

[2.0.9] Proposition: The *units* in $\mathbb{Z}_p$ are

$$\mathbb{Z}_p^\times \;=\; \{x \in \mathbb{Q}_p :\; |x| = 1\}$$

The non-zero ideals in the ring $\mathbb{Z}_p$ are $p^n \cdot \mathbb{Z}_p$ for $0 \le n \in \mathbb{Z}$. We have $\mathbb{Z} \cap p^n\mathbb{Z}_p = p^n\mathbb{Z}$, and the natural maps

$$\mathbb{Z}/p^n\mathbb{Z} \;\longrightarrow\; \mathbb{Z}_p/p^n\mathbb{Z}_p$$

are isomorphisms.

Proof: For $x, y \in \mathbb{Z}_p$ such that $xy = 1$, of course $1 = |1| = |xy| = |x| \cdot |y|$. Since $|x| \le 1$ and $|y| \le 1$, both have norm 1. Thus,

$$\mathbb{Z}_p^\times \;\subset\; \{x \in \mathbb{Z}_p \;:\; |x| = 1\}$$

On the other hand, take $|x| = 1$ and $y \in \mathbb{Z}$ such that $|x - y| < 1$. Then

$$|y| \;=\; |y - x + x| \;=\; \max\{|y-x|,\, |x|\} \;=\; |x| \;=\; 1$$

since the norms are unequal. Thus, y is an integer not divisible by p. An exercise using Hensel's lemma shows y has an inverse in $\mathbb{Z}_p$. Then

$$x \;=\; x - y + y \;=\; y \cdot (1 + \frac{x-y}{y})$$

yields a convergent series expression for an inverse to x, namely

$$x^{-1} \;=\; y^{-1} \cdot \left((1 - \frac{x-y}{y} + \left(\frac{x-y}{y}\right)^2 - \left(\frac{x-y}{y}\right)^3 + \ldots \right)$$

which converges since $|(x-y)/y| < 1$. Thus, x^{-1} exists in $\mathbb{Z}_p$. The natural maps $\mathbb{Z} \to \mathbb{Z}_p$ and $p^n\mathbb{Z} \to p^n\mathbb{Z}_p$ give a map

$$\mathbb{Z}/p^n\mathbb{Z} \;\longrightarrow\; \mathbb{Z}_p/p^n\mathbb{Z}_p$$

For $x \in \mathbb{Z} \cap p^n\mathbb{Z}_p$, we have $|x| \le p^{-n}$, so p^n divides x. That is, $x \in p^n\mathbb{Z}$, as claimed. Thus, these natural maps are *injections*. For $y \in \mathbb{Z}_p$, consider the coset $y + p^n\mathbb{Z}_p$. Using the density of $\mathbb{Z}$ in $\mathbb{Z}_p$, let $x \in \mathbb{Z}$ with $|y - x| < p^{-n}$. Then $y - x \in p^n\mathbb{Z}_p$, so

$$y + p^n\mathbb{Z}_p \;=\; (y - x) + x + p^n\mathbb{Z}_p \;=\; p^n\mathbb{Z}_p + x + p^n\mathbb{Z}_p \;=\; x + p^n\mathbb{Z}_p$$

proving the surjectivity of the natural map.

Given a non-zero ideal I in $\mathbb{Z}_p$, let M be the sup of the p-adic norms of elements of I. We claim that the sup is attained at some element of I, and that such a largest element generates I. Indeed, there are only finitely-many values of the p-adic norm on $\mathbb{Z}_p$ lying in any interval $[\delta, 1]$ for $\delta > 0$, so the sup is attained. For x in I of maximum norm, for any $y \in I$, $|y/x| \le 1$, so $y/x \in \mathbb{Z}_p$. That is, $y \in \mathbb{Z}_p \cdot x$. This shows that $I = \mathbb{Z}.x$. ///

[2.0.10] Remark: One should prove local compactness of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ from this metric viewpoint as an exercise.

[2.0.11] Example: Let b be an integer relatively prime to a prime p. Let x_1 be an inverse of b mod p. Then the recursion

$$x_{n+1} \;=\; x_n - x_1^{-1} b x_n$$

produces a sequence of integers x_n such that

$$bx_n \;=\; 1 \bmod p^n$$

Indeed, letting $f(x) = kx - 1$, Hensel's Lemma gives the recursion

$$x_{n+1} = x_n - f(x_n) \cdot f'(x_1)^{-1} \;=\; x_n - (bx_n - 1) \cdot b^{-1} \;=\; x_n - (bx_n - 1) \cdot x_1$$

using only the inverse mod p, not any higher power of p. For $b \ne 0 \bmod p$ this gives a sequence of integers x_n such that

$$x_{n+1} \;=\; x_n \bmod p^n \qquad \text{and} \qquad b \cdot x_n \;=\; 1 \bmod p^n$$

---

3. *Elementary/clumsy definitions of adeles $\mathbb{A}$ and ideles $\mathbb{J}$*

We give the quick-but-opaque common definitions of *adeles* $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ and *ideles* $\mathbb{J} = \mathbb{J}_\mathbb{Q}$ of $\mathbb{Q}$. The sense and motivation and roles are discussed subsequently, in a slightly more sophisticated context.

For a finite set S of primes of $\mathbb{Z}$, let

$$\mathbb{A}_S \;=\; \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$$

with the product topology. The full adele ring $\mathbb{A}$ is the *ascending union*, sometimes called a *restricted product*, of the rings $\mathbb{A}_S$: [12]

$$\mathbb{A} \;=\; \bigcup_S \mathbb{A}_S \;=\; \{(x_\infty, x_2, \ldots, x_p, \ldots) \in \mathbb{R} \times \prod_p \mathbb{Q}_p \;:\; \text{all but finitely-many } x_p \text{ are in } \mathbb{Z}_p\}$$

A basis for a topology on $\mathbb{A}$ is given by the union of bases for the topologies of the subsets $\mathbb{A}_S$. That is, adeles are tuples of real numbers and p-adic numbers all-but-finitely-many of which are local integers.

Similarly, for a finite set S of primes of $\mathbb{Z}$, let

$$\mathbb{J}_S \;=\; \mathbb{R}^\times \times \prod_{p\in S} \mathbb{Q}_p^\times \times \prod_{p\notin S} \mathbb{Z}_p^\times$$

with the product topology. The full idele group $\mathbb{J}$ is the ascending union, sometimes called a *restricted product*, of the groups $\mathbb{J}_S$: [13]

$$\mathbb{J} \;=\; \bigcup_S \mathbb{J}_S \;=\; \{(x_\infty, x_2, \ldots, x_p, \ldots) \in \mathbb{R}^\times \times \prod_p \mathbb{Q}_p^\times \;:\; \text{all but finitely-many } x_p \text{ are in } \mathbb{Z}_p^\times\}$$

A basis for a topology on $\mathbb{J}$ is given by the union of bases for the topologies of the subsets $\mathbb{J}_S$.

[3.0.1] Remark: At this point, one should check that, while $\mathbb{J} \subset \mathbb{A}$, the subspace topology on $\mathbb{J}$ from $\mathbb{A}$ is strictly coarser than the topology on $\mathbb{J}$.

---

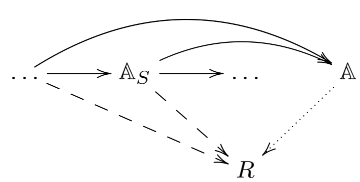
4. *Uniqueness of objects characterized by mapping properties*

By *object* we mean an instance of a type of mathematical entity, such as group, ring, topological space, or something combining these or other structures. By *map* we mean a *suitable* map among the specified objects: for groups a map is a group homomorphism, for topological spaces a map is a continuous function.

We make some very mild assumptions: the notion of *map* includes closure under composition, associativity of the composition of maps, and an *identity map* id_X of every suitable object X to itself, with the expected requirements that $\mathrm{id}_X \circ f = f$ for maps $f : Y \to X$, and $f \circ \mathrm{id}_X = f$ for maps $f : X \to Y$.

The collection of objects, together with the maps, is a *category*. [14]

[12] The term *restricted product* is *only* used in this clumsy definition of adeles and ideles. Since this notion does not directly refer to anything else for comparison, it is useless except as a reminder that the adeles are not the whole *product* of $\mathbb{R}$ and all the $\mathbb{Q}_p$'s. More unfortunately, that viewpoint fails to emphasize that $\mathbb{A}$ is a *colimit* of the rings $\mathbb{A}_S$, meaning that for every collection of maps $\mathbb{A}_S \to R$ to another topological ring R, compatible with inclusions, there is a unique map $\mathbb{A} \to R$ giving a commutative diagram



The colimit characterization shows that there is *no choice* in the topology. We will take up this viewpoint below.

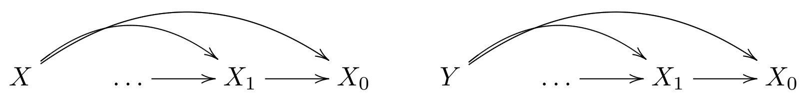
[13] Again, $\mathbb{J}$ is the *colimit* of the groups $\mathbb{J}_S$. Again, the *colimit* characterization of $\mathbb{J}$ shows that there is *no choice* of correct topology on $\mathbb{J}$: it is uniquely determined.

[14] We are doing *naive* category theory, as opposed to *formal* or *axiomatic* category theory, in the same way that ordinary mathematics uses *naive* set theory, not *formal* or *axiomatic* set theory.

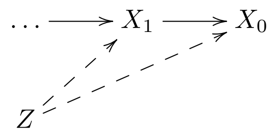
Given objects X_i with maps

$$\dots \longrightarrow X_1 \longrightarrow X_0$$

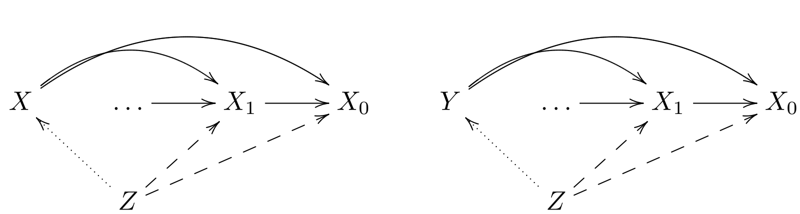
let object X and maps $X \to X_i$ and object Y and maps $Y \to X_i$ fit into diagrams



such that, for all families of maps $Z \to X_i$ such that all triangles commute in

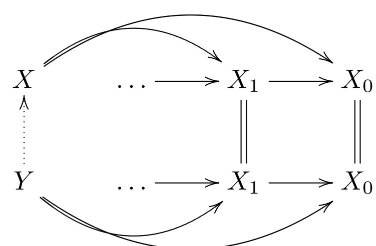


there are unique maps $Z \to X$ and $Z \to Y$ giving commutative diagrams

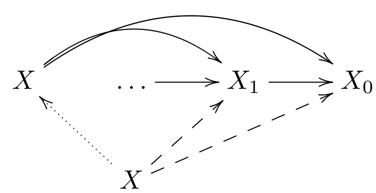


Then

[4.0.1] Claim: There is a *unique* isomorphism $Y \to X$ giving a commutative diagram

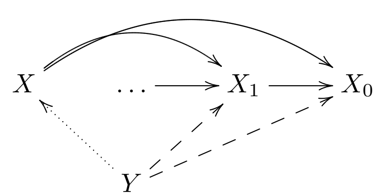


Proof: First, we prove that the only map of a projective limit to itself compatible with all projections is the identity map. That is, using $X \to X_i$ in the role of : $Z \to X_i$, we find a unique map $X \to X$ making all triangles commute in

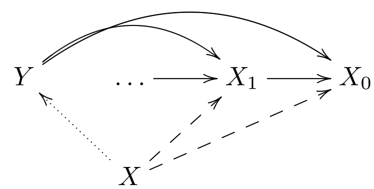


The identity map $\mathrm{id}_X : X \to X$ fits the role, so by uniqueness the *only* such map is id_X.

