(*May 18, 2016*)

Transition: Eisenstein series on adele groups

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/12_2_transition_Eis.pdf]

A simple Eisenstein series, for $z \in \mathfrak{H}$, $\Gamma = SL_2(\mathbb{Z})$, is

$$E_s(z) \;=\; \sum_{\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\gamma z)^s \;=\; \tfrac{1}{2} \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}} \qquad \Big(\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\}\Big)$$

This Eisenstein series can be rewritten to exhibit the role of p-adic groups $SL_2(\mathbb{Q}_p)$, thereby exhibiting Hecke operators as *integral operators*, parallel to the relevance of the representation theory of $SL_2(\mathbb{R})$ to invariant differential operators. [1] The explicit nature of Eisenstein series allows vivid illustration of some basic mechanisms.

For $SL_2(\mathbb{Q})$, one can survive without this viewpoint. However, for SL_2 over number fields with non-trivial class groups, for SL_n with $n > 2$, and for more general groups, the neo-classical elementary ideas sufficient for $SL_2(\mathbb{Q})$ fall short. Even in the simplest case of $SL_2(\mathbb{Z})$, understanding the role of the p-adic groups $SL_2(\mathbb{Q}_p)$ greatly simplifies and clarifies many classical issues.

That is, this shift in viewpoint is *helpful*, not merely a stylistic choice.

---

1. *Moving automorphic forms from domains to groups*

Automorphic forms on *domains* such as the upper half-plane $\mathfrak{H}$ are usually introduced first because it is easier to discuss them. However, this picture hides important features about the *representation theory* of $SL_2(\mathbb{R})$. It is easy to convert automorphic forms on $\mathfrak{H}$, both waveforms and holomorphic modular forms, to automorphic forms on $SL_2(\mathbb{R})$. [2]

[1.1] Γ-invariant functions and $SO_2(\mathbb{R})$-invariance Choice of (maximal) compact subgroup $SO(2,\mathbb{R})$ in $SL_2(\mathbb{R})$ amounts to choice of basepoint $i \in \mathfrak{H}$, as $SO_2(\mathbb{R})$ is the isotropy subgroup of i. Via the isomorphism $SL_(\mathbb{R})/SO_2(\mathbb{R}) \to \mathfrak{H}$ by $gSO_2(\mathbb{R}) \to g(i)$, and function f on $\mathfrak{H}$ can be converted to a function F on $SL_2(\mathbb{R})$ by

$$F(g) \;=\; f(g(i))$$

[1] Despite contrary assertions in the literature, rewriting Eisenstein series, as opposed to more general automorphic forms, on adele groups does *not* use *Strong Approximation*. Strong Approximation *does* make precise the relation between *general* automorphic forms on adele groups and automorphic forms on SL_2 and even on SL_n, but rewriting these Eisenstein series does not need this comparison. Indeed, Strong Approximation does not hold in the simplest form for general semi-simple or reductive groups, but this does harm anything. Strong approximation *does* show that there can be no *other* extension of the Eisenstein series to the adele group beyond that described here.

[2] This is an example of conversion to automorphic forms on *reductive* or *semi-simple real Lie groups*, of which $SL_2(\mathbb{R})$ is a small example.

The function F is right $SO_2(\mathbb{R})$-invariant:

$$F(gk) \;=\; f(gk(i)) \;=\; f(g(i)) \;=\; F(g) \qquad\qquad (\text{for } g \in SL_2(\mathbb{R}) \text{ and } k \in SO_2(\mathbb{R}))$$

Oppositely, any such right-$SO_(\mathbb{R})$-invariant function F on $SL_2(\mathbb{R})$ descends to a function f on the quotient $\mathfrak{H}$ by

$$f(z) \;=\; F(g_z) \qquad\qquad (\text{for any } g_z \in SL_2(\mathbb{R}) \text{ with } g_z(i) = z)$$

For f also left invariant by $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$, the corrersponding function F on $SL_2(\mathbb{R})$ is left Γ-invariant, and vice-versa:

$$F(\gamma g) \;=\; f(\gamma g(i)) \;=\; f(g(i)) \;=\; F(g) \qquad\qquad (\text{for } \gamma \in \Gamma \text{ and } g \in SL_2(\mathbb{R}))$$

[1.2] Automorphy condition and $SO_2(\mathbb{R})$-equivariance To motivate extension of the treatment of left Γ-*invariant* function on $\mathfrak{H}$, of course holomorphic modular forms f on $\mathfrak{H}$ are not quite left Γ-*invariant*, but satisfy an *automorphy condition*

$$f(\gamma z) \;=\; j(\gamma, z) \cdot f(z) \qquad\qquad (\text{where } j(\alpha\beta, z) = j(\alpha, \beta z) \cdot j(\beta, z))$$

Apart from the trivial cocycle $j(g, z) = 1$, the simplest example of *cocycle* is $j(g, z) = (cz + d)^{2k}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This *extends* to be a cocycle defined not merely on $\Gamma \times \mathfrak{H}$ but on $SL_2(\mathbb{R}) \times \mathfrak{H}$. Associate the function F on $SL_2(\mathbb{R})$ given by

$$F(g) \;=\; j(g, i)^{-1} \cdot f(g(i))$$

[1.2.1] Claim: The function F is left Γ-invariant on $SL_2(\mathbb{R})$, and right $SO_2(\mathbb{R})$-equivariant by

$$F(gk) \;=\; j(k, i)^{-1} \cdot F(g)$$

and $k \to j(k, i)^{-1}$ is a *group homomorphism* $SO_2(\mathbb{R}) \to \mathbb{C}^\times$.

Proof: This is a direct computation. For $\gamma \in \Gamma$,

$$F(\gamma g) \;=\; j(\gamma g, i)^{-1} \cdot f(\gamma g(i)) \;=\; \Big(j(\gamma, g(i))\, j(g, i)\Big)^{-1} j(\gamma, g(i)) \cdot f(g(i))$$

$$=\; j(g, i)^{-1} \cdot j(\gamma, g(i))^{-1} \cdot j(\gamma, g(i)) \cdot f(g(i)) \;=\; j(g, i)^{-1} \cdot f(g(i)) \;=\; F(g)$$

For $k \in SO_2(\mathbb{R})$,

$$F(gk) \;=\; j(gk, i)^{-1} \cdot f(gk(i)) \;=\; \Big(j(g, k(i))\, j(k, i)\Big)^{-1} \cdot f(g(i)) \;=\; \Big(j(g, i)\, j(k, i)\Big)^{-1} \cdot f(g(i))$$

$$=\; j(k, i)^{-1} \cdot j(g, i)^{-1} \cdot f(g(i)) \;=\; j(k, i)^{-1} \cdot F(g)$$

Finally, for $k, h \in SO_2(\mathbb{R})$,

$$j(hk, i) \;=\; j(h, k(i)) \cdot j(k, i) \;=\; j(h, i) \cdot j(k, i)$$

proving that $k \to j(k, i)$ is a group homomorphism on $SO_2(\mathbb{R})$. ///

[1.2.2] Remark: *Half-integral weight* automorphic forms have a cocycle on $\Gamma \times \mathfrak{H}$ which does *not* extend to $SL_2(\mathbb{R}) \times \mathfrak{H}$. The obstruction to this extension *defines* a two-fold covering group of $SL_2(\mathbb{R})$, the *metaplectic* group, the group where half-integral weight automorphic forms live. This and other complications account for our emphasis on integral-weight holomorphic modular forms, and on waveforms, in these notes.

[1.2.3] Remark: The seemingly different analytic conditions, *holomorphy* and being an *eigenfunction* for the invariant Laplacian, both become eigenfunction conditions on the group $SL_2(\mathbb{R})$. We will return to this a little later.

[1.3] Conversion to automorphic forms on $GL_2(\mathbb{R})$

To eventually accommodate Hecke operators, and for many other reasons, the group GL_2 is better than $SL_2(\mathbb{R})$.

