(*March 16, 2014*)

# *Example of characterization by mapping properties: the product topology*

*Paul Garrett* garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/12b_product_top.pdf]



Often the *internal structure* of a thing is irrelevant. Its *interactions* with *other* objects are relevant. Often details of the *construction* of a thing are irrelevant. We should *not* first *construct* the thing, and only gradually admit that it does what we had planned. Instead, we should tell what *external interactions* we demand or expect, and worry about internal details later.

Perhaps surprisingly, often characterizations in terms of maps to and from other objects of the same sort uniquely determine the thing. Even more surprisingly, often this uniqueness follows merely from the shape of the diagrams of the maps, not from any subtler features of the maps or objects.

As practice in using mapping-property characterizations, we first reconsider *familiar* objects in this light, before applying this approach to unfamiliar objects in unfamiliar circumstances.

For example, the nature of the *product topology* on products of topological spaces is illuminated by this approach. One might have (at some point) wondered *why the product topology is so coarse.* That is, on infinite products the product topology is strictly coarser than the *box topology.* [1] The answer is that the question itself is misguided, since *the product topology is what it has to be.* That is, there is no genuine *choice* in the construction. This answer itself needs explanation.

---

## 1. *Characterization and uniqueness of products*

A *product* of non-empty topological spaces $X_\alpha$ for $\alpha$ in an index set $A$ is a topological space $X$ with (*projection*) maps [2] $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from some other topological space *factors through* the $p_\alpha$ *uniquely*, in the sense that there is a unique $f : Z \to X$ such that $f_\alpha = p_\alpha \circ f$ for all $\alpha$. Pictorially, *all triangles commute* in the diagram [3]

$$
\begin{tikzcd}
Z \arrow[rr, dotted, "f"] \arrow[dr, dashed, "f_\alpha"'] \arrow[drrr, dashed, "f_\beta"'] & & X \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & & \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}
$$

**[1.0.1] Remark:** Why not some *other* diagrams, instead? This is a fair question. Indeed, reversing *all* the

---

[1] Recall that the *product topology* on the Cartesian product $\prod_{\alpha \in A} X_\alpha$ of topological spaces $X_\alpha$ is usually defined to be the topology with basis consisting of sets $\prod_{\alpha \in A} U_\alpha$ with $U_\alpha$ open in $X_\alpha$, and only finitely-many of the $U_\alpha$'s different from $X_\alpha$. The latter constraint struck me as disappointing. Dropping that condition gives the *box topology*, so named for the obvious reason.

[2] Throughout this discussion *map* means *continuous map*, although much of what is said will not use continuity.

[3] Often arrows are drawn with a solid line to show that it is unchanging or not dependent on other maps. For us, *dashed* arrows will be newly-given, and *dotted* arrows the maps whose existence is guaranteed by a universal mapping property.

arrows turns out to be reasonable, defining a *coproduct*. Other choices of arrow directions turn out to be silly. We examine these possibilities after talking about products.

**[1.0.2] Claim:** There is at most one product of given spaces $X_\alpha$, up to *unique* isomorphism.

That is, given two products, $X$ with projections $p_\alpha$ and $Y$ with projections $q_\alpha$, there is a unique isomorphism $i : X \to Y$ respecting the projections in the sense that

$$p_\alpha \;=\; q_\alpha \circ i \qquad \text{(for all } \alpha\text{)}$$

That is, there is a unique isomorphism $i : X \to Y$ such that all triangles commute in the diagram

$$\begin{tikzcd}
X \arrow[rrrr, dotted, "i", "\approx"'] \arrow[dr, dashed, "p_\alpha"'] \arrow[drrr, dashed, "p_\beta"] & & & & Y \arrow[dlll, "q_\alpha"'] \arrow[dl, "q_\beta"] \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}$$

*Proof:* First, we prove that products *have no (proper) endomorphisms*, meaning that the *identity map* is the only map $\varphi : X \to X$ making all triangles commute in the diagram

$$\begin{tikzcd}
X \arrow[rrrr, dotted, "\varphi"] \arrow[dr, dashed, "p_\alpha"'] \arrow[drrr, dashed, "p_\beta"] & & & & X \arrow[dlll, "p_\alpha"'] \arrow[dl, "p_\beta"] \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}$$

Indeed, using $X$ and the $p_\alpha$ in the role of $Z$ and $f_\alpha$ in the defining property of the product, we have a unique $\varphi$ making all triangles commute in

$$\begin{tikzcd}
X \arrow[drr, dashed, "\varphi"] \arrow[ddr, "p_\alpha"'] \arrow[ddrrr, "p_\beta"] & & & & \\
& & X \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & & \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}$$

The identity map certainly meets the requirement on $\varphi$, so, by uniqueness, the identity map on $X$ is the *only* such map $\varphi$.