Now the main part of the proof. Let $Y \to X_i$ take the role of $Z \to X_i$. Then there is a unique $Y \to X$ such that all triangles commute in



To show that q is an isomorphism, reverse the roles of X and Y. Then there is a unique $X \to Y$ such that all triangles commute in



The composites $Y \to X \to Y$ and $X \to Y \to X$ are maps compatible with projections, so must be id_Y and id_X, by the first part of this argument. That is, these are mutually inverse maps, so the map $X \to Y$ is an isomorphism. ///

[4.0.2] **Remark:** As usual in these categorical arguments, requirements on the maps are packaged (or hidden) in the quantification over all objects Z and all families of maps $Z \to X_i$.

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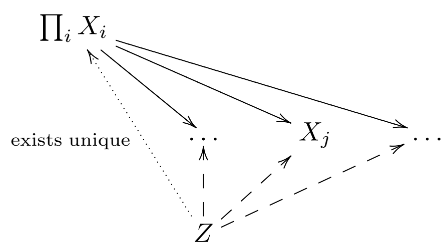
5. *Existence of limits*

Existence of *limits* of various types of objects can be proven by constructing the limits as subobjects of *products* of those objects. Most often, products of concrete [15] objects are the cartesian products of the underlying sets, with additional structure corresponding to the situation.

[5.1] Limits of sets As a warm-up, we consider sets with no further structures. We probably believe that the mapping-property *product* $\prod_i X_i$ of a family $\{X_i : i \in I\}$ of sets can be *constructed* as the usual Cartesian product, namely, as

$$\prod_{i \in I} X_i \;=\; \{\text{functions } f \text{ on } I \text{ so that } f(i) \in X_i\}$$

This does indeed partly beg the question, by making it depend on a too-vague notion of *function*, but never mind. The *projections* $p_j : \prod_i X_i \to X_j$ are $p_j(f) = f(j)$. We first demonstrate that the Cartesian product of sets does have the universal mapping property: given an arbitrary set Z and arbitrary set maps $f_j : Z \to X_j$, there should be a unique map $f : Z \to \prod_i X_i$ such that $f_j = p_j \circ f : Z \to X_j$ for all j. That is, there should be a commutative diagram



Taking $f(z) \in \prod_i X_i$ to be $f(z)(j) = f_j(z)$ gives *existence* of the map $Z \to \prod_i X_i$. And for any *other* map $g : Z \to \prod_i X_i$ with $f_j = p_j \circ g$, evaluating the latter condition at $z \in Z$ gives

$$f_j(z) \;=\; (p_j \circ g)(z) \;=\; p_j(g(z)) \;=\; g(z)(j)$$

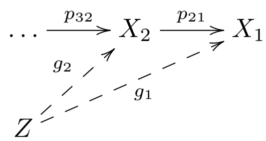
which agrees with f, proving that the Cartesian product of sets is the mapping-property product.

[15] An object is *concrete* if it is a *set* with additional structure, like a group, ring, vector space, topological space, and so on. Thus, concrete objects really do have *underlying sets*.

Limits of sets can be constructed as subsets of products, and the construction can be *discovered*, rather than merely verified, as follows. For a limit with a simple indexing poset of the form

$$\ldots \xrightarrow{p_{32}} X_2 \xrightarrow{p_{21}} X_1$$

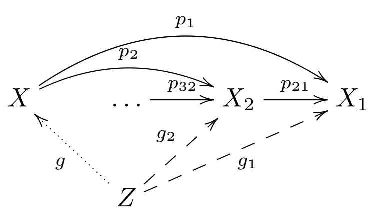
given a compatible family of maps $g_j : Z \to X_i$, that is, giving a commutative diagram



there is a unique map $g : Z \to \prod_i X_i$ to the *product*, by the universal property of that product. The compatibility property $p_{j+1,j}(g_{j+1}(z)) = g_j(z)$ on the maps $Z \to X_i$ make the image $g(Z) \subset \prod_i X_i$ lie inside the subset

$$X \;=\; \{\{x_1, x_2, \ldots\} \;:\; p_{j+1,j}(x_{j+1}) = x_j \text{ for all } j\} \;\;\subset\;\; \prod_j X_j$$

Unsurprisingly, on this subset the projections $p_j : \prod_i X_i \to X_j$ become *compatible* in the sense that $p_{j+1,j} \circ p_{j+1} = p_j : X \to X_j$. That is, we have a commutative diagram



That is, $X = \lim_i X_i$ can be *constructed* as compatible families, much as Hensel's lemma creates compatible sequences:

$$X \;=\; \{\{x_j \in X_j : p_{j+1,j}(x_{j+1}) = x_j \text{ for all } j\} \;=\; \{\ldots \to x_2 \to x_1\}$$

and $p_i : X \to X_i$ by $p_i\{x_j\} = x_i$. This all shows that limits of sets exist, by exhibiting a construction as a subset of the Cartesian product.

[5.2] Limits of concrete objects The previous argument shows that limits exist whenever products exist, and we showed that Cartesian products of sets are universal-mapping-property products. This correctly suggests that when the objects are sets with additional structure, first, products exist by constructing them as the set-product with corresponding additional structure, and then limits exist by constructing them as subsets of the product with additional compatibility requirements.

Since we explicitly demonstrate below that otherwise-constructed objects are limits, we stop the general discussion here.

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6. $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ as limits

[6.1] p-adic integers $\mathbb{Z}_p$ as limit The metric-completion definition of the p-adic integers $\mathbb{Z}_p$ really is the limit of the quotients $\mathbb{Z}/p^n$:

[6.1.1] Theorem: The limit $\lim_n \mathbb{Z}/p^n$ is the p-adic metric completion $\mathbb{Z}_p$ of $\mathbb{Z}$, with projections

$$\mathbb{Z}_p \xrightarrow{\text{quot}} \mathbb{Z}_p/p^n\mathbb{Z}_p \xrightarrow{\text{isom}} \mathbb{Z}/p^n$$

where the isomorphism $\mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n$ is the natural one exhibited earlier.

[6.1.2] Remark: We give two proofs, one emphasizing the *limit* viewpoint, the other emphasizing the *metric* viewpoint.

Proof: *(First)* The maps $q_n : \mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n$ are a compatible family of (continuous!) maps to the limitands in $\lim_n \mathbb{Z}/p^n$, so induce a map of $\mathbb{Z}_p$ to the limit. For each non-zero element $x \in \mathbb{Z}_p$, there is some exponent n such that the image of x in $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is non-zero, so $\mathbb{Z}_p$ *injects* to the limit. Thus, we might guess that $\mathbb{Z}_p$ *is* the limit, and try to verify this. [16] Let $f_n : Z \to \mathbb{Z}/p^n$ be a compatible family of maps from another object For fixed $z \in Z$, for each n choose $x_n \in \mathbb{Z}$ such that [17]

$$x_n + p^n\mathbb{Z}_p \;=\; f_n(z)$$

We claim that the sequence x_n is a *Cauchy* sequence in $\mathbb{Z}_p$, so by completeness we could take a limit

$$f(z) \;=\; \lim_n x_n \in \mathbb{Z}_p$$

Cauchy-ness follows from the compatibility of the f_n's, and, then, from the necessary compatibility of the integer representatives x_n. This defines a map $f : Z \to \mathbb{Z}_p$ compatible with the f_n's and the projections. We need to show there is a *unique* such f to prove that $\mathbb{Z}_p$ is the limit. For two maps f and g compatible with the projections and f_n's,

$$0 \;=\; p_n\left(f(z) - g(z)\right) \in \mathbb{Z}/p^n \approx \mathbb{Z}_p/p^n\mathbb{Z}_p$$

Taking the intersection over n gives the uniqueness $f(z) = g(z)$, proving $\mathbb{Z}_p$ is the limit. ///

Proof: *(Second)* We will prove that the limit is a completion of $\mathbb{Z}$ with respect to a metric which agrees on $\mathbb{Z}$ with the p-adic metric, and is complete, so (by the uniqueness of *completions*) naturally isomorphic to the completion $\mathbb{Z}_p$ of $\mathbb{Z}$. We need a bit of general discussion of products and limits of metric spaces:

[6.1.3] Claim: A *countable* product $\prod_i X_i$ of metric spaces X_i is *metrizable*. If every X_i is *complete*, then the product is complete.

[6.1.4] Remark: We will use explicit but non-canonical expressions for the metric on the product. For example, letting $d_i(,)$ be the metric on X_i, the expression

$$d(\{x_i\}, \{y_i\}) \;=\; \sum_{n\geq 1} 2^{-n}\,\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a metric on the product. The oddness of this expression is put into context by several observations. First, the powers of 2 appearing can be replaced by *any* sequence of positive real numbers whose sum converges. Second, the expressions

$$\frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

have the effect of specifying metrics giving the same topology on the X_i which are *bounded* by 1. The 1 in the denominator can be replaced by any positive real number, still giving the same topology. For notational ease, let us replace each $d_i(,)$ by $d_i(,)/(1 + d_i(,))$, so that we effectively assume that the metric on each of the factors is *already* bounded by 1, to avoid carrying along the more complicated expressions.

[16] The argument so far applies as well to $\mathbb{Z}$ itself, which does indeed inject to the limit, but is a proper subobject.

[17] That there exists such x_n is the content of the assertion that the natural map $\mathbb{Z}/p^n \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ is an isomorphism.

[6.1.5] Remark: The extreme ambiguity of the constants reminds us that many different metrics can give the same topology. Thus, a topology cannot possibly specify a canonical metric, in general. The uniqueness of the *topology* on a product has no canonical metric analogue.

Proof: *(of claim)* It is easy and not so interesting to verify that $d(,)$ gives a metric on the product. It is more interesting to see that it gives the product topology. [18] The trick is that a condition

$$d(\{x_i\}, \{y_i\}) \;<\; \varepsilon$$

gives no condition whatsoever on x_i and y_i for i large enough such that $2^{-i} < \varepsilon/2$, since

$$\frac{1}{2^i} + \frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \frac{1}{2^{i+3}} + \ldots \;=\; \frac{2^i}{1-\frac{1}{2}} = 2\cdot 2^{-i} \;<\; \varepsilon$$

and since $d_i(x_i, y_i)/(1 + d_i(x_i, y_i)) < 1$ for all inputs. For fixed $x = \{x_i\}$ in the product, the collection of $y = \{y_i\}$'s within distance $\varepsilon > 0$ of x includes the whole $X_i \times X_{i+1} \times \ldots$ for some large-enough index i.

More precisely, given $d(x, y) < \varepsilon$, necessarily $d_i(x_i, y_i) < 2^i \cdot \varepsilon$ for all i. Thus, given a collection of open balls in the individual X_i's, almost all [19] of them being the whole X_i, we can choose $\varepsilon > 0$ such that the resulting ball in the $d(,)$ metric is *contained in* the product of those balls.

Conversely, given x in the product and $\varepsilon > 0$, taking n large enough such that $2^{-n} < \varepsilon/2$, the product

$$(\varepsilon\text{-ball at } x_1) \times (\varepsilon\text{-ball at } x_2) \times \ldots (\varepsilon\text{-ball at } x_n) \times X_{n+1} \times X_{n+2} \times \ldots$$

is contained in the ε-ball at x. Thus, the two topologies are the same.