Already, the subgroup

$$GL_2^+(\mathbb{R}) \;=\; \{g \in GL_2(\mathbb{R}) \;:\; \det g > 0\}$$

preserves the upper half-plane $\mathfrak{H}$, and

$$GL_2^+(\mathbb{Z}) \;=\; \{g \in GL_2(\mathbb{Z}) \;:\; \det g > 0\} \;=\; SL_2(\mathbb{Z})$$

since $g \in GL_2(\mathbb{Z})$ has determinant ± 1. Now the isotropy group of $i \in \mathfrak{H}$ is

$$Z \;=\; \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;:\; a \in \mathbb{R}^\times\}$$

As with $SL_2(\mathbb{R})$, we have

$$\mathfrak{H} \;\approx\; GL_2^+(\mathbb{R})/(\text{isotropy subgroup of } i) \;=\; GL_2^+(\mathbb{R})/Z_{\mathbb{R}}^+ GL_2^+(\mathbb{Z})$$

For left $GL_2^+(\mathbb{Z})$-invariant functions f on $\mathfrak{H}$, the associated function F on $GL_2^+(\mathbb{R})$ is

$$F(g) \;=\; f(g(i))$$

and F is left $GL_2^+(\mathbb{Z})$-invariant, right $SO_2(\mathbb{R})$-invariant, and Z-invariant. Since Z is the *center* of $GL_2(\mathbb{R})$, the function F is both right and left Z-invariant.

For f not left $GL_2^+(\mathbb{Z})$-*invariant*, but only meeting a cocycle condition $f(\gamma z) = j(\gamma, z) \cdot f(z)$ with a cocycle extending to $GL_2^+(\mathbb{R}) \times \mathfrak{H}$, such as

$$j(g, z) \;=\; (cz + d)^{2k} \cdot (\det g)^{-k} \qquad\qquad (\text{with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}))$$

for weight $2k$ holomorphic modular forms, the corresponding function F on $GL_2^+(\mathbb{R})$ is again

$$F(g) \;=\; j(g, i)^{-1} \cdot f(g(i))$$

as for $SL_2(\mathbb{R})$. Some cocycles are trivial on Z, some are not.

Further, the positive-determinant condition can be removed when the function f on $\mathfrak{H}$ is extended to be a function on $\mathfrak{H} \cup \overline{\mathfrak{H}}$, since $GL_2(\mathbb{R})$ stabilizes the union of both half-planes, producing functions F on $GL_2(\mathbb{R})$ that are left $GL_2(\mathbb{Z})$-invariant, right $O_2(\mathbb{R})$-equivariant by some *representation* of $O_2(\mathbb{R})$, and Z-equivariant by some character $\mathbb{Z} \to \mathbb{C}^\times$.

2. *Iwasawa decompositions of* $GL_2(\mathbb{R})$ *and* $GL_2(\mathbb{Q}_p)$

That $g \in GL_2(\mathbb{R})$ can be written as a product $g = pk$ of upper-triangular p and orthogonal matrices k was known for a long time before K. Iwasawa's work in the 1940s, but after Iwasawa's work this and other related classical facts are understood as instances of a very general pattern that applies, not only to *real* or *complex* matrix groups, but also to related *p-adic* matrix groups.

The pattern of an Iwasawa decomposition in a real or complex or p-adic matrix group is essentially [3]

$$\text{whole group} \;=\; (\text{upper-triangular elements}) \cdot (\text{maximal compact subgroup})$$

The subgroup P of upper-triangular matrices is a *parabolic subgroup*. The general definition of *parabolic* is not essential here, nor is determination or certification that various subgroups are *maximal* compact. Rather, we are acknowledging that our present examples fit into a larger pattern.

[2.1] Iwasawa decomposition for $GL_2(\mathbb{R})$ Let $G_\infty = GL_2(\mathbb{R})$. Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$, choose the element $k \in K_\infty = O_2(\mathbb{R})$ to right multiply by to put gk^{-1} into the group P_∞ of upper-triangular real matrices, by rotating the lower half $(c\ d)$ of g in $\mathbb{R}^2$ into the form $(0\ *)$. Indeed, realizing that matrices in K_∞ are of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (\text{for } \theta \in \mathbb{R})$$

in particular with the squares of the left column summing to 1, choose

$$k^{-1} \;=\; \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & * \\ \frac{-c}{\sqrt{c^2+d^2}} & * \end{pmatrix}$$

with the right column of course completely determined by the left, so that

$$gk^{-1} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & * \\ \frac{-c}{\sqrt{c^2+d^2}} & * \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ c \cdot \frac{d}{\sqrt{c^2+d^2}} + d \cdot \frac{-c}{\sqrt{c^2+d^2}} & * \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

as desired. For application to Eisenstein series, we only need the *diagonal* entries of gk^{-1}, and we easily further compute the lower-right entry

$$gk^{-1} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ \frac{-c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix}$$

For $g \in SL_2(\mathbb{R})$, the determinant-one condition gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \cdot K_\infty \;\subset\; P_\infty \cdot K_\infty$$

[3] Probably the group should be *reductive* or *semi-simple*, whose formal definitions do not concern us for the moment. Rather, we note that this class of groups includes important examples: $GL_n(\mathbb{R})$, $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $GL_n(\mathbb{C})$, as well as orthogonal groups, unitary groups, and other matrix groups defined by preservation of additional structure. The class does *not* include upper-triangular matrices in $GL_n(\mathbb{R})$, although this subgroup is important in its own right. This class of groups does also include p-adic versions of groups over $\mathbb{R}$ and $\mathbb{C}$.

For general $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; \begin{pmatrix} \frac{\det g}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \cdot K_\infty \;\subset\; P_\infty \cdot K_\infty$$

[2.2] Iwasawa decomposition for $GL_2(\mathbb{Q}_p)$ Even though the substance of a p-adic Iwasawa decomposition is quite different from that of archimedean Iwasawa decompositions, the commonalities are very useful.

The standard maximal compact subgroup K_v of $G_v = GL_2(\mathbb{Q}_v)$ with v a finite prime, in sharp contrast to the orthogonal group as maximal compact subgroup of $GL_2(\mathbb{R})$, is

$$K_v \;=\; GL_2(\mathbb{Z}_v) \;=\; \{p\text{-adic integral matrices with determinants in } \mathbb{Z}_v^\times\} \qquad (v \text{ is finite prime } p)$$

It is not so hard to prove that this is *compact*, but a little harder to prove that it is *maximal* compact. But, for the moment, we don't use either property, especially not the maximality.

The essential aspect of $\mathbb{Q}_p$ is that, for any two $x, y \in \mathbb{Q}_p^\times$, either $x/y \in \mathbb{Z}_p$ or $y/x \in \mathbb{Z}_p$, since

$$\mathbb{Z}_p \;=\; \{z \in \mathbb{Q} \;:\; |z|_p \le 1\}$$

This contrasts violently with the classical situation of rational numbers x, y, where the classical (=archimedean!) notion of "size" is very far from a sufficient determiner of divisibility. Thus, given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$,

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & (\text{for } |c|_v \le |d|_v) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} & (\text{for } |c|_v \ge |d|_v) \end{cases}$$

In both cases, the right-multiplying matrix is in $K_v = GL_2(\mathbb{Z}_v)$: in the first case $|c/d|_v \le 1$ implies $c/d \in \mathbb{Z}_v$, and in the second $|d/c|_v \le 1$ implies $d/c \in \mathbb{Z}_v$. In the first case, the resulting product is already in the group P_v of upper-triangular matrices in G_v. In the second, further right-multiplication by the long Weyl element $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ puts the product into P_v. This proves that the p-adic Iwasawa decomposition succeeds.