The more serious part of the claim is the completeness. Given a Cauchy sequence $\{x^{(k)}\}$ in the product with the funny metric, since the metric on the product dominates the (bounded-by-1) metrics on the factors, for each index i the i^{th} components $\{x_i^{(k)}\}$ are Cauchy in X_i, so have a limit x_i in X_i. It is irresistible to suspect that the point $x = \{x_i\}$ in the product is the limit of the original Cauchy sequence. Indeed, given $\varepsilon > 0$, take n large enough such that $2^{-n} < \varepsilon/2$. Let N be large enough such that for $k \geq N$ for each of the *finitely*-many $i \leq n$

$$d(x_i^{(k)}, x_i) < \varepsilon$$

Then it easily follows that for $k \geq N$

$$d(\{x_i^{(k)}\}, \{x_i\}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \ldots + \frac{\varepsilon}{2^n} + \frac{2^{-(n+1)}}{1-\frac{1}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \ldots + \frac{\varepsilon}{2^n} + \frac{\varepsilon}{2} = \varepsilon$$

That is, the original Cauchy sequence does indeed have the anticipated limit. Thus, with this funny metric, the product is *complete*. ///

Keep in mind that projective limits are subsets of the corresponding products, and can be given the metric from the product by restriction. Since *closed* subsets of complete metric spaces are complete, [20] this proves the completeness of the projective limit in this case.

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[18] It is especially interesting to see that this metric gives the product topology if one still thinks of the product topology as being disappointingly coarse. That is, it might seem unlikely that it could arise from a metric, but it does.

[19] This standard usage of *almost all* means *all but finitely-many.*

[20] That topologically closed subsets of complete metric spaces are again complete is straightforward: a Cauchy sequence in the subset does have a limit in the whole space by the completeness of the larger space, and the limit point lies in the subset, by closedness.

Now we can complete the metric-space proof that the projective limit definition of $\mathbb{Z}_p$ is the same as the metric one. The discussion just completed shows that $\lim_n \mathbb{Z}/p^n$ does have a structure of complete metric space. What remains is to show that this is the same as that on $\mathbb{Z}_p$. To do so, we show that we can choose a particular form of the metric on the limit such that the restrictions of the two metrics to $\mathbb{Z}$ are identical. We also must show that $\mathbb{Z}$ is *dense* in $\lim_n \mathbb{Z}/p^n$, and then we're done, by the uniqueness of metric completions.

Returning to the funny expressions for metrics on a countable product, give $\mathbb{Z}/p^n$ the natural metric that *any* discretely topologized set can be given, namely

$$d_n(x,y) \;=\; \begin{cases} 1 & (\text{for } x \neq y) \\ 0 & (\text{for } x = y) \end{cases}$$

Being a little clever in choice of constants, we try

$$d(\{x_n\},\{y_n\}) \;=\; \sum_{n\geq 1} p^{-n} \cdot d(x_n,y_n)$$

Unlike points in the whole product $\Pi_n \mathbb{Z}/p^n$, points in the limit have a compatibility condition. Thus, given $\{x_n\} \neq \{y_n\}$ in the limit, there is a unique index N such that $x_n = y_n$ for $n \leq N$ and $x_n \neq y_n$ for $n > N$. With this N,

$$d(\{x_n\},\{y_n\}) \;=\; \sum_{1\leq n\leq N} p^{-n}\cdot 0 + \sum_{n>N} p^{-n}\cdot 1 \;=\; \frac{p^{-(N+1)}}{1-\frac{1}{p}} \;=\; p^{-N}\cdot \frac{1}{p-1}$$

When $\{x_n\}$ and $\{y_n\}$ come from integers x, y, the integer N is the maximal one such that p^N divides $x-y$, so

$$d(x,y) \;=\; |x-y|_p \cdot \frac{1}{p-1}$$

That is, up to the easily reparable constant $1/(p-1)$, this contrived metric agrees on $\mathbb{Z}$ with the p-adic one.

Finally, we should show that $\mathbb{Z}$ is dense in the limit. Denseness is an intrinsic topological property, not necessarily metric, but since we've already contrived the metric we may as well use it again. Given a compatible sequence $\{x_n\}$ in the limit, and given $\varepsilon > 0$, let n be large enough such that $p^{-(n+1)} < \varepsilon$. Let x be an integer such that $x = x_n \bmod p^n$. Then the same sort of calculation gives

$$d(x,\{x_n\}) \;\leq\; \sum_{1\leq i\leq n} p^{-i}\cdot 0 + \sum i > np^{-i}\cdot 1 \;=\; \frac{p^{-(n+1)}}{1-\frac{1}{p}} \;=\; p^{-n}\cdot\frac{1}{p-1} < 2\varepsilon$$

Thus, $\mathbb{Z}$ is dense in the limit, so the limit is its metric completion. This proves (a second time) that the limit is the same as the metric completion definition of $\mathbb{Z}_p$. ///

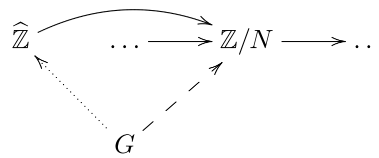
[6.2] $\widehat{\mathbb{Z}}$ as limit The poset of positive integers ordered by divisibility requires more complicated diagrams than powers of a single prime, the latter giving the much-simpler poset of non-negative integers ordered by size.

A common definition is

$$\widehat{\mathbb{Z}} \;=\; \prod_{p \text{ prime}} \mathbb{Z}_p \qquad\qquad (\text{product topology})$$

Also,

[6.2.1] Theorem: $\widehat{\mathbb{Z}}$ is the limit of the quotients $\mathbb{Z}/N$ with $\mathbb{Z}/N \to \mathbb{Z}/M$ for $M|N$ by $\ell + N\mathbb{Z} \to \ell + M\mathbb{Z}$. That is, for all topological groups G and compatible families of continuous group homomorphisms $G \to \mathbb{Z}/N$, there is a unique continuous group homomorphism $G \to \widehat{\mathbb{Z}}$ giving a commutative diagram



[6.2.2] Remark: This can be proven in imitation of the proof for $\mathbb{Z}_p$, but is also a corollary of a more general fact, that *projective limits commute with products*. This general assertion holds in any category with products and projective limits, and so is proven diagrammatically. Indeed, projective limits and products are examples of the same more-general construction. Then, since by Sun-Ze's theorem the quotients Z/N are products of $\mathbb{Z}/p^e$ with prime powers p^e appearing in the prime factorization of N,

$$\lim_N \mathbb{Z}/N \;=\; \lim_{e_2,e_3,e_5,\ldots} (\mathbb{Z}/2^{e_2}3^{e_3}\ldots) \;\approx\; \prod_p \Big(\lim_e \mathbb{Z}/p^e\Big) \;\approx\; \prod_p \mathbb{Z}_p \;=\; \widehat{\mathbb{Z}}$$

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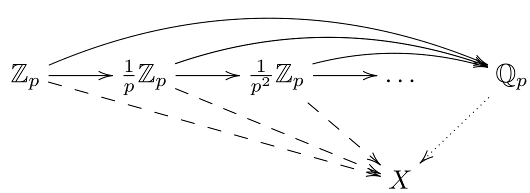
7. $\mathbb{Q}_p$ *and* $\mathbb{A}$ *as colimits*

We compare characterization of the adeles $\mathbb{A}$ and the simpler case of $\mathbb{Q}_p$. The definition of $\mathbb{Q}_p$ as p-adic completion of $\mathbb{Q}$ makes it clear (as earlier, above) that it is the field of fractions of $\mathbb{Z}_p$.

The *ascending union* $\mathbb{Q}_p = \bigcup_n p^{-n}\mathbb{Z}_p$ expresses $\mathbb{Q}_p$ as a *colimit* of topological (additive) *groups*: that is, for every compactible collection of continuous group homomorphisms $p^{-n}\mathbb{Z}_p \to X$ there is a unique continuous group homomorphism $\mathbb{Q}_p \to X$ giving a commutative diagram



expresses $\mathbb{Q}_p$ as a topological group, not as a *ring*. Nevertheless, the limitands in the colimit behave reasonably, in the sense that

$$p^{-m}\,\mathbb{Z}_p \cdot p^{-n}\,\mathbb{Z}_p \;\subset\; p^{-(m+n)}\,\mathbb{Z}_p$$

so that the ring structure on $\mathbb{Z}_p$ (as limit) and the obvious multiplicative properties of powers of p give the colimit a ring structure.

Similarly, the ring of *finite adeles*

$$\mathbb{A}_{\text{fin}} \;=\; \bigcup_{\text{finite } S} \mathbb{A}_{\text{fin},S} \qquad\qquad (\text{with } \mathbb{A}_{\text{fin},S} = \textstyle\prod_{p\in S}\mathbb{Q}_p \times \prod_{p\notin S}\mathbb{Z}_p)$$

is expressible as

$$\mathbb{A}_{\text{fin}} \;=\; = \;\bigcup_n \frac{1}{n}\widehat{\mathbb{Z}} \;=\; \text{colim}_n \frac{1}{n}\widehat{\mathbb{Z}}$$

indexing by positive integers ordered by *divisibility*. This gives a colimit expression for $\mathbb{A}$ as topological group. Similarly, the full adele ring

$$\mathbb{A} \;=\; \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$$

is similarly

$$\mathbb{A}_{\mathrm{fin}} \;=\;=\; \bigcup_n \frac{1}{n}\widehat{\mathbb{Z}} \;=\; \mathrm{colim}_n \frac{1}{n}\widehat{\mathbb{Z}}$$

indexing by positive integers ordered by *divisibility*. This gives a colimit expression for $\mathbb{A}$ as topological group.

[7.0.1] Remark: To talk about $\mathbb{R}$ at the same time as all the other completions $\mathbb{Q}_p$, often one talks as though there were a *prime* ∞ whose corresponding completion is $\mathbb{Q}_\infty = \mathbb{R}$. This has no content apart from allowing a uniform language. It is more proper to speak of *places* rather than *primes* as a generalized notion that includes both genuine primes and the standard metric that yields $\mathbb{R}$, but insisting on using *place* is pointless. Thus, *the infinite prime* is just the index ∞ that allows us to talk about $\mathbb{R}$ in the same manner we talk about $\mathbb{Q}_p$.

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8. *Abelian solenoids* $(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$ *and* $\mathbb{A}/\mathbb{Q}$

Both p-adic *numbers* $\mathbb{Q}_p$ and *adeles* $\mathbb{A}$ are discovered inside automorphism groups of *families* of ordinary circles. These appearances are far more important than *ad hoc* definitions of $\mathbb{Q}_p$ as a metric completion and $\mathbb{A}$ as restricted product. That is, p-adic numbers and the adeles appear *inevitably* in modestly complicated natural structures, as parts of automorphism groups.

Projective limits of various families of circles are *solenoids*, since there is a wound-up copy of $\mathbb{R}$ inside.

This discussion of automorphisms of families of circles is a warm-up to the more complicated situation of automorphisms of families of objects acted upon by *non-abelian* groups.

An important theme is that when a group G acts *transitively*[21] on a set X, then X is in bijection with G/G_x, where G_x is the isotropy subgroup[22] in G of a chosen base point x in X, by $gG_x \to gx$. The point is that such sets X are really *quotients of the group G*. Topological and other structures also correspond, under mild hypotheses. Isomorphisms $X \approx G/G_x$ are informative and useful, as seen subsequently.