Specifics about the diagonal entries in the Iwasawa decomposition will be needed in rewriting Eisenstein series:

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix} \;=\; \begin{pmatrix} a - \frac{bc}{d} & * \\ 0 & d \end{pmatrix} & (\text{for } |c|_v \le |d|_v) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -\frac{d}{c} & 1 \\ 1 & 0 \end{pmatrix} \;=\; \begin{pmatrix} -\frac{ad}{c} + b & * \\ 0 & c \end{pmatrix} & (\text{for } |c|_v \ge |d|_v) \end{cases}$$

That is,

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; \begin{pmatrix} a - \frac{bc}{d} & * \\ 0 & d \end{pmatrix} \cdot K_v & (\text{for } |c|_v \le |d|_v) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; \begin{pmatrix} -\frac{ad}{c} + b & * \\ 0 & c \end{pmatrix} \cdot K_v & (\text{for } |c|_v \ge |d|_v) \end{cases}$$

[2.2.1] Remark: It is very convenient that the shape of the parabolic P_v is the same for both archimedean and non-archimedean v, while the shape of the (maximal) compact subgroup K_v is significantly different.

3. *Rewriting the GL_2 Eisenstein series*

With $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\}$, the simplest waveform-type Eisenstein series on the *domain* $\mathfrak{H}$ is the usual

$$E_s(z) \;=\; \sum_{\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\gamma z)^s \qquad\qquad (\text{for } z \in \mathfrak{H} \text{ and } \mathrm{Re}(s) > 1)$$

This Eisenstein series on the domain $\mathfrak{H}$ is easily converted to a left Γ-invariant, right $SO_2(\mathbb{R})$-invariant function on the *group* $GL_2^+(\mathbb{R})$, still called an *Eisenstein series*, by

$$E_s^{\text{group}}(g) \;=\; E_s(g(i)) \qquad\qquad (\text{for } g \in GL_2^+(\mathbb{R}))$$

The basepoint $i \in \mathfrak{H}$ is chosen since $K_\infty = SO_2(\mathbb{R})$ is its isotropy group, which immediately explains the right $SO_2(\mathbb{R})$-invariance.

In fact, E_s can be extended to an automorphic form on the union of upper and lower half-planes by $E_s(-z) = E_s(z)$, noting that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(z) \;=\; -z$$

Thus, now letting $\Gamma = GL_2(\mathbb{Z})$ and $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\}$, we have an expression for the extended Eisenstein series that is of the same form:

$$E_s(z) \;=\; \sum_{\Gamma_\infty\backslash\Gamma} \big|\mathrm{Im}\,\gamma z\big|^s \qquad\qquad (\text{for } z \in \mathfrak{H} \cup \overline{\mathfrak{H}})$$

The Eisenstein series on the whole group $G_\infty = GL_2(\mathbb{R})$ is

$$E_s^{\text{group}}(g) \;=\; E_s(g(i)) \qquad\qquad (\text{for } g \in GL_2(\mathbb{R}))$$

[3.1] The localized rewrite As above, let v be an index for *completions* [4] of $\mathbb{Q}$, with $\mathbb{Q}_v$ the v^{th} completion, and $\mathbb{Z}_v$ the v-adic integers for v a prime, with $\mathbb{Q}_\infty = \mathbb{R}$, and $|\cdot|_\infty$ the usual *real* absolute value. The latter is distinguished from genuine primes by the convention of calling it the *infinite prime*, in analogy with a more legitimate use of this in the function-field setting.

For each place v, finite and infinite, using the v^{th} Iwasawa decomposition $G_v = P_v K_v$, define a function φ_v on G_v by

$$\varphi_v\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot k\Big) \;=\; \Big|\frac{a}{d}\Big|_v^s \qquad \begin{cases} \text{for } k \in GL_2(\mathbb{Z}_v) & (\text{for } v \text{ non-archimedean}) \\ \text{for } k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in O(2) & (\text{for } v \text{ archimedean}) \end{cases}$$

where in all cases $a, d \in \mathbb{Q}_v^\times$ and $b \in \mathbb{Q}_v$. By design, each φ_v is invariant under the center Z_v of G_v. Define a function on the adele group $G_\mathbb{A} = GL_2(\mathbb{A})$ by

$$\varphi \;=\; \bigotimes_v \varphi_v \qquad\qquad (\text{meaning } \varphi(\{g_v\}) = \textstyle\prod_v \varphi_v(g_v), \text{ where } g_v \in G_v)$$

[4] Another tradition for refering to p-adic completions, as well as to $\mathbb{R}$, is as *places* of $\mathbb{Q}$. Also, sometimes ∞ is called the *infinite prime*, while the actual primes p are *finite primes*.

[3.1.1] Claim: *(The product formula)* For $x \in \mathbb{Q}^\times$,

$$\prod_{v \le \infty} |x|_v \;=\; 1 \qquad\qquad \text{(product over all } p\text{-adic norms as well as } \mathbb{R})$$

Proof: Since the assertion is *multiplicative*, it suffices to prove the product formula for units and for primes. All norms of ± 1 are 1, and all norms but the archimedean and p^{th} of a prime p are 1. Since $|p|_\infty = p$ and $|p|_p = \frac{1}{p}$, we have the product formula. ///

The product formula shows shows that φ is left $P_\mathbb{Q}$-invariant: using the Iwasawa decomposition locally everywhere, with $k \in \prod_v K_v$,

$$\varphi(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} k) \;=\; |Aa/Dd|^s \;=\; |A/D|^s \cdot |a/d|^s \;=\; |A/D|^s \cdot \varphi(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} k)$$

Let

$$Z_\mathbb{A} \;=\; \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;:\; a, a^{-1} \in \mathbb{A}\}$$

[3.1.2] Theorem: For $g_\infty \in GL_2(\mathbb{R})$, acting as usual on $\mathfrak{H} \cup \overline{\mathfrak{H}}$,

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty)$$

This expression gives a left $G_\mathbb{Q}$-invariant, $Z_\mathbb{A}$-invariant function (still denoted E_s) on the *adele group* $G_\mathbb{A} = GL_2(\mathbb{A})$:

$$E_s^{\text{adelic}}(g) \;=\; \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g)$$

Proof of this will occupy the rest of this section.

[3.1.3] Remark: The notations E_s^{group} and E_s^{adelic} are not standard, but have the obvious descriptive utility. In fact, we will revert to writing E_s for the Eisenstein series on $G_\mathbb{A} = GL_2(\mathbb{A})$.

[3.2] Disambiguation

The immediate question arises of evaluation of $\varphi(\gamma \cdot g_\infty)$. One point is that $G_\mathbb{Q} = GL_2(\mathbb{Q})$ should *not* be considered as *only* a subgroup of $G_\infty = GL_2(\mathbb{R})$, but also a subgroup of *every* $G_v = GL_2(\mathbb{Q}_v)$. Thus, $G_\mathbb{Q}$ is best considered as imbedded *on the diagonal* in $G_\mathbb{A}$. Adding a temporary notational burden for clarity, for each place v let $j_v : GL_2(\mathbb{Q}) \to GL_2(\mathbb{Q}_v)$ be the natural injective map, and let

$$j \;=\; \prod_v j_v \;:\; GL_2(\mathbb{Q}) \longrightarrow \prod_v GL_2(\mathbb{Q}_v)$$

be the natural *diagonal map* to the product. Then

$$\varphi(\gamma \cdot g_\infty) \;=\; \varphi_\infty(j_\infty(\gamma) \cdot g_\infty) \cdot \prod_{v<\infty} \varphi_\infty(j_v(\gamma))$$

That is, in the definition of φ, the archimedean $g_\infty \in G_\infty$ does not interact with the non-archimedean groups $G_v = GL_2(\mathbb{Q}_v)$.

[3.3] Comparison to classical formulation

The familiar Eisenstein series $E_s(z)$ can be obtained from the above by *reverting* to a form that does not refer to anything p-adic or adelic. That is, we claim that

$$\varphi_\infty(g_\infty) \;=\; \big|\text{Im}\,(g_\infty \cdot i)\big|^s \qquad\qquad (\text{for } g_\infty \in GL_2(\mathbb{R}))$$

with $GL_2(\mathbb{R})$ acting on $\mathfrak{H} \cup \overline{\mathfrak{H}}$. The argument is about the *Iwasawa decomposition*, namely, that any element of $GL_2(\mathbb{R})$ can be written as a product of upper-triangular and orthogonal matrices. Indeed, given a matrix in $GL_2(\mathbb{R})$, right multiplication by an orthogonal matrix can be viewed as *rotating* the bottom row. This suggests the appropriate orthogonal group element: formulaically,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ \frac{-c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \frac{c^2+d^2}{\sqrt{c^2+d^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix}$$

Thus,

$$\varphi_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi_\infty \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} = \left| \frac{1/\sqrt{c^2+d^2}}{\sqrt{c^2+d^2}} \right|^s = \left| \frac{1}{c^2+d^2} \right|^s$$

On the other hand, a familiar computation gives

$$\mathrm{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (i) = \frac{1}{2i} \Big(\frac{ai+b}{ci+d} - \frac{-ai+b}{-ci+d} \Big) = \frac{ad-bc}{c^2+d^2} = \frac{1}{c^2+d^2}$$

Since $\gamma \in GL_2(\mathbb{Z})$ maps to $GL_2(\mathbb{Z}_v)$ at all finite places v,

$$\varphi_v(\gamma) = 1 \qquad \text{(for } \gamma \in GL_2(\mathbb{Z}) \text{ and finite place } v)$$

Thus,

$$\sum_{\gamma \in P_{\mathbb{Z}} \backslash GL_2(\mathbb{Z})} \varphi(\gamma \cdot g_\infty) = \sum_{\gamma \in P_{\mathbb{Z}} \backslash GL_2(\mathbb{Z})} \varphi_\infty(\gamma \cdot g_\infty) \cdot 1 = \sum_{\gamma \in P_{\mathbb{Z}} \backslash GL_2(\mathbb{Z})} \Big(\mathrm{Im}\, \gamma g_\infty \cdot i \Big)^s$$

Taking the popular choice

$$g_\infty = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \qquad \text{(with } x \in \mathbb{R} \text{ and } y > 0)$$

produces $E_s(x+iy)$ on $\mathfrak{H}$, as claimed.

[3.4] Well-definedness on $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$ We should show that $\varphi(\gamma g_\infty)$ depends only upon the coset $P_{\mathbb{Q}}\gamma$. Any $\gamma \in GL_2(\mathbb{Q})$ is in $GL_2(\mathbb{Z}_v)$ for almost all v, since the entries are in $\mathbb{Z}_v$ for almost all v, and the determinant is a v-adic unit for almost all v, so the inverse is v-integral also. Thus, in an infinite product $\prod_{v<\infty} \varphi_v(\gamma)$, all but finitely-many factors are 1.

Let χ_v be the character on upper-triangular v-adic matrices P_v given by

$$\chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \left| \frac{a}{d} \right|_v^s \qquad \text{(with } a, d \in \mathbb{Q}_v^\times \text{ and } b \in \mathbb{Q}_v)$$

The usual maximal compact [5] subgroups K_v of the groups $GL_2(\mathbb{Q}_v)$ are

$$K_v = \begin{cases} GL_2(\mathbb{Q}_v) & \text{(for } v \text{ finite)} \\ O(2) & \text{(for } v \text{ real)} \end{cases}$$

The description of φ_v can be rewritten more succinctly as

$$\varphi_v(pk) = \chi_v(p) \qquad \text{(for } p \in P_v \text{ and } k \in K_v)$$

[5] We will not use the fact that these are *maximal* among compact subgroups, despite refering to them as such.

For $g_\infty \in G_\mathbb{R}$, $\gamma \in G_\mathbb{Q}$, and $\beta \in P_\mathbb{Q}$, keeping in mind that $G_\mathbb{Q}$ maps to *all* groups G_v, not just to G_∞,

$$\varphi(\beta \cdot \gamma \cdot g_\infty) \;=\; \varphi_\infty(\beta \cdot \gamma \cdot g_\infty) \cdot \prod_{v<\infty} \varphi_v(\beta \cdot \gamma)$$

At the archimedean place, let $\gamma g_\infty = pk$ be an Iwasawa decomposition in G_v, with $p \in P_v$ and $k \in K_v$. We see the left equivariance of φ_v by χ_v, namely,

$$\varphi_v(\beta\gamma g_\infty) \;=\; \varphi_v(\beta pk) \;=\; \chi_v(\beta \cdot p) \;=\; \chi_v(\beta) \cdot \chi_v(p) \;=\; \chi_v(\beta) \cdot \varphi_v(pk) \;=\; \chi_v(\beta) \cdot \varphi_v(\gamma g_\infty)$$

Similarly, but now without g_∞ playing any role, at a finite place v, let $\gamma = pk$ be an Iwasawa decomposition in G_v, with $p \in P_v$ and $k \in K_v$. We see the left equivariance of φ_v by χ_v:

$$\varphi_v(\beta\gamma) \;=\; \varphi_v(\beta pk) \;=\; \chi_v(\beta \cdot p) \;=\; \chi_v(\beta) \cdot \chi_v(p) \;=\; \chi_v(\beta) \cdot \varphi_v(\gamma)$$

Putting all these local equivariances together,

$$\varphi(\beta \cdot \gamma \cdot g_\infty) \;=\; \prod_v \chi_v(\beta) \cdot \varphi_v(\gamma \cdot g_\infty) \qquad (\text{for } \beta \in P_\mathbb{Q},\ \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty)$$

By the product formula,

$$\prod_v \chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;=\; \prod_v \Big|\frac{a}{d}\Big|_v^s \;=\; 1 \qquad (\text{for } a/d \in \mathbb{Q}^\times)$$

That is, we have the left invariance

$$\varphi(\beta \cdot \gamma \cdot g_\infty) \;=\; \varphi(\gamma \cdot g_\infty) \qquad (\text{for } \beta \in P_\mathbb{Q},\ \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty)$$

[3.5] Bijection of cosets

Changing from the Γ and Γ_∞ notation, let $G_\mathbb{Z} = GL_2(\mathbb{Z})$ and $P_\mathbb{Z} = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G_\mathbb{Z}\}$, we claim that

$$P_\mathbb{Q}\backslash G_\mathbb{Q} \;\approx\; P_\mathbb{Z}\backslash G_\mathbb{Z}$$

More precisely, we claim that the obvious map $P_\mathbb{Z}\backslash G_\mathbb{Z} \to P_\mathbb{Q}\backslash G_\mathbb{Q}$ by $P_\mathbb{Z} g \to P_\mathbb{Q} g$ is a *surjection.* It is an *injection* because $P_\mathbb{Z} = G_\mathbb{Z} \cap P_\mathbb{Q}$. That is, we claim that that every coset $P_\mathbb{Q} h$ with $h \in G_\mathbb{Q}$ has a representative in $G_\mathbb{Z} = GL_2(\mathbb{Z})$. The argument attaches meaning to both these coset spaces, and will thereby give the bijection.