[8.1] The p-solenoid $\lim_n \mathbb{R}/p^n\mathbb{Z}$ As reported by MacLane in his autobiography, around 1942 Eilenberg talked to him at a conference in Michigan about families of circles related to each other by repeated *windings*, and about understanding the *limiting object*. We make this precise. The point is that a surprisingly complicated physical object is made from families of *circles*.

As a model for the circle S^1 take $S^1 = \mathbb{R}/\mathbb{Z}$. Eilenberg and MacLane considered a family of circles and maps

$$\ldots \xrightarrow{\times p} \mathbb{R}/\mathbb{Z} \xrightarrow{\times p} \mathbb{R}/\mathbb{Z} \xrightarrow{\times p} \mathbb{R}/\mathbb{Z}$$

with each circle mapped to the next by multiplying wrapping p times, more precisely, by multiplication by p on the quotients $\mathbb{R}/\mathbb{Z}$, namely

$$x + \mathbb{Z} \;\longrightarrow\; px + \mathbb{Z} \qquad\qquad (\text{for } x \in \mathbb{R})$$

---

[21] The spirit of *action* of group G on X is that G moves around elements of the set X: an *action* of G on a set X is a map $G \times X \to X$ such that $1_G \cdot x = x$ for all $x \in X$, and $(gh)x = g(hx)$ for $g, h \in G$ and $x \in X$. A group G acts *transitively* on X when, for all $x, y \in X$, there is g in G such that $gx = y$.

[22] Recall that the *isotropy subgroup* G_x of a point x in a set X on which G acts is the subgroup of G *fixing* x, that is, G_x is the subgroup of $g \in G$ such that $gx = x$.

Since $p\mathbb{Z} \subset \mathbb{Z}$ this is well-defined. That is, each circle is a *p-fold cover* of the circle to its immediate right in the sequence. This sequence of circles with p-fold covering maps is the *p-solenoid*. We might ask *what is the limiting object*

$$??? \qquad \ldots \xrightarrow{\times p} \mathbb{R}/\mathbb{Z} \xrightarrow{\times p} \mathbb{R}/\mathbb{Z} \xrightarrow{\times p} \mathbb{R}/\mathbb{Z}$$

A topologically equivalent model is a more convenient. Consider the sequence

$$\ldots \xrightarrow{\varphi_{43}} \mathbb{R}/p^3\mathbb{Z} \xrightarrow{\varphi_{32}} \mathbb{R}/p^2\mathbb{Z} \xrightarrow{\varphi_{21}} \mathbb{R}/p\mathbb{Z} \xrightarrow{\varphi_{10}} \mathbb{R}/\mathbb{Z}$$

where $\varphi_{n,n-1} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^{n-1}\mathbb{Z}$ is induced from the identity map on $\mathbb{R}$ in the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \\ \downarrow {\scriptstyle \bmod p^n} & & \downarrow {\scriptstyle \bmod p^{n-1}} \\ \mathbb{R}/p^n\mathbb{Z} & \overset{\varphi_{n,n-1}}{\dashrightarrow} & \mathbb{R}/p^{n-1}\mathbb{Z} \end{array}$$

That is,

$$\varphi_{n,n-1} \;:\; x + p^n\mathbb{Z} \longrightarrow x + p^{n-1}\mathbb{Z}$$

This second model has advantages: the maps $\varphi_{n,n-1}$ are locally distance-preserving on the circles, moving to the left in the sequence of circles the circles get larger, and there is the single copy of $\mathbb{R}$ mapping locally isometrically to all the circles.

What is the limit of these circles?

One plausible, naive guess would be that since $\bigcap_n p^n\mathbb{Z} = \{0\}$, that perhaps the limit is $\mathbb{R}/\{0\} \approx \mathbb{R}$. It is not.

A concrete model of the limit X in a diagram

$$X \qquad \ldots \xrightarrow{\varphi_{32}} X_2 \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0 \qquad (\varphi_2 : X \to X_2,\ \varphi_1 : X \to X_1,\ \varphi_0 : X \to X_0)$$

is the collection of all sequences $x_o, x_1, x_2, \ldots$ such that the transition maps $\varphi_{n,n-1}$ map them to each other, that is,

$$\varphi_{n,n-1}(x_n) \;=\; x_{n-1} \qquad\qquad \text{(for all indices } n\text{)}$$

We may write a *compatible* family of elements as

$$\ldots \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow x_o$$

The projection to the i^{th} limitand $\mathbb{R}/p^i\mathbb{Z}$ maps the compatible family to $x_i \in \mathbb{R}/p^i\mathbb{Z}$.

For the abelian solenoid $\lim_n \mathbb{R}/p^n\mathbb{Z}$, x_o is in the circle $\mathbb{R}/\mathbb{Z}$. Take a representative $x_o \in \mathbb{R}$. There are exactly p choices of $x_1 \in \mathbb{R}/p\mathbb{Z}$, with representatives $x_o, x_o+1, x_o+2, \ldots, x+p-1$. Similarly, for each such choice $x_1 = x_o + \ell_1$, there are p compatible choices for x_2, expressible as $x_2 = x_o + \ell_2$ with $\ell_2 = \ell_1 \bmod p$. The sequence of integers ℓ_i with compatibilities $\ell_{i+1} = \ell_i \bmod p^i$ is an element ℓ of $\mathbb{Z}_p$.

Thus, we are tempted to say that the limit is $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}_p$. However, the way that $\ell \in \mathbb{Z}_p$ determines an element of the limit depends somewhat on $x_o \in \mathbb{R}/\mathbb{Z}$, and this is a slightly misleading picture, and can be improved as follows. [23]

[23] What we literally have at this point is that the p-solenoid is a *fiber bundle* over $\mathbb{R}/\mathbb{Z}$ with fibers $\mathbb{Z}_p$. We can do better.

[8.2] Automorphisms of solenoids

Make sense of *automorphisms* of a limit X by looking at automorphisms of the *diagram*. With a large-enough group G of automorphisms to act *transitively* on X, X is a *quotient* of G:

$$X \approx G/G_x \;=\; \{G_x\text{-cosets in } G\} \;=\; \{gG_x : g \in G\} \qquad\qquad (\text{with } \textit{isotropy subgroup } G_x \subset G \text{ of } x \in X)$$

This is an isomorphism of G-*spaces*, meaning (topological) spaces A, B on which G acts continuously. As expected, a map of G-spaces is a set map $\psi : A \to B$ such that

$$\psi(g \cdot a) \;=\; g \cdot \psi(a)$$

for $a \in A$, $g \in G$, where on the left the action is of G on A, and on the right it is the action on B.

The solenoid is itself a group, being a limit of groups, but is also a quotient of *more familiar* and *simpler* objects.

One kind of *automorphism* f of the p-solenoid is a collection of maps $f_n : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z}$ such that all squares commute in

$$\begin{array}{ccccccccc}
\ldots \xrightarrow{\varphi_{43}} & \mathbb{R}/p^3\mathbb{Z} & \xrightarrow{\varphi_{32}} & \mathbb{R}/p^2\mathbb{Z} & \xrightarrow{\varphi_{21}} & \mathbb{R}/p\mathbb{Z} & \xrightarrow{\varphi_{10}} & \mathbb{R}/\mathbb{Z} \\
& \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\
\ldots \xrightarrow{\varphi_{43}} & \mathbb{R}/p^3\mathbb{Z} & \xrightarrow{\varphi_{32}} & \mathbb{R}/p^2\mathbb{Z} & \xrightarrow{\varphi_{21}} & \mathbb{R}/p\mathbb{Z} & \xrightarrow{\varphi_{10}} & \mathbb{R}/\mathbb{Z}
\end{array}$$

There are obvious families of maps f_n. For example, since all circles are quotients of $\mathbb{R}$ in a compatible fashion, a simple sort of family of maps f_n is obtained by letting $r \in \mathbb{R}$ act, by

$$f_n(x_n + p^n\mathbb{Z}) \;=\; x_n + r + p^n\mathbb{Z}$$

Orbits of this action are highly-wound-up copies of $\mathbb{R}$, earning the name *solenoid*.

Another simple family of maps is obtained by sequences of *integers* y_n and maps

$$f_n(x_n + p^n\mathbb{Z}) \;=\; x_n + y_n + p^n\mathbb{Z}$$

requiring that the sequence y_n be chosen so that the squares in the diagram commute. That is, we must have

$$(x_n + y_n + p^n\mathbb{Z}) + p^{n-1}\mathbb{Z} \;=\; x_{n-1} + y_{n-1} + p^{n-1}\mathbb{Z}$$

Since already

$$(x_n + p^n\mathbb{Z}) + p^{n-1}\mathbb{Z} \;=\; x_{n-1} + p^{n-1}\mathbb{Z}$$

it is necessary and sufficient that

$$(y_n + p^n\mathbb{Z}) + p^{n-1}\mathbb{Z} \;=\; y_{n-1} + p^{n-1}\mathbb{Z}$$

That is, the compatible sequence of integers

$$\ldots \longrightarrow y_n \longrightarrow \ldots \longrightarrow y_2 \longrightarrow y_1$$

gives an element in the limit $\mathbb{Z}_p$.

Still without worrying about the topology,

[8.2.1] Claim: The product group $\mathbb{R} \times \mathbb{Z}_p$ acts *transitively* on the p-solenoid. The point $\ldots \to 0 \to 0 \to 0$ in the solenoid has isotropy group which is the diagonally imbedded copy of the integers

$$\mathbb{Z}^\Delta \;=\; \{(\ell, -\ell) \in (\mathbb{Z} \times \mathbb{Z}) \subset \mathbb{R} \times \mathbb{Z}_p \;:\; \ell \in \mathbb{Z}\}$$

Proof: Given a compatible family

$$\ldots \to x_3 + p^3\mathbb{Z} \to x_2 + p^2\mathbb{Z} \to x_1 + p\mathbb{Z} \to x_0 + \mathbb{Z}$$

of elements $x_n + p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z}$, act by $r \in \mathbb{R}$ as above such that $x_0 + r = 0 \in \mathbb{R}/\mathbb{Z}$. Since the x_n's are compatible, it must be that $(r+x_1) \bmod 1 = (x_0+r) = 0$, $(x_2+r) \bmod p = (x_1+r)$, $(x_3+r) \bmod p^2 = x_2+r$, and so on. That is, every $x_n + r \in \mathbb{Z}$, and the sequence $y_n = x_n + r$ gives a compatible family

$$\ldots \to y_3 + p^3\mathbb{Z} \to y_2 + p^2\mathbb{Z} \to y_1 + p\mathbb{Z} \to y_0 + \mathbb{Z}$$

which gives an element in $\mathbb{Z}_p$. That is, the further action by $-y_n$ on the solenoid will send every element to

$$(x_n + r) - y_n \;=\; (x_n + r) - (x_n + r) \;=\; 0$$

proving transitivity.