The coset space $P_\mathbb{Q}\backslash G_\mathbb{Q}$ is in bijection with the set of lines in $\mathbb{Q}^2$, respecting the right multiplication by $G_\mathbb{Q}$, because $G_\mathbb{Q}$ is transitive on these lines, and $P_\mathbb{Q}$ is the stabilizer of the line $\{(0\ *)\}$. Next, each line in $\mathbb{Q}^2$ meets $\mathbb{Z}^2$ in a free rank-one $\mathbb{Z}$-module generated by a primitive vector (x, y), meaning that $\gcd(x, y) = 1$. Call such a $\mathbb{Z}$-module a *primitive* $\mathbb{Z}$*-line* in $\mathbb{Q}^2$. The collection of lines in $\mathbb{Q}^2$ is thus in bijection with primitive $\mathbb{Z}$-lines in $\mathbb{Z}^2$, by sending a line to its intersection with $\mathbb{Z}^2$. The group $SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$ is already transitive on primitive $\mathbb{Z}$-lines: for $\gcd(x, y) = 1$, let $b, d \in \mathbb{Z}$ be such that

$$bx + dy \;=\; \gcd(x, y) \;=\; 1$$

Then

$$(x\ \ y) \cdot \begin{pmatrix} y & b \\ -x & d \end{pmatrix} \;=\; (0\ \ 1)$$

That is, any primitive vector can be mapped to $(0\ 1)$, so the action of $SL_2(\mathbb{Z})$ is transitive on primitive *vectors*, hence on primitive $\mathbb{Z}$*-lines.* Thus, certainly the slightly larger group $GL_2(\mathbb{Z})$ is transitive on primitive vectors. The stabilizer subgroup of the primitive $\mathbb{Z}$-line spanned by $(0, 1)$ in $GL_2(\mathbb{Z})$ is

$$P_\mathbb{Z} \;=\; \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;:\; a, d \in \mathbb{Z}^\times,\ b \in \mathbb{Z}\}$$

This proves the bijection of coset spaces.

[3.6] The essential conclusion From

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Z} \backslash G_\mathbb{Z}} \varphi_\infty(\gamma \cdot g_\infty) \qquad\qquad (\text{with } g_\infty \in GL_2(\mathbb{R}))$$

from the bijection of coset spaces, and from the well-definedness of φ left modulo $P_\mathbb{Q}$, we have the desired re-expression of the Eisenstein series

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty) \qquad\qquad (\text{with } g_\infty \in GL_2(\mathbb{R}))$$

This is the first main point, and there are further advantages to the viewpoint. Computation of the *constant term* is the first illustration, after the preparation of the next section.

---

4. *Bruhat decomposition for* GL_2

Expressing the coset space $P_\mathbb{Z} \backslash G_\mathbb{Z}$ in terms of *rational* matrices $P_\mathbb{Q} \backslash G_\mathbb{Q}$, rather than *integral* matrices, simplifies natural choices of representatives immediately, as we will see. Further, more significantly, this makes visible the *unwinding* and *Euler factorization* of the Bruhat-cell summands of the constant term, as we see in the next section.

[4.1] Bruhat decomposition The *Bruhat decomposition* for $GL_2(k)$ for *any* field k is [6]

$$G_k \;=\; P_k \;\sqcup\; P_k w N_k \qquad\qquad (\text{with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix})$$

This purely algebraic fact has natural extensions to GL_n and the classical groups. For $G = GL_2$, the Bruhat decomposition is easy to prove: of course, the *little cell* $P_\mathbb{Q}$ consists of matrices with $c = 0$, and we must claim that the *big cell* $P_k w N_k$ is exactly

$$P_k w N_k \;=\; \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k) \;:\; c \neq 0 \}$$

Indeed, given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$,

$$g \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \;\in\; P_k w$$

That proves the Bruhat decomposition in this simple case.

This expression for G_k exhibits a remarkably simple set of representatives for $P_k \backslash G_k$ when *rational* rather than only *integral* entries are allowed:

$$P_k \backslash G_k \;=\; P_k \backslash P_k \cup P_k \backslash \big(P_k w N_k \big) \;\approx\; \{1\} \cup w \cdot \big(w^{-1} P_k w \cap N_k \big) \backslash N_k \;\approx\; \{1\} \cup w N_k$$

---

[6] The validity of this decomposition for $GL_2(k)$ or $GL_2(k)$, and other specific groups, was known long before F. Bruhat's work in the 1950s. Nevertheless, it is good practice to refer to the GL_2 case in terms indicated the extension to the general case.

where the bijection of cosets is

$$P_k wn \;=\; w\cdot(w^{-1}P_k w\cdot n) \;\longleftarrow\; w\cdot(w^{-1}P_k w\cap N_k)\cdot n$$

Indeed, with H, N subgroups of a larger group, $H\cdot n = H\cdot n'$ for $n\in N$, if and only if $n'n^{-1}\in H$. Also, $n'n^{-1}\in N$, so

$$H\cdot n = H\cdot n' \iff (H\cap N)\cdot n = (H\cap N)\cdot n' \qquad\qquad (\text{for } n, n'\in N)$$

[4.2] Another comparison

Another computation verifies our rewrite of the Eisenstein series, paying attention to the Bruhat-cell parametrization. In principle, this computation is unnecessary, but it is informative.

Using the Bruhat decomposition, the sum defining the Eisenstein series is

$$E_s(g_\infty) \;=\; \sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi(\gamma\cdot g_\infty) \;=\; \varphi(g_\infty) \;+\; \sum_{\gamma\in wN_{\mathbb{Q}}} \varphi(\gamma\cdot g_\infty)$$

That is, the summands can be parametrized by $\{1\}$ and $N_{\mathbb{Q}}\approx\mathbb{Q}$. This will be important in computing the *constant term* in the next section.

As φ_∞ is right $O(2)$-invariant and center-invariant, by the Iwasawa decomposition $G_\infty = P_\infty\cdot K_\infty$ we can take

$$g_\infty \;=\; \begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} \qquad\qquad (\text{with } x\in\mathbb{R} \text{ and } y>0)$$

With this g_∞, the term $\gamma = 1$ is

$$\varphi(g_\infty) \;=\; \varphi_\infty(g_\infty)\cdot\prod_{v<\infty}\varphi_v(1) \;=\; |y|^s\cdot 1 \;=\; |y|^s$$

For the big cell contribution to the sum, we need to compute the archimedean part

$$\varphi_\infty(w\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}) \qquad\qquad (\text{for } t\in\mathbb{Q})$$

and the non-archimedean

$$\varphi_v(w\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}) \qquad\qquad (\text{for finite } v, \text{ with } t\in\mathbb{Q})$$

In the archimedean case,

$$w\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} \;=\; \begin{pmatrix} 0 & -1\\ y & x+t\end{pmatrix}$$

Right multiplying by a suitable orthogonal matrix rotates the bottom row to put the result in $P_{\mathbb{R}}$, namely,

$$\begin{pmatrix} 0 & -1\\ y & x+t\end{pmatrix}\begin{pmatrix} \frac{x+t}{\sqrt{(x+t)^2+y^2}} & \frac{y}{\sqrt{(x+t)^2+y^2}}\\ \frac{-y}{\sqrt{(x+t)^2+y^2}} & \frac{x+t}{\sqrt{(x+t)^2+y^2}}\end{pmatrix} \;=\; \begin{pmatrix} \frac{y}{\sqrt{(x+t)^2+y^2}} & *\\ 0 & \sqrt{(x+t)^2+y^2}\end{pmatrix}$$

Thus,

$$\varphi_\infty(w\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}) \;=\; \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s$$

For finite v, adjust the given matrix by right multiplication by $GL_2(\mathbb{Z}_v)$ to make the result upper-triangular. For $t \in \mathbb{Z}_v$, the matrix is already in $GL_2(\mathbb{Z}_v)$, so

$$\varphi_v\Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big) \;=\; 1 \qquad\qquad (\text{for finite } v, \text{ with } t \in \mathbb{Q} \cap \mathbb{Z}_v)$$

For $t \notin \mathbb{Z}_v$, necessarily $t^{-1} \in \mathbb{Z}_v$. Thus, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$$

can be multiplied by $\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}$ in $GL_2(\mathbb{Z}_v)$ to obtain

$$\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \;=\; \begin{pmatrix} t^{-1} & * \\ 0 & t \end{pmatrix}$$

Thus,

$$\varphi_v\Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big) \;=\; |t|_v^{-2s} \qquad\qquad (\text{for finite } v, \text{ with } t \notin \mathbb{Q} \cap \mathbb{Z}_v)$$

That is,

$$\varphi_v\Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big) \;=\; \begin{cases} 1 & (\text{for } |t|_v \le 1) \\ |t|_v^{-2s} & (\text{for } |t|_v \ge 1) \end{cases}$$

Combining the archimedean and non-archimedean,

$$\varphi\Big(w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Big) \;=\; \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s \cdot \prod_{v<\infty} \begin{cases} 1 & (\text{for } |t|_v \le 1) \\ |t|_v^{-2s} & (\text{for } |t|_v > 1) \end{cases}$$

Since $\mathfrak{o}$ is a principal ideal domain with units ± 1, we can easily parametrize $t \in \mathbb{Q}$ in a fashion conforming to evaluation of the displayed expression. Namely, write $t = d/c$ with c, d relatively prime, modulo ± 1. Note that for relatively prime integers c, d

$$\begin{cases} 1 & (\text{for } |d/c|_v \le 1) \\ |d/c|_v^{-2s} & (\text{for } |d/c|_v > 1) \end{cases} \;=\; \begin{cases} 1 & (\text{for } p_v \text{ not dividing } c) \\ |d/c|_v^{-2s} & (\text{for } p_v \text{ dividing } c) \end{cases} \;=\; \begin{cases} 1 & (\text{for } p_v \text{ not dividing } c) \\ |c|_v^{2s} & (\text{for } p_v \text{ dividing } c) \end{cases}$$

That is, by the product formula,

$$\prod_{v<\infty} \begin{cases} 1 & (\text{for } |d/c|_v \le 1) \\ |d/c|_v^{-2s} & (\text{for } |d/c|_v > 1) \end{cases} \;=\; \prod_{v<\infty} |c|_v^{2s} \;=\; \frac{1}{|c|_\infty^{2s}}$$

Then, once again, we recover the expected:

$$\Big|\frac{y}{(x+\frac{d}{c})^2+y^2}\Big|_\infty^s \cdot \frac{1}{|c|_\infty^{2s}} \;=\; \Big|\frac{y}{(cx+d)^2+(cy)^2}\Big|_\infty^s \;=\; \frac{y^s}{|cz+d|^{2s}}$$

Again, in principle the above computation is unnecessary, but it is informative to see the details of the reversion to a classical form.

5. *Application: constant term of* GL_2 *Eisenstein series*

An immediate use of the localized rewrite of the Eisenstein series is computation of the constant term presenting each Bruhat cell's contribution as an *Euler product*, by *unwinding* the integral defining the constant term.

[5.1] Transition for the constant term In these simplest situations, the *constant term* of any kind of modular/automorphic form is the zero-th Fourier component, separating variables in the $x+iy$ coordinates: in most elementary, though far from optimally explanatory, terms,

$$c_P f(iy) \;=\; (\text{constant term of } f)(iy) \;=\; \int_0^1 f(x+iy)\,dx$$

Since $f(x+iy)$ is periodic in x, we obtain the same outcome integrating over *any* interval $[a, a+1]$ in place of $[0,1]$. In fact, the integral is over the *quotient* $\mathbb{R}/\mathbb{Z}$

$$c_P f(iy) \;=\; \int_{\mathbb{R}/\mathbb{Z}} f(x+iy)\,dx$$

Further, the integral can be written as an integral over a quotient of a *subgroup* of G_∞, namely, with $N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$,

$$c_P f(iy) \;=\; \int_{N_{\mathbb{Z}}\backslash N_{\mathbb{R}}} f(n\cdot iy)\,dn$$

where the Haar measure on $N_{\mathbb{R}}$ is really just the usual measure on $\mathbb{R}$.

[5.1.1] Claim: $\mathbb{R} + \mathbb{Q}^\Delta + \widehat{\mathbb{Z}} = \mathbb{A}$.

Proof: An adele x fails to be in $\mathbb{Z}_v$ only for v in a finite set S of finite places $v = p$. At such v, we can write

$$x_v \;=\; \frac{a_{-\ell}}{p^\ell} + \ldots + \frac{a_{-1}}{p} + a_o + a_1 p^1 + \ldots \qquad\qquad (\text{with } a_i \in \mathbb{Z})$$

The truncated sum

$$y_v \;=\; \frac{a_{-\ell}}{p^\ell} + \ldots + \frac{a_{-1}}{p} + a_o$$

is a rational number, and is v'-integral at all other finite v'. Thus,

$$y \;=\; \sum_{v\in S} y_v$$

is a rational number, and $x - y$ is everywhere locally integral, where $y \in \mathbb{Q}^\Delta$. That is, $x - y \in \mathbb{R} + \widehat{\mathbb{Z}}$. ///

[5.1.2] Remark: The point of the corollary is *not* to get from $\mathbb{A}/\mathbb{Q}$ back to $\mathbb{R}/\mathbb{Z}$, but to get from the classical quotient $\mathbb{R}/\mathbb{Z}$ to $\mathbb{A}/\mathbb{Q}$.

[5.1.3] Remark: In fact, *(additive approximation)* asserts that $\mathbb{R} + \mathbb{Q}^\Delta$ is *dense* in $\mathbb{A}$, where $\mathbb{Q}^\Delta$ is the *diagonal* copy. Equivalently, the diagonal copy of $\mathbb{Q}$ in the *finite* adeles $\mathbb{A}_{\text{fin}}$ is dense.

Thus, $\mathbb{A}/\mathbb{Q}$ has representatives in $\mathbb{R} + \widehat{\mathbb{Z}}$. Two elements $r, r' \in \mathbb{R}$ have the same image in $\mathbb{A}/\mathbb{Q}$ if and only if $r - r' \in \mathbb{Q} \cap (\mathbb{R} + \widehat{\mathbb{Z}})$. The latter intersection is the diagonal copy of $\mathbb{Z}$, since a rational number that is in $\mathbb{Z}_v$ for all $v < \infty$ is in $\mathbb{Z}$. Thus, $\mathbb{R}/\mathbb{Z}$ injects to $\mathbb{A}/\mathbb{Q}$.

Applying this to the coordinate in the subgroup N of $G_{\mathbb{A}} = GL_2(\mathbb{A})$,

$$N_{\mathbb{Z}} \cdot \Big(N_{\mathbb{R}} \cdot (N_{\mathbb{A}} \cap K_{\text{fin}}) \Big) \;=\; N_{\mathbb{A}}$$

and

$$N_{\mathbb{Z}} \backslash \big(N_{\mathbb{R}} \cdot (N_{\mathbb{A}} \cap K_{\text{fin}}) \big) \;=\; N_{\mathbb{Q}} \backslash N_{\mathbb{A}}$$

For right K_{fin}-invariant f on the adele group, for $n_\infty \in N_\infty$ and $n_o \in N_{\text{fin}}$,

$$f(n_\infty \cdot g_\infty) \;=\; f(n_\infty \cdot g_\infty \cdot n_o) \;=\; f(n_\infty \cdot n_o \cdot g_\infty)$$

because G_∞ and G_{fin} *commute* as subgroups of $G_{\mathbb{A}}$. Thus, evaluating the constant term on g_∞,

$$c_P f(g_\infty) \;=\; \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} f(n_\infty \cdot g_\infty)\, dn_\infty \;=\; \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} \int_{N_{\mathbb{A}} \cap K_{\text{fin}}} f(n_\infty \cdot g_\infty \cdot n_o)\, dn_o\, dn_\infty$$

$$= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} \int_{N_{\mathbb{A}} \cap K_{\text{fin}}} f(n_\infty \cdot n_o \cdot g_\infty)\, dn_o\, dn_\infty \;=\; \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} f(n \cdot g_\infty)\, dn$$