To determine the isotropy group of a point, let r be a real number and y_n an integer modulo p^n, such that the 0-element

$$\ldots 0 \to 0 \to 0 \to 0$$

is mapped to itself. That is, require that

$$0 + r + y_n \;\in\; 0 + p^n\mathbb{Z}$$

for all n. First, this implies that $r \in \mathbb{Z}$. Then y_n, which is only determined modulo 2^n anyway, is completely determined modulo p^n by

$$y_n + p^n\mathbb{Z} \;=\; -r + p^n\mathbb{Z}$$

That is, $y_n = -r \bmod p^n$. These conditions are visibly *sufficient*, as well, to fix the 0. Thus, the isotropy group truly is the diagonal copy of $\mathbb{Z}$. ///

Again, for a group G acting transitively on a set X, for fixed $x \in X$, there is a bijection

$$X \longleftrightarrow G/G_x \;=\; \{gG_x \;:\; g \in G\} \qquad \text{by} \qquad gx \longleftrightarrow gG_x$$

Indeed, the map from G to X by $g \to gx$ is a *surjection*, since G is transitive. This map factors through G/G_x and is *injective*, since $gx = hx$ if and only if $h^{-1}gx = x$, if and only if $h^{-1}g \in G_x$, if and only if $gG_x = hG_x$ as desired. The topological features are discussed in a supplemental note. Thus, we have the corrected expression for the limit in terms of $\mathbb{R} \times \mathbb{Z}_p$:

[8.2.2] Corollary: The p-solenoid is

$$\lim_n \mathbb{R}/p^n\mathbb{Z} \;\approx\; (\mathbb{R} \times \mathbb{Z}_p)/\mathbb{Z}^\Delta$$

where $\mathbb{Z}^\Delta$ is the diagonal copy.

[8.3] Bigger diagrams and more automorphisms

Bigger diagrams *with the same limit* make visible more automorphisms of the common limit object. For the p-solenoid, we will find a copy of the *p-adic rational numbers* $\mathbb{Q}_p$ acting on the p-solenoid, rather than merely the *p-adic integers* $\mathbb{Z}_p$. This is a big change: $\mathbb{Q}_p$ is *non-compact*, $\mathbb{Z}_p$ is *compact*. We had already seen that

$$p\text{-solenoid} \;=\; \lim_n \mathbb{R}/p^n\mathbb{Z} \;\approx\; (\mathbb{R} \times \mathbb{Z}_p)/\mathbb{Z}^\Delta$$

with $\mathbb{Z}^\Delta$ the diagonal copy of $\mathbb{Z}$. Having found a larger group of automorphisms, we will have

$$p\text{-solenoid} \approx (\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[1/p]^\Delta$$

The diagonal copy $\mathbb{Z}[1/p]^\Delta$ of

$$\mathbb{Z}[1/p] = \mathbb{Z} + \frac{1}{p}\cdot\mathbb{Z} + \frac{1}{p^2}\cdot\mathbb{Z} + \frac{1}{p^3}\cdot\mathbb{Z} + \ldots$$

(rational numbers with denominators restricted to be powers of p) is *discrete* in $\mathbb{R}\times\mathbb{Q}_p$.

Different (related, of course) diagrams can give the same limit object. We find a larger diagram with no *bottom* object, but giving the same limit. So far, the p-solenoid X is a limit fitting into a diagram

$$X \qquad \ldots \longrightarrow \mathbb{R}/p^2\mathbb{Z} \longrightarrow \mathbb{R}/p\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$

Given a point x on the solenoid, let x_n be its projection to $\mathbb{R}/p^n\mathbb{Z}$, and think of x as the a compatible family of points on the respective circles, written

$$x \;=\; (\ldots \to x_2 \to x_1 \to x_0)$$

We already found that $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ acts on the p-solenoid. As earlier, given a point x on X, an element $r\in\mathbb{R}$ on all circles simultaneously, to make 0^{th} projection $0\in\mathbb{R}/\mathbb{Z}$. That is, the *new* values

$$x \;=\; (\ldots \to x_2 \to x_1 \to x_0 = 0)$$

are in $\mathbb{Z}$, and form a compatible family inside

$$\ldots \xrightarrow{\text{mod } p^3} \mathbb{Z}/p^3 \xrightarrow{\text{mod } p^2} \mathbb{Z}/p^2 \xrightarrow{\text{mod } p} \mathbb{Z}/p \xrightarrow{\text{mod } 1} \mathbb{Z}/1$$

In the diagram defining the p-solenoid there is no compulsion to stop at the circle $\mathbb{R}/\mathbb{Z}$. We can continue to the right with ever-shrinking circles, as in

$$\ldots \longrightarrow \mathbb{R}/p^2\mathbb{Z} \longrightarrow \mathbb{R}/p\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{p}\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{p^2}\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{p^3}\mathbb{Z} \longrightarrow \ldots$$

[8.3.1] Claim: The limit of this larger diagram is naturally isomorphic to the limit of the original diagram.

Proof: Let X be a projective limit fitting into a commutative diagram

$$X \qquad \ldots \longrightarrow X_1 \longrightarrow X_0$$

and consider also an enlarged diagram with projective limit Y:

$$Y \qquad \ldots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \ldots \longrightarrow X_{-n} \longrightarrow \ldots$$

We claim that there is a natural isomorphism $X \to Y$, induced from the commutative diagram

$$\begin{array}{ccccccccc} X & \ldots \longrightarrow & X_1 & \longrightarrow & X_0 & & & & \\ & & \| & & \| & & & & \\ Y & \ldots \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & \ldots \longrightarrow & X_{-n} & \longrightarrow \ldots \end{array}$$

A map from Y to the projective limit X is determined uniquely by a compatible family of maps from Y to the X_n with $n \ge 0$, provided by the projections of Y to the X_n with $n \ge 0$. To get a map from X to Y is to give a compatible family of maps from X to all the X_n, now with $n\in\mathbb{Z}$. For $n \ge 0$ the projections of X to

X_n work. For $-n < 0$, there are many possibilities. For example, map X to X_0 and then map to X_{-n} by the transition maps in the diagram for Y.

Thus, we obtain *unique* maps $f : Y \to X$ and $g : X \to Y$ compatible with all the projections and equalities. Then $f \circ g : Y \to Y$ is a self-map of Y preserving all the projections, so, by the uniqueness of the projective limit, is the identity map. Similarly, $g \circ f$ is the identity on X. Thus, $X \approx Y$. ///

The larger diagram for the same object makes more automorphisms visible, as follows.

Given a point

$$x \;=\; (\ldots \to x_1 \to x_0 \to x_{-1} \to \ldots)$$

in the larger diagram, since there is no *bottom circle* to normalize, we have the further auxiliary choice of an integer n, and let $\mathbb{R}$ act to rotate $x_n \in \mathbb{R}/p^n\mathbb{Z}$ to 0. Take $\mathbb{Z} \ni -n \le 0$, and let $\mathbb{R}$ act by $x_i \to x_i + r$ for all indices i, with r chosen to rotate x_{-n} to 0 in $\mathbb{R}/p^{-n}\mathbb{Z}$. Thus, since the arrows are group homomorphisms,

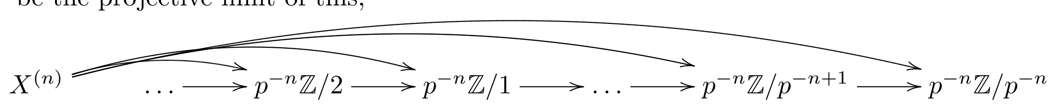
$$\ldots \to x_1 \to x_0 \to x_{-1} \to \ldots \to x_{-n} = 0 \to 0 \to 0 \to \ldots$$

That is, at and after the $-n^{th}$ place, all the x_i have become 0.

Thus, $x_{-n} = 0 \in p^{-n}\mathbb{Z}/p^{-n}\mathbb{Z}$, and there are exactly p choices for x_{-n+1}, namely $p^{-n}\mathbb{Z} \bmod p^{-n+1}\mathbb{Z}$. For each of these p choices, there are p choices of x_{-n+2}, and so on. The choice $x_{-n} = 0$ on the n^{th} circle $\mathbb{R}/p^{-n}\mathbb{Z}$ means that x_{-n+i} is in $p^{-n}\mathbb{Z}$ modulo $p^{-n+i}\mathbb{Z}$. The collection of all such compatible families for a *fixed* choice of $-n$ fits together as

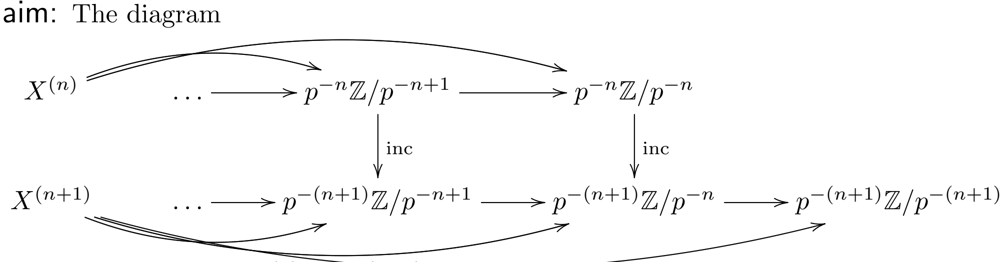
$$\ldots \to p^{-n}\mathbb{Z}/p^2 \to p^{-n}\mathbb{Z}/p \to p^{-n}\mathbb{Z}/1 \to p^{-n}\mathbb{Z}/\frac{1}{p}\mathbb{Z} \to p^{-n}\mathbb{Z}/\frac{1}{p^2}\mathbb{Z} \to \ldots \to p^{-n}\mathbb{Z}/p^{-n}\mathbb{Z} = \{0\}$$

Let $X^{(n)}$ be the projective limit of this,



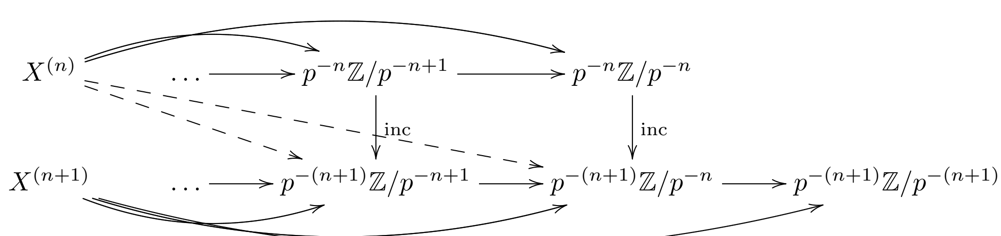
The family of these diagrams fits together, giving an ascending chain of larger-and-larger limits. Indeed,

[8.3.2] Claim: The diagram



induces a unique *injective* map $X^{(n)} \to X^{(n+1)}$ compatible with all the projections (where the vertical maps are the obvious inclusions).

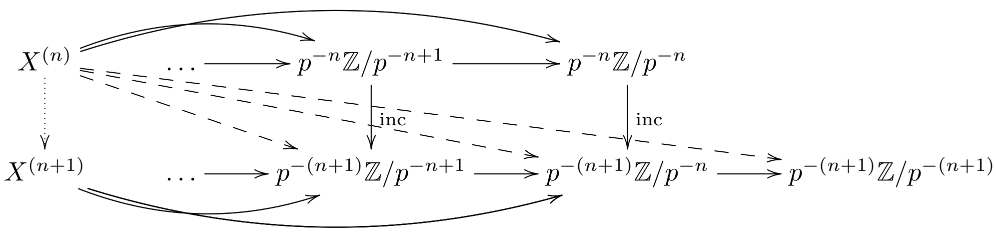
Proof: A map to a projective limit is a compatible family of maps to the limitands. By composition with the inclusions, we obtain the dashed arrows



Since the initial diagram commutes, there is a map

$$p^{-n}\mathbb{Z}_p = X^{(n)} \to p^{-(n+1)}\mathbb{Z}/p^{-(n+1)}$$

given by composition with *any choice of* inclusion map from the top row to the bottom. Thus, there is a unique induced dotted arrow in the commuting diagram



The induced map is injective, because an element in a projective limit of groups or topological groups is 0 if and only if all its projections are 0.

Given non-zero $x \in X^{(n)}$, at least one projected image $x_i \in p^{-n}\mathbb{Z}/p^{-n+i}$ is non-zero. The inclusion to $p^{-(n+1)}\mathbb{Z}/p^{-n+i}$ is still non-zero, so the image under the induced map to $X^{(n+1)}$ cannot be 0. Thus, the (abelian) group homomorphism $X^{(n)} \to X^{(n+1)}$ has trivial kernel, so is injective. ///

Thus, we have a family of inclusions

$$\begin{array}{ccccccc} X^{(0)} & \xrightarrow{\text{inc}} & X^{(1)} & \xrightarrow{\text{inc}} & X^{(2)} & \xrightarrow{\text{inc}} & \ldots \\ \| & & \| & & \| & & \\ \mathbb{Z}_p & & \frac{1}{p}\cdot\mathbb{Z}_p & & \frac{1}{p^2}\cdot\mathbb{Z}_p & & \end{array}$$

of groups acting on the p-solenoid. The action of $X^{(n+1)}$ matches that of $X^{(n)}$ when restricted to $X^{(n)}$, giving an action on the p-solenoid of the *colimit*

$$\mathbb{Q}_p \;=\; \bigcup_{n=0}^{\infty} p^{-n}\mathbb{Z}_p = \bigcup_{n=0}^{\infty} X^{(n)} \;=\; \text{colim}_n X^{(n)}$$

Now determine the *isotropy subgroup* of the point 0 in the p-solenoid, under the action of $\mathbb{R}\times\mathbb{Q}_p$. Recall that $r \in \mathbb{R}$ acts by

$$r\cdot(\ldots \to x_i + p^i\mathbb{Z} \to \ldots) = \ldots \to r + x_i + p^i\mathbb{Z} \to \ldots$$

Similarly, each $y \in \mathbb{Q}_p$ is of the form (for some n, depending on y)

$$\ldots \longrightarrow y_i + p^i\mathbb{Z} \longrightarrow \ldots \longrightarrow y_{-n+1} + p^{-n+1}\mathbb{Z} \longrightarrow y_{-n} = 0 + p^{-n}\mathbb{Z} \longrightarrow 0 \to 0 \to \ldots$$

with y_{-n+i} lying in $p^{-n}\mathbb{Z}/p^{-n+i}\mathbb{Z}$. The action on x in the p-solenoid is

$$y\cdot x \;=\; \big(\ldots \longrightarrow y_i + x_i + p^i\mathbb{Z} \longrightarrow \ldots\big)$$

Already $\mathbb{R}\times\mathbb{Z}_p$ was transitive, so certainly $\mathbb{R}\times\mathbb{Q}_p$ is transitive. The isotropy group of the point $x = 0$ in the p-solenoid is the collection of $r \in \mathbb{R}$ and $y \in \mathbb{Q}_p$ such that

$$r + y_i \in p^i\mathbb{Z} \quad (\text{for all } i \in \mathbb{Z})$$

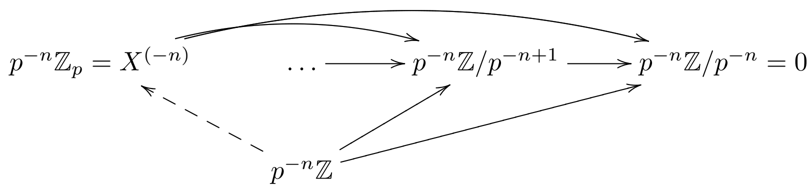
where for each $y \in \mathbb{Q}_p$ there is an integer $n \geq 0$ such that all y_i lie in $p^{-n}\mathbb{Z}$. For fixed y with associated n, taking $i = -n$, since $y_{-n} \in p^{-n}\mathbb{Z}$,

$$r \;\in\; -y_{-n} + p^{-n}\mathbb{Z} \;=\; p^{-n}\mathbb{Z}$$

For all indices $0 \geq i \in \mathbb{Z}$, by the isotropy condition,

$$y_{-n+i} \;=\; -r \quad (\text{in } p^{-n}\mathbb{Z}/p^{-n+i}\mathbb{Z})$$

For all $n \geq 0$, we have the diagonal copy of $p^{-n}\mathbb{Z}$ imbedded in $X^{(n)} = p^{-n}\mathbb{Z}_p$ induced from



That is, for all $n \geq 0$, inside the isotropy group

$$\big(p^{-n}\mathbb{Z}\big)^\Delta = \{(\delta, -\delta) : \delta \in p^{-n}\mathbb{Z}\} \subset \mathbb{R} \times \mathbb{Q}_p$$

Taking the ascending union, the diagonal copy of $\mathbb{Z}[1/p]$ is the isotropy group. Thus, as $\mathbb{R} \times \mathbb{Q}_p$ -spaces,

$$p\text{-solenoid} \;=\; \lim_n \mathbb{R}/p^n\mathbb{Z} \;\approx\; (\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[1/p]^\Delta$$

To see the *discreteness* of the diagonal copy of $\mathbb{Z}[1/p]$ in the product $\mathbb{R} \times \mathbb{Q}_p$, recall that $\mathbb{Z}_p$ is *open* in $\mathbb{Q}_p$, because

$$\mathbb{Z}_p \;=\; \{x \in \mathbb{Q}_p \;:\; |x|_p < p\}$$

Thus, $\mathbb{R} \times \mathbb{Z}_p$ is open in $\mathbb{R} \times \mathbb{Q}_p$, and

$$(\mathbb{R} \times \mathbb{Z}_p) \cap \mathbb{Z}[1/p]^\Delta \;=\; \mathbb{Z}^\Delta$$

The *projection* of $\mathbb{Z}^\Delta$ to $\mathbb{R}$ is discrete, so $\mathbb{Z}[1/p]^\Delta$ is discrete in $\mathbb{R} \times \mathbb{Q}_p$. ///

[8.4] Projections $(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[1/p] \longrightarrow \mathbb{R}/p^i\mathbb{Z}$ Limits are not genuinely described without giving the projection maps.

[8.4.1] Claim: The quotient

$$\Big(\mathbb{R} \times \mathbb{Q}_p\Big)\Big/\mathbb{Z}[1/p]^\Delta \;\longrightarrow\; \Big(\mathbb{R} \times \mathbb{Q}_p\Big)\Big/\Big(\mathbb{Z}[1/p]^\Delta + p^i\mathbb{Z}_p\Big)$$

is naturally isomorphic to $\mathbb{R}/p^i\mathbb{Z}$.

Proof: First, $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$: given $x \in p^{-n}\mathbb{Z}_p$, approximate $p^n x$ well by $y \in \mathbb{Z}$, depending on n. Then $p^{-n}y$ is close to x.

Thus, given $r \times x$ in $\mathbb{R} \times \mathbb{Q}_p$, adjust by $\mathbb{Z}[1/p]$ to put x into $p^i\mathbb{Z}_p$, and then act by $p^i\mathbb{Z}_p$ to give a representative of the form $r \times 1$. The only allow further possible adjustments are exactly by by

$$\mathbb{Z}[1/p] \cap p^i\mathbb{Z}_p \;=\; p^i\mathbb{Z}$$

That is, $(\mathbb{R} \times \mathbb{Q}_p)/(\mathbb{Z}[1/p] + p^i\mathbb{Z}_p)$ has representatives in $\mathbb{R} \times \{1\}$, and two representatives give the same image exactly when they differ by $p^i\mathbb{Z}$. ///

[8.5] The big solenoid $\lim_N \mathbb{R}/N\mathbb{Z} \approx \mathbb{A}/\mathbb{Q}$ Replacing $\{p^n : 1 \leq n \in \mathbb{Z}\}$ by integers divisible only by a specified set of primes, possibly all primes, an analogous discussion gives an analogous result. In particular, allowing divisibility by all primes gives the biggest solenoid of this type,

$$\text{big solenoid} \;=\; \lim_N \mathbb{R}/N\mathbb{Z} \;\approx\; \mathbb{A}/\mathbb{Q}^\Delta$$

because

$$\mathbb{A} \;=\; \bigcup_N (\mathbb{R}\times \frac{1}{N}\widehat{\mathbb{Z}}) \;=\; \mathrm{colim}_N(\mathbb{R}\times \frac{1}{N}\widehat{\mathbb{Z}})$$

Further, the diagonal copy of $\mathbb{Q}$ in $\mathbb{A}$ is *discrete*: $\widehat{\mathbb{Z}}$ is open in the finite adeles, so $\mathbb{R}\times\widehat{\mathbb{Z}}$ is open in $\mathbb{A}$, and $(\mathbb{R}\times\widehat{\mathbb{Z}})\cap\mathbb{Q}^\Delta = \mathbb{Z}^\Delta$, which is discrete when projected to $\mathbb{R}$.

The projections are

$$\mathbb{A}/\mathbb{Q} \;\longrightarrow\; \mathbb{A}/(\mathbb{Q}+N\widehat{\mathbb{Z}}) \;\approx\; \mathbb{R}/N\mathbb{Z}$$

Indeed, $\mathbb{Q}$ is dense in $\mathbb{A}_{\mathrm{fin}}$, so, given $r+x$ in $\mathbb{A}=\mathbb{R}\times\mathbb{A}_{\mathrm{fin}}$, adjust by $\mathbb{Q}$ to put x into $N\widehat{\mathbb{Z}}$, and then act by $N\widehat{\mathbb{Z}}$ to make the finite-prime component trivial. To avoid any further adjustments disturbing this normalization, the only allow further possible adjustments are exactly by

$$\mathbb{Q}\cap N\widehat{\mathbb{Z}} \;=\; N\mathbb{Z}$$

That is, $\mathbb{A}/(\mathbb{Q}+N\widehat{\mathbb{Z}})$ has representatives in $\mathbb{R}\times\{1\}$, and two representatives give the same image exactly when they differ by $N\mathbb{Z}$. Thus,

$$\mathbb{A}/\mathbb{Q} \;\longrightarrow\; \mathbb{A}/(\mathbb{Q}+N\widehat{\mathbb{Z}}) \;=\; (\mathbb{R}+N\widehat{\mathbb{Z}})\,/\,(\mathbb{Q}+N\widehat{\mathbb{Z}}) \;\approx\; \mathbb{R}/N\mathbb{Z}$$

gives the projections.

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9. *Non-abelian solenoids and* $SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})$

The non-abelian solenoid of interest is the projective limit $\lim_N \Gamma_N\backslash\mathfrak{H}$, with transition maps $\Gamma_N\cdot z\longrightarrow\Gamma_M\cdot z$ for $M|N$. It is better to use $\mathfrak{H}\approx SL_2(\mathbb{R})/SO(2,\mathbb{R})$, and consider over-lying objects $\Gamma(p^n)\backslash SL_2(\mathbb{R})$ whose quotient on the right by $SO(2,\mathbb{R})$ is $\Gamma(p^n)\backslash\mathfrak{H}$, as this allows $SL_2(\mathbb{R})$ to act on the right on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$. Write

$$G_\infty \;=\; SL_2(\mathbb{R})$$

Thus, consider the non-abelian solenoid $\lim_N \Gamma_N\backslash G_\infty$, with transition maps $\Gamma_N\cdot g\longrightarrow\Gamma_M\cdot g$ for $M|N$.

As with the abelian solenoids, to simplify the indexing poset, we carry out the discussion for the somewhat simpler solenoid $\lim_n \Gamma(p^n)\backslash G_\infty$ with prime p.

Model elements of this projective limit by sequences

$$\ldots\longrightarrow g_2\longrightarrow g_1\longrightarrow g_o$$

meeting the compatibility condition $\Gamma(p^{n-1})\cdot g_n = \Gamma(p^{n-1})\cdot g_{n-1}$ for all n. That is, $g_n=\gamma_n\cdot g_o$ for a sequence $g_n\in\Gamma(1)$ with $\Gamma(p^{n-1})\cdot\gamma_n = \Gamma(p^{n-1})\cdot\gamma_{n-1}$.