[5.2] Unwinding

With

$$E_s(g_\infty) \;=\; \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma \cdot g_\infty)$$

the *constant term* of E_s along P is the adelic integral

$$c_P E_s(g) \;=\; \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s(ng)\, dn$$

Parametrizing $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$ via the Bruhat decomposition makes computation nearly trivial:

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s(ng)\, dn \;=\; \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma n g)\, dn \;=\; \sum_{w \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}} / N_{\mathbb{Q}}} \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in P_{\mathbb{Q}} \backslash P_{\mathbb{Q}} w N_{\mathbb{Q}}} \varphi(\gamma n g)\, dn$$

By the Bruhat decomposition, $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}} / N_{\mathbb{Q}}$ has exactly two representatives, $1, w$, and the constant term becomes, upon *unwinding* the second sum-and-integral,

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng)\, dn \;+\; \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in N_{\mathbb{Q}}} \varphi(w \gamma n g)\, dn \;=\; \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng)\, dn \;+\; \int_{N_{\mathbb{A}}} \varphi(wng)\, dn$$

Because φ is left $N_{\mathbb{A}}$-invariant, the first of the two summands is

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng)\, dn \;=\; \varphi(g) \cdot \text{vol}\,(N_{\mathbb{Q}} \backslash N_{\mathbb{A}}) \qquad \text{(the small Bruhat cell contribution)}$$

Since the integral in the second summand unwound, it factors over primes

$$\int_{N_{\mathbb{A}}} \varphi(wng)\, dn \;=\; \prod_{v \le \infty} \int_{N_v} \varphi_v(wng_v)\, dn$$

[5.3] Essential features of p-adic integrals

We do not need many formulaic details about integrals on $\mathbb{Q}_p$, since we will only consider $\mathbb{Z}_p^\times$-invariant integrands $f(x)$, that is, $f(\eta \cdot x) = f(x)$ for all $\eta \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Q}_p$.

It is reasonable to normalize the additive Haar measure on $\mathbb{Q}_p$ so that the compact, open subgroup $\mathbb{Z}_p$ has total measure 1.

The p^n distinct compact, open subgroups $a + p^n\mathbb{Z}_p \subset \mathbb{Z}_p$ are translates of each other, and mutually disjoint, so the total measure of $a + p^n\mathbb{Z}_p$ is p^{-n} for $a \in \mathbb{Z}_p$. By translation-invariance, the total measure of $a + p^n\mathbb{Z}_p$ is p^{-n} for *any* $a \in \mathbb{Q}_p$.

For $\eta \in \mathbb{Z}_p^\times$,

$$\eta \cdot (a + p^n\mathbb{Z}_p) \;=\; \eta a + p^n\mathbb{Z}_p$$

Thus, multiplication by units preserves Haar measure. [7] The expression

$$\mathbb{Z}_p^\times \;=\; \mathbb{Z}_p - p\mathbb{Z}_p$$

shows that the measure of $\mathbb{Z}_p^\times$ is $1 - \frac{1}{p}$. Similarly, the measure of $p^n\mathbb{Z}_p^\times$ is $(1 - \frac{1}{p}p^{-n}$. Thus, with continuous $\mathbb{Z}_p^\times$-invariant f,

$$\int_{\mathbb{Z}_p} f(x)\,dx \;=\; \sum_{\ell=0}^\infty \int_{p^\ell\mathbb{Z}_p^\times} f(x)\,dx \;=\; \sum_{\ell=0}^\infty f(p^\ell)\cdot\int_{p^\ell\mathbb{Z}_p^\times} 1\,dx \;=\; \sum_{\ell=0}^\infty f(p^\ell)\cdot(1-\frac{1}{p})\,p^{-\ell}$$

[5.4] Evaluation of local factors: non-archimedean case

For $g \in G_\infty$, so that $g_v = 1$, the finite-prime local factors in the Euler product for the big Bruhat cell are readily evaluated, as follows. Above, we computed

$$\varphi_v(w\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}) \;=\; \begin{cases} 1 & (\text{for } |t|_v \le 1)\\ |t|_v^{-2s} & (\text{for } |t|_v > 1)\end{cases}$$

With the v-adic factor corresponding to prime p, the v-adic local factor is

$$\int_{|t|_v\le 1} 1\,dt \;+\; \int_{|t|_v>1} |t|_v^{-2s}\,dt \;=\; 1 + \sum_{\ell=1}^\infty |p^{-\ell}|_v^{-2s}\cdot\int_{p^{-\ell}\mathbb{Z}_p^\times} 1\,dt \;=\; 1 + \sum_{\ell=1}^\infty (p^\ell)^{-2s}\cdot p^{\ell-1}(p-1)$$

$$=\; 1 + \big(1-\frac{1}{p}\big)\frac{p^{1-2s}}{1-p^{1-2s}} \;=\; \frac{1-p^{1-2s}+p^{1-2s}-p^{-2s}}{1-p^{1-2s}} \;=\; \frac{1-p^{-2s}}{1-p^{1-2s}} \;=\; \frac{\zeta_v(2s-1)}{\zeta_v(2s)}$$

where $\zeta_v(s)$ is the v^{th} Euler factor of the zeta function. Thus, the finite-prime part of the big-cell summand is $\zeta(2s-1)/\zeta(2s)$.

[5.5] Evaluation of local factors: archimedean case

The archimedean factor of the big-cell summand of the constant term is

$$\int_{\mathbb{R}} \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s\,dt \;=\; \int_{\mathbb{R}} \frac{y^s}{\big((x+t)^2+y^2\big)^s}\,dt \;=\; y^s\cdot\int_{\mathbb{R}} \frac{1}{\big(t^2+y^2\big)^s}\,dt \;=\; y^{1+s}\cdot\int_{\mathbb{R}} \frac{1}{\big((ty)^2+y^2\big)^s}\,dt$$

$$=\; y^{1-s}\cdot\int_{\mathbb{R}} \frac{1}{(t^2+1)^s}\,dt \;=\; \frac{y^{1-s}}{\Gamma(s)}\cdot\int_{\mathbb{R}}\int_0^\infty e^{u(1+t^2)}\,u^s\,\frac{du}{u}\,dt \;=\; \frac{y^{1-s}}{\Gamma(s)}\cdot\int_{\mathbb{R}}\int_0^\infty e^{u+t^2}\,u^{s-\frac{1}{2}}\,\frac{du}{u}\,dt$$

$$=\; \frac{y^{1-s}\,\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \;=\; y^{1-s}\cdot\frac{\pi^{-(s-\frac{1}{2})}\,\Gamma(s-\frac{1}{2})}{\pi^{-s}\,\Gamma(s)} \;=\; y^{1-s}\cdot\frac{\zeta_\infty(2s-1)}{\zeta_\infty(2s)} \qquad (\text{with } \zeta_\infty(s) = \pi^{-s/2}\,\Gamma(s/2))$$

---

[7] Quite generally, a *compact* group of automorphisms of a topological group must preserve the Haar measure on the latter.

[5.6] Conclusion of constant-term computations Thus, with $\xi(s)$ the completed zeta function $\xi(s) = \zeta_\infty(s) \cdot \zeta(s)$, the constant term of E_s is

$$c_P E_s(x+iy) \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}$$

The present point is that rewriting the Eisenstein series as an automorphization of a *product of local data* makes the computation of the constant term far more natural, and more genuinely representative of the corresponding computation for larger groups.

---

6. *Application: Hecke operators on GL_2 Eisenstein series*

The rewritten Eisenstein series will show that the Hecke operators are not *global* things, but are *local*, just acting on the local components φ_v. Indeed, the local components φ_v are eigenfunctions for the *local* version of Hecke operators, with eigenvalues depending on the parameter s.

[6.1] Classical description of Hecke operators The p^{th} Hecke operator T_p on weight-0 automorphic forms f for $\Gamma = GL_2(\mathbb{Z})$ is

$$T_p f(z) \;=\; \sum_{\gamma \in \Gamma\backslash\Theta_p} f(\gamma \cdot z) \qquad\qquad (\text{where } \Theta_p = \text{integer matrices with } \det = p)$$

with action [8] by linear fractional transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \dfrac{az+b}{cz+d}$.

[6.2] Hecke operators on rewritten Eisenstein series Directly computing, with $g_\infty \in G_\infty$,

$$T_p E_s(g_\infty) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} E_s(j_\infty(\delta) \cdot g_\infty) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi\Big(\gamma \cdot j_\infty(\delta) \cdot g_\infty\Big)$$

Let $j_o = \prod_{v<\infty} j_v$. Replace γ by $\gamma \cdot \delta^{-1}$ in $G_{\mathbb{Q}}$, to obtain

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi\Big(\gamma \cdot j_o(\delta^{-1}) \cdot g_\infty\Big)$$

The finite-prime factor $j_o(\delta^{-1})$ commutes with the archimedean-prime factor g_∞, so this is

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi\Big(\gamma \cdot g_\infty \cdot j_o(\delta^{-1})\Big) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi_\infty(j_\infty(\gamma) \cdot g_\infty) \cdot \prod_{v<\infty} \varphi_v(j_v(\gamma \cdot \delta^{-1}))$$

At all finite places v' but $v \sim p$, δ^{-1} is in the local maximal compact $K_{v'} = GL_{v'}(\mathbb{Z}_{v'})$, so $\varphi_{v'}(\gamma \cdot \delta^{-1}) = \varphi_{v'}(\gamma)$ for $v' \neq v$. Thus, suppressing j_v in the notation,

$$T_p E_s(g_\infty) \;=\; \sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi(\gamma \cdot g_\infty) \cdot \frac{\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(\gamma \cdot \delta^{-1})}{\varphi_v(\gamma)}$$

---

[8] That is, the weight-0 situation allows us to avoid worry over what to do with the *determinant* in GL_2. In the classical holomorphic case, the so-called *slash operator* is a normalization that accommodates this, usually without explanation or motivation.

Thus, the p^{th} Hecke operator's effect is *local* at $v \sim p$. Further, this situation correctly suggests that we should hope that φ_v is an *eigenfunction* for the effect of T_p, with eigenvalue depending on the complex parameter s.

[6.3] Hecke operators as integral operators Continue to let v correspond to prime p. The function φ_v on $G_v = GL_2(\mathbb{Q}_v)$ is left P_v-equivariant by χ_v, and right K_v-invariant. Each of the right translates $g \to \varphi_v(g \cdot \delta^{-1})$ with $g \in G_v$ retains the left P_v, χ_v-equivariance, but cannot be expected to retain right K_v-invariance.

Nevertheless, we claim that the sum over $\delta \in \Gamma\backslash\Theta_p$ recovers the right K_v-invariance. That is, apparently, $\Gamma\backslash\Theta_p$, or its image projected to G_v, is stable under right multiplication by K_v. As it stands, this doesn't make sense, since Θ_p itself (projected to G_v) is not literally stable under right multiplication by K_v.

As on other occasions, the necessary claim suggests itself: let $\widetilde{\Theta}_v$ be the v-adic analogue of Θ_p, namely, elements of $G_v = GL_2(\mathbb{Q}_v)$ with entries in $\mathbb{Z}_v$ and determinant of p-adic ord 1. Then we *must* claim that the natural map $\Theta_p^{-1} \to \widetilde{\Theta}_v^{-1}/K_v$ induces a *bijection*

$$\Theta_p^{-1}/\Gamma \longrightarrow \widetilde{\Theta}_v^{-1}/K_v$$

Equivalently, inverting, $\Theta_p \to K_v\backslash\widetilde{\Theta}_v$ induces a bijection

$$\Gamma\backslash\Theta_p \longrightarrow K_v\backslash\widetilde{\Theta}_v$$

Indeed, $K_v\backslash\widetilde{\Theta}_v$ has the same representatives as $\Gamma\backslash\Theta_p$, namely,[9]

$$\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \qquad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{with } b \in \mathbb{Z},\ 0 \le b < p)$$

Thus, giving K_v total measure 1, using the right K_v-invariance of φ_v,

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(g \cdot \delta^{-1}) \;=\; \sum_{h \in \widetilde{\Theta}^{-1}/K_v} \varphi_v(g \cdot h) \;=\; \int_{\widetilde{\Theta}^{-1}} \varphi_v(g \cdot h)\, dh$$

Letting η be the characteristic function of $\widetilde{\Theta}_v^{-1}$, this integral is an integral operator attached to the right translation action of G_v on functions on G_v:

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(g \cdot \delta^{-1}) \;=\; \int_{G_v} \eta(h)\, \varphi_v(g \cdot h)\, dh \;=\; (\eta \cdot \varphi_v)(g)$$

The integral expression makes right K_v-invariance clear, by changing variables in the integral:

$$\int_{G_v} \eta(h)\, \varphi_v(gk \cdot h)\, dh \;=\; \int_{G_v} \eta(k^{-1}h)\, \varphi_v(g \cdot h)\, dh \;=\; \int_{G_v} \eta(h)\, \varphi_v(g \cdot h)\, dh \qquad (\text{for } k \in K_v)$$

[9] The v-adic argument is easier than that over $\mathbb{Z}$: given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\widetilde{\Theta}_v$, if $\gcd(a,c) = 1$, then either a or c is in $\mathbb{Z}_v^\times$. Thus, left multiplication by either $\begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & -a/c \end{pmatrix}$ puts g into the form $\begin{pmatrix} 1 & b' \\ 0 & d' \end{pmatrix}$. Necessarily $\mathrm{ord}_v d' = 1$, so further left multiplication gives the form $\begin{pmatrix} 1 & b'' \\ 0 & p \end{pmatrix}$. Since $\mathbb{Z}_v/p\mathbb{Z}_v \approx \mathbb{Z}/p\mathbb{Z}$, further left multiplication by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in K_v$ gives the indicated representatives parametrized by b. When $\gcd(a,c) = p$, a similar argument gives the form $\begin{pmatrix} p & b' \\ 0 & 1 \end{pmatrix}$, and now the b' entry can be made 0.

since η is left and right K_v-invariant.

[6.4] Hecke eigenvalues Now we can prove that φ_v is an eigenvector for the localized version of T_p (with $v \sim p$), and compute its eigenvalue. The v-adic *Iwasawa decomposition* is $G_v = P_v \cdot K_v$. Thus, up to constant multiples, there is a *unique* left P_v, χ_v-equivariant, right K_v-invariant function on G_v. Thus, every such is a multiple of φ_v.

In particular, with η the characteristic function of $\widetilde{\Theta}_v^{-1}$, necessarily $\eta \cdot \varphi_v = \lambda_s \cdot \varphi_v$ for some $\lambda_s \in \mathbb{C}$. To determine λ_s, it suffices to evaluate at $g = 1$, using $\varphi_v(1) = 1$. Thus,

$$\lambda_s \;=\; \int_{G_v} \eta(h)\, \varphi_v(h)\, dh \;=\; \int_{\widetilde{\Theta}_v^{-1}/K_v} \varphi_v(h)\, dh \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(\delta^{-1})$$

$$= \sum_b \chi_v\Big(\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}^{-1} \Big) \;+\; \chi_v\Big(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Big) \;=\; p \cdot \chi_v \begin{pmatrix} 1 & * \\ 0 & p^{-1} \end{pmatrix} \;+\; \chi_v \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \; p \cdot \Big| \frac{1}{p^{-1}} \Big|_v^s + \Big| \frac{p^{-1}}{1} \Big|_v^s \;=\; p^{1-s} + p^s$$

This is the p^{th} Hecke eigenvalue of E_s:

$$T_p E_s \;=\; (p^{1-s} + p^s) \cdot E_s$$