[9.0.1] Claim:

$$\lim_n \Gamma(1)/\Gamma(p^n)\approx \lim_n SL_2(\mathbb{Z}/p^n)\approx SL_2(\mathbb{Z}_p)$$

and this group has a natural action on $\lim_n \Gamma(p^n)\backslash G_\infty$.

Proof: First, because $\Gamma(p^n)$ is normal in $\Gamma(1)=SL_2(\mathbb{Z})$, the quotient group $\Gamma(1)/\Gamma(p^n)$ acts on $\Gamma(p^n)\backslash G_\infty$ by

$$\gamma\cdot\big(\Gamma(p^n)\cdot g\big) \;=\; \gamma\Gamma(p^n)\gamma^{-1}\cdot\gamma g \;=\; \Gamma(p^n)\cdot\gamma g$$

For $\gamma\in\Gamma(p^n)$, the action is trivial since γ is absorbed: $\gamma\cdot\Gamma(p^n)=\Gamma(p^n)$. Thus, group elements

$$\ldots\to\gamma_2\to\gamma_1\to\gamma_0 \qquad\qquad (\gamma_n\in\Gamma(1)/\Gamma(p^n))$$

with the compatibility

$$\gamma_{n+1} \cdot \Gamma(p^n) \;=\; \gamma_n \cdot \Gamma(p^n)$$

give an automorphism of the limit by

$$(\ldots \to \gamma_2 \to \gamma_1 \to \gamma_0) \cdot (\ldots \to g_2 \to g_1 \to g_0) \;=\; (\ldots \to \gamma_2 g_2 \to \gamma_1 g_1 \to \gamma_0 g_0)$$

It remains to check that

$$\lim_n \Gamma(1)/\Gamma(p^n) \;\approx\; \lim_n SL_2(\mathbb{Z}/p^n) \approx SL_2(\mathbb{Z}_p)$$

Surjectivity of $\Gamma(1) \to SL_2(\mathbb{Z}/p^n)$ is an exercise.

The kernel of this homomorphism is $\Gamma(p^n)$. The elements of the projective limit are compatible families

$$\ldots \xrightarrow{\text{mod } p^3} \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \xrightarrow{\text{mod } p^2} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \xrightarrow{\text{mod } p} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

Thus, that each of the four sequences of entries is a compatible family of elements in the projective limit $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$. That is, $\lim_n SL_2(\mathbb{Z}/p^n) \approx SL_2(\mathbb{Z}_p)$. ///

Similarly as for the abelian solenoids, for each $g_o \in G_\infty$, an element $\gamma \in SL_2(\mathbb{Z}_p)$ given by a compatible sequence $\ldots \to \gamma_1 \to \gamma_o$, specifies an element in $\lim_n \Gamma(p^n)\backslash G_\infty$ by $\ldots \to \gamma_1 g_o \to \gamma_o g_o$. However, as occurred already with the abelian solenoids, this does not present the limit as nicely as is possible.

[9.1] Automorphisms of the solenoid In the same fashion as $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ appeared in the automorphisms of the p-solenoid, the limit $SL_2(\mathbb{Z}_p) = \lim_n \Gamma(p^n)\backslash\Gamma(1)$ gives *some* automorphisms of the limit, by

$$(\ldots \gamma_1 \to \gamma_o) \cdot (\ldots \to x_1 \to x_o) \;=\; (\ldots \gamma_1 x_1 \to \gamma_o x_o) \qquad (\text{for } (\ldots \to x_1 \to x_o) \in \lim_n \Gamma(p^n)\backslash G_\infty)$$

As $\mathbb{R}$ acted on the abelian solenoid, $g \in G_\infty$ acts by

$$g \cdot (\ldots \longrightarrow x_1 \longrightarrow x_o) \;=\; (\ldots \longrightarrow x_1 g^{-1} \longrightarrow x_o g^{-1}) \qquad (\text{for } (\ldots \to x_1 \to x_o) \in \lim_n \Gamma(p^n)\backslash G_\infty)$$

with inverse so that this is an associative action written on the left. Thus, $G_\infty \times SL_2(\mathbb{Z}_p)$ acts on this solenoid. Given $\ldots \to x_1 \to x_o$ in the projective limit, act by $g = x_o$ to put this into the form $\ldots \to x_2 \to x_1 \to 1$. The compatibility implies that $x_i \in \Gamma(1)$ and $\Gamma(p^i)x_{i+1} = \Gamma(p^i)x_i$. Thus, $\ldots \to x_2 \to x_1 \to 1$ gives an element $\gamma \in SL_2(\mathbb{Z}_p)$. This shows that $G_\infty \times SL_2(\mathbb{Z}_p)$ acts *transitively*.

The isotropy group of $\ldots \to 1 \to 1 \to 1$ consists of (g, γ) with $\gamma = (\ldots \to \gamma_2 \to \gamma_1 \to \gamma_o)$ in $SL_2(\mathbb{Z}_p)$ with $\gamma_i \in \Gamma(1)$, and $g \in G_\infty$, such that

$$(\ldots \to \gamma_2 g^{-1} \to \gamma_1 g^{-1} \to \gamma_o g^{-1}) \;=\; (\ldots \to \gamma_2 \to \gamma_1 \to \gamma_o) \cdot g \cdot (\ldots \to 1 \to 1 \to 1) \;=\; \cdot g \cdot (\ldots \to 1 \to 1$$

Thus, $\gamma_i g^{-1} \in \Gamma(p^i)$ for all i. In particular, $g \in \Gamma(1)$. Then $\gamma_i \in \Gamma(p^i)g$. Since γ_i is really in $\Gamma(1)/\Gamma(p^i)$, we can say $\gamma_i = g$ for all i. Thus, the isotropy subgroup is a diagonal copy $SL_2(\mathbb{Z})^\Delta$ of $SL_2(\mathbb{Z})$ in $G_\infty \times SL_2(\mathbb{Z}_p)$, and

$$\lim_n \Gamma(p^n)\backslash G_\infty \;\approx\; (G_\infty \times SL_2(\mathbb{Z}_p)) \,/\, SL_2(\mathbb{Z})^\Delta$$

Or, purely notationally, using *right* actions rather than left,

$$\lim_n \Gamma(p^n)\backslash G_\infty \;\approx\; SL_2(\mathbb{Z})^\Delta\backslash(G_\infty \times SL_2(\mathbb{Z}_p))$$

This is more convenient when looking at the induced translation action on *functions* on the solenoid.

[9.2] Larger diagram, more automorphisms Next, we find a larger cofinal diagram to see more automorphisms. We allow movement *outside* the group $SL_2(\mathbb{Z})$, although not far.

A *p-power congruence subgroup* is a subgroup Γ of $SL_2(\mathbb{Q})$ which contains some $\Gamma(p^n)$ with finite index. That is, for some $0 \le n \in \mathbb{Z}$

$$\Gamma \supset \Gamma(p^n) \qquad [\Gamma : \Gamma(p^n)] < \infty$$

[9.2.1] Claim: For $g \in SL_2(\mathbb{Z}[\frac{1}{p}])$, the action $\Gamma \to g\Gamma g^{-1}$ stabilizes the set of p-power congruence subgroups of $SL_2(\mathbb{Q})$.

Proof: Let Γ be a p-power congruence subgroup. Given $g \in SL_2(\mathbb{Z}[\frac{1}{p}])$, we must show that $g\Gamma g^{-1}$ contains some $\Gamma(p^\ell)$ with finite index. That is, we want to show that Γ contains some $g^{-1}\Gamma(p^\ell)g$ with finite index. Since the subgroups $\Gamma(p^\ell)$ are of finite index in each other, it suffices to verify the finite-index property for *sufficiently small* $\Gamma(p^\ell)$.

Let m be large enough such that there are integral matrices A, B such that

$$g = 1_2 + p^{-m}A \qquad g^{-1} = 1_2 + p^{-m}B$$

For $\gamma = 1_2 + p^n N$ in $\Gamma(p^n)$, where N is an integral matrix,

$$g^{-1}\gamma g = (1 + p^{-m}B)\,(1 + p^n N)\,(1 + p^{-m}A)$$

$$= 1 + p^{-m}B + p^{-m}A + p^{-2m}BA + p^n N + p^{n-m}BN + p^{n-m}NA + p^{n-2m}BNA$$

The first four summands sum to 1, since $g\,g^{-1} = 1$, so this is

$$1 + p^n N + p^{n-m}NA + p^{n-m}BN + p^{n-2m}BNA$$

For $n > 2m$, we have $g^{-1}\gamma g \in \Gamma(p^{n-2m})$, so

$$g^{-1}\,\Gamma(p^n)\,g \subset \Gamma(p^{n-2m}) \subset \Gamma \qquad (n \text{ large enough such that } \Gamma(p^{n-2m}) \subset \Gamma)$$

That is, for large enough n such that $\Gamma(p^{n-2m}) \subset \Gamma$, we have the desired containment

$$\Gamma(p^n) \subset g\Gamma(p^{n-2m})g^{-1} \subset g\Gamma g^{-1}$$

For the finite-index condition,

$$[g\Gamma g^{-1} : \Gamma(p^n)] = [\Gamma : g^{-1}\Gamma(p^n)g] = [\Gamma : \Gamma(p^{n-2m})] \cdot [\Gamma(p^{n-2m}) : g^{-1}\Gamma(p^n)g]$$

since the indices are unaltered by conjugation.

$$[\Gamma(p^{n-2m}) : g^{-1}\Gamma(p^n)g] = [g\Gamma(p^{n-2m})g^{-1} : \Gamma(p^n)] \le [g\Gamma(p^{n-4m})g^{-1} : \Gamma(p^n)] < \infty$$

by the same computation as above. This proves that conjugation by elements of $SL_2(\mathbb{Z}[\frac{1}{p}])$ stabilizes the set of p-power congruence subgroups. ///

Consider the larger family of limitands $\Gamma\backslash G_\infty$ where Γ is a p-power congruence subgroup, with the natural maps

$$\Gamma\backslash G_\infty \longrightarrow \Gamma'\backslash G_\infty \qquad (\text{for } \Gamma \subset \Gamma')$$

[9.2.2] Claim: The family $\Gamma\backslash G_\infty$ of quotients with p-power congruence subgroup Γ has *cofinal* subfamily of the quotients $\Gamma(p^n)\backslash G_\infty$ by *principal* congruence subgroups $\Gamma(p^n)$, giving a natural isomorphism

$$\lim_{p\text{-power }\Gamma} \Gamma\backslash G_\infty \approx \lim_n \Gamma(p^n)\backslash G_\infty$$

Proof: Each such Γ contains some $\Gamma(p^n)$, so there is a surjection

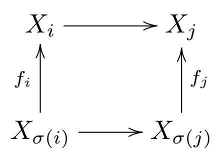
$$\Gamma(p^n)\backslash G_\infty \longrightarrow \Gamma\backslash G_\infty$$

That is, the collection of quotients by principal congruence subgroups is *cofinal*. Cofinal limits are naturally isomorphic. ///

[9.2.3] Corollary: An element $g \in SL_2(\mathbb{Z}[\frac{1}{p}])$ acts on the limit of the p-power congruence quotients $\lim_\Gamma \Gamma\backslash G_\infty$ by an action induced from the compatible family of *isomorphisms*

$$g : \Gamma\backslash G_\infty \longrightarrow g\Gamma g^{-1}\backslash G_\infty \quad \text{by} \quad g\cdot(\Gamma\cdot x) = g\Gamma g^{-1}\cdot(g\cdot x) \qquad (\text{for } x \in G_\infty)$$

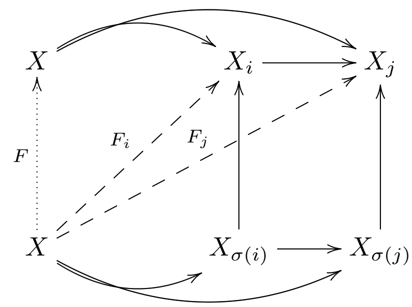
Proof: A map of a limit $X = \lim_{i\in I} X_i$ to itself is given by a compatible family of maps $X \to X_i$ to the limitands. One way to give such a family is as follows. Let $p_j : X \to X_j$ be the projection to the j^{th} limitand. Let σ be a an order-preserving permutation of the index set I, and suppose we are given a family of *isomorphisms* $f_i : X_\sigma(i) \longrightarrow X_i$ *compatible* in the sense that for $i > j$ there is a commutative diagram



Define a family of maps $F_i : X \to X_i$ by

$$F_i = f_i \circ p_{\sigma(i)}$$

giving a commutative diagram



with uniquely induced map $F : X \to X$. This idea applies to the p-power congruence subgroups with the natural isomorphisms

$$\Gamma\backslash G_\infty \longrightarrow g\Gamma g^{-1}\backslash G_\infty \quad \text{by} \quad \Gamma\cdot x \longrightarrow g\Gamma g^{-1}\cdot gx$$

Thus, $SL_2(\mathbb{Z}[\frac{1}{p}])$ acts on the projective limit. ///

The natural actions of $G_\infty \times SL_2(\mathbb{Z}_p)$ and of $SL_2(\mathbb{Z}[\frac{1}{p}])$ on $\lim_n \Gamma(p^n)\backslash G_\infty$, the latter also expressible as the limit over p-power congruence subgroups, can be combined.

[9.2.4] Claim: $SL_2(\mathbb{Z}[1/p])$ is dense in $SL_2(\mathbb{Q}_p)$.

Proof: First, $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$: given $y \in \mathbb{Q}_p$, some $p^n y$ is in $\mathbb{Z}_p$. Using the density of $\mathbb{Z}$ in $\mathbb{Z}_p$, approximate $p^n y$ well by $x_o \in \mathbb{Z}$, with closeness of approximation depending on n, of course. Then $p^n x_o$ approximates y well. Approximate the entries in given $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_p)$ closely by $h_o \in SL_2(\mathbb{Z}[1/p])$ with respective entries a_o, b_o, c_o, d_o in $\mathbb{Z}[1/p]$, modifyied to keep the determinant-one condition. ///

Thus, an element x of $\lim_\Gamma \Gamma\backslash G_\infty$ (limit over p-congruence subgroups) is a compatible family $\{x_\Gamma\}$ indexed by Γ, rather than by non-negative integers as for the sub-family $\Gamma = \Gamma(p^n)$. Given $h \in SL_2(\mathbb{Q}_p)$, for each Γ

let $h_\Gamma \in SL_2(\mathbb{Z}[1/p])$ approximate h well enough so that $h = h_\Gamma \cdot k_\Gamma$ with $k_\Gamma \in SL_2(\mathbb{Z}_p)$ close enough to 1 so that k_Γ acts *trivially* on $\Gamma\backslash G_\infty$. Then

$$h \cdot (\ldots \longrightarrow \Gamma x_\Gamma \longrightarrow \ldots) \;=\; (\ldots \longrightarrow (h_\Gamma k_\Gamma)\cdot \Gamma x_\Gamma \longrightarrow \ldots) \;=\; (\ldots \longrightarrow h_\Gamma \cdot \Gamma x_\Gamma \longrightarrow \ldots)$$

$$=\; (\ldots \longrightarrow h_\Gamma \Gamma h_\Gamma^{-1} \cdot h_\Gamma x_\Gamma \longrightarrow \ldots)$$

Since $G_\infty \times SL_2(\mathbb{Z}_p)$ was already transitive on the limit, certainly $G_\infty \times SL_2(\mathbb{Q}_p)$ is transitive. The isotropy subgroup of the basepoint $(\ldots \to \Gamma \cdot 1 \to \ldots)$ is the collection of $g \times h \in G_\infty \times SL_2(\mathbb{Q}_p)$ such that

$$(\ldots \longrightarrow \Gamma\cdot 1 \longrightarrow \ldots) \;=\; (g\times h)\cdot(\ldots \longrightarrow \Gamma\cdot 1 \longrightarrow \ldots) \;=\; (\ldots \longrightarrow h_\Gamma \Gamma h_\Gamma^{-1} \cdot h_\Gamma g^{-1} \longrightarrow \ldots)$$

That is, for every p-congruence Γ, we require $h_\Gamma g^{-1} \in h_\Gamma \Gamma h_\Gamma^{-1}$. That is, $h_\Gamma \in g \cdot \Gamma$ for all Γ. Since h_Γ is ambiguous by Γ, we may as well take $h_\Gamma = g$ for all Γ. In particular, $g \in SL_2(\mathbb{Z}[1/p])$, and the isotropy group is the diagonal copy $SL_2(\mathbb{Z}[1/p])^\Delta$ of $SL_2(\mathbb{Z}[1/p])$ in $G_\infty \times SL_2(\mathbb{Q}_p)$. The sided-ness of the action can be reversed, if desired. Thus, we have proven

[9.2.5] Theorem:

$$\lim_n \Gamma(p^n)\backslash G_\infty \;=\; \lim_{p-\text{congruence }\Gamma} \Gamma\backslash G_\infty \;\approx\; SL_2(\mathbb{Z}[1/p])^\Delta \backslash \Big(G_\infty \times SL_2(\mathbb{Q}_p)\Big)$$

[9.3] Projections $SL_2(\mathbb{Z}[1/p]) \backslash (G_\infty \times SL_2(\mathbb{Q}_p)) \longrightarrow \Gamma(p^n)\backslash G_\infty$

These complete the description of the limit. There is no necessity of giving projections to the more general p-congruence subgroups' quotients, although one can do so. Let

$$K(p^n) \;=\; \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}_p) \;:\; \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \bmod p^n\}$$

We claim that the quotient on the right by $K(p^n)$

$$SL_2(\mathbb{Z}[1/p]) \backslash (G_\infty \times SL_2(\mathbb{Q}_p)) \;\longrightarrow\; SL_2(\mathbb{Z}[1/p]) \backslash (G_\infty \times SL_2(\mathbb{Q}_p)) \,/\, K(p^n)$$

produces an object isomorphic to $\Gamma(p^n)\backslash G_\infty$. Indeed, we can adjust given $g\times h$ in $G\infty SL_2(\mathbb{Q}_p)$ to move h into $K(p^n)$, since $SL_2(\mathbb{Z}[1/p])$ is dense in $SL_2(\mathbb{Q}_p)$. Further, right multiplication by $K(p^n)$ gives a representative of the form $g \times 1$. Two such representatives $g\times 1$ and $g'\times 1$ are equivalent left-modulo $SL_2(\mathbb{Z}[1/p])$ and right modulo $K(p^n)$ exactly when there is $\gamma \in SL_2(\mathbb{Z}[1/p])$ and $k \in K(p^n)$ such that

$$\gamma \cdot (g\times 1)\cdot k \;=\; g' \times 1$$

Projecting γ to $SL_2(\mathbb{Q}_p)$, we have $\gamma \cdot k = 1$. Thus, γ is in the intersection $SL_2(\mathbb{Z}[1/p]) \cap K(p^n) = \Gamma(p^n)$. ///

[9.4] Bigger solenoid and adele-group action

The projective limit over *all* $\Gamma_N = \Gamma(N)$ rather than just $\Gamma(p^n)$ for fixed prime p is treated analogously. First, we would note that $\lim_N SL_2(\mathbb{Z}/N) = SL_2(\widehat{\mathbb{Z}})$ acts, and

$$\lim_N \Gamma_N\backslash G_\infty \;\approx\; SL_2(\mathbb{Z}) \backslash (G_\infty \times SL_2(\widehat{\mathbb{Z}}))$$

being aware by this point that it is natural to take the *diagonal* copy of $SL_2(\mathbb{Z})$ rather than just the copy inside G_∞.

Looking at the larger diagram in which Γ ranges over *all* congruence subgroups of $SL_2(\mathbb{Q})$, that is, all subgroups containing some Γ_N with finite index, and noting that $\mathbb{A}_{\text{fin}} = \mathbb{Q}\cdot\widehat{\mathbb{Z}}$, we see an action of $SL_2(\mathbb{A}_{\text{fin}})$.

Since $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}}$ and $SL_2(\mathbb{A}) = SL_2(\mathbb{R}) \times SL_2(\mathbb{A}_{\text{fin}})$, and realizing that the diagonal copy of $SL_2(\mathbb{Q})$ is natural,

$$\lim_{\Gamma} \Gamma\backslash G_\infty \;\approx\; SL_2(\mathbb{Q}) \backslash (G_\infty \times SL_2(\mathbb{A}_{\text{fin}})) \;=\; SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})$$

To describe the projection to $\Gamma_N\backslash G_\infty$ for principal congruence Γ_N, let

$$K(N) \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\widehat{\mathbb{Z}}) \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$$

There is the natural quotient map

$$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) \;\longrightarrow\; SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) \,/\, K(N)$$

and we claim that

$$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) \,/\, K(N) \;\approx\; \Gamma_N\backslash G_\infty$$

Indeed, $SL_2(\mathbb{Q})$ is dense (see below) in $SL_2(\mathbb{A}_{\text{fin}})$, so the finite-prime part of an element of $SL_2(\mathbb{A})$ can be moved to $K(N)$, and then made trivial by right adjustment by $K(N)$. For $g \times 1$ and $g' \times 1$ in $G_\infty \times SL_2(\mathbb{A}_{\text{fin}})$, if for $\gamma \in SL_2(\mathbb{Q})$ and $k \in K(N)$ we have

$$\gamma \cdot (g \times 1) \cdot k \;=\; g' \times 1$$

then the projection of γ to $SL_2(\mathbb{A}_{\text{fin}})$ must be k^{-1}. Thus, projected to $SL_2(\mathbb{A}_{\text{fin}})$, we have $\gamma \in SL_2(\mathbb{Q}) \cap K(N) = \Gamma_N$.

[9.5] Density of $\mathbb{Q}$ in $\mathbb{A}_{\text{fin}}$ Although $\mathbb{Q}$ is *discrete* in $\mathbb{A}$, dropping the factor $\mathbb{R}$ makes the copy of $\mathbb{Q}$ *dense*, demonstrated as follows.

Given finite adele x, the set of primes p so that the p-component x_p of x is not in $\mathbb{Z}_p$ is *finite*. Since $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$, and since $1/p$ is in $\mathbb{Z}_q$ for primes $q \neq p$, we can adjust x by a finite sum of elements from various copies of $\mathbb{Z}[1/p]$ to make x lie in $\mathbb{Z}_p$ for *all* p. That is, $x \in \widehat{\mathbb{Z}}$.

The diagrammatic characterization of $\widehat{\mathbb{Z}}$ already shows that the natural image of $\mathbb{Z}$ is dense, since its closure in $\widehat{\mathbb{Z}}$ has all the properties of a projective limit of the quotients $\mathbb{Z}/N$.

This density result is the larger part of the argument that $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_{\text{fin}})$.

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