Next, let $Y$ be another product, with projections $q_\alpha$. Letting $Y$ take the role of $Z$ and $q_\alpha$ the role of $f_\alpha$ in the defining property of the product $X$ (with its $p_\alpha$'s), we have a *unique* $q : Y \to X$ such that all triangles commute in

$$\begin{tikzcd}
 & Y \arrow[dr, dotted, "q"] \arrow[dd, dashed, "q_\alpha"'] \arrow[ddrr, dashed, "q_\beta"] & & & \\
 & & X \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & & \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}$$

Reversing the roles of $X$ and $Y$ (and their projections), we also have a *unique* $p : X \to Y$ such that (this time in symbols rather than the picture) $p_\alpha = q_\alpha \circ p$ for all $\alpha$.

Then $p \circ q : Y \to Y$ is an endomorphism of $Y$ respecting the projections $q_\alpha$, since

$$q_\alpha \circ (p \circ q) \;=\; (q_\alpha \circ p) \circ q \;=\; p_\alpha \circ q \;=\; q_\alpha$$

Thus, $p \circ q$ must be the identity on $Y$. Similarly, $q \circ p$ is the identity on $X$. Thus, $p$ and $q$ are mutual inverses, so both are isomorphisms. *Uniqueness* was obtained along the way. ///

**[1.0.3] Remark:** No features of topological spaces nor of continuous maps were used in this proof. Instead, the quantification over other topological spaces $Z$ and maps $f_\alpha$ indirectly described events. Thus, a similar result holds for any prescribed collection of things with prescribed maps between them that allow an *associative composition*, etc. In particular, for our present purposes we have shown that the product is unique up to *continuous* isomorphism, not just *set* isomorphism.

**[1.0.4] Remark:** *Existence* of a product will be proven by a *construction*.

**[1.0.5] Remark:** Since there is *at most* one product, the issue of construction is less ambiguous and simpler than if there were several possibilities. Further, the mapping properties give hints about a construction.

---

# 2. *Construction of products of sets*

Before addressing the *topology* on the product, we first construct it as a *set*. Of course, we expect that it is the usual Cartesian product, but it is interesting to see that this follows from the mapping properties, rather than unenlighteningly *verifying* that the Cartesian product fits (which we do at the end).

The uniqueness proof just given applies immediately to *sets without topologies*, as well. That is, the same diagrams with objects sets and maps arbitrary set maps, *define* a product $X$ with projections $p_\alpha$) of non-empty *sets* $X_\alpha$, and the same argument proves that there is at most one such thing up to unique isomorphism. We anticipate that the product is the usual Cartesian product, and the projections are the usual things, but we want to see how this is *discovered*.

To investigate properties of a product $X$ of non-empty sets $X_\alpha$ (with projections $p_\alpha$), consider various sets $Z$ and maps $f_\alpha : Z \to X_\alpha$ to see what we learn about $X$ as by-products. In this austere setting, there are not very many choices, but, on the other hand, there are not many *wrong* choices.

For example, given $x \in X$, we have the collection of all projections' values $p_\alpha(x)$, with unclear relations (or none at all) among these values. For example, to see whether things get mashed down, and as an exercise in technique, we try to prove

**[2.0.1] Claim:** For $x \neq y$ both in $X$, there is at least one $\alpha \in A$ such that $p_\alpha(x) \neq p_\alpha(y)$.

*Proof:* Suppose that $p_\alpha(x) = p_\alpha(y)$ for all $\alpha \in A$. Let $S = \{s\}$ be a set with one element, and define $f_\alpha : S \to X_\alpha$ by

$$f_\alpha(s) \;=\; p_\alpha(x) \qquad (= p_\alpha(y), \text{ also})$$

By definition, there is unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$ for all $\alpha$. Both $f$ defined by $f(s) = x$ and also $f$ defined by $f(s) = y$ have this property. By uniqueness of $f$, we have $x = f(s) = y$. ///

From the other side, we can wonder whether *all* possible collections of values of projections can occur. Intuitively (and secretly knowing the answer in advance) we might doubt that there are any constraints, and we demonstrate this by exhibiting maps.

**[2.0.2] Claim:** Given choices $x_\alpha \in X_\alpha$ for all $\alpha \in A$, there is $x$ in the product such that $p_\alpha(x) = x_\alpha$ for all $\alpha$. (By the previous claim this $x$ is *unique*.)

*Proof:* Again, let $S = \{s\}$ be a set with a single element, and define $f_\alpha : S \to X_\alpha$ by $f_\alpha(s) = x_\alpha$. There is a unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$. That is,

$$x_\alpha \;=\; f_\alpha(s) \;=\; (p_\alpha \circ f)(s) \;=\; p_\alpha(f(s))$$

The element $x = f(s)$ is the desired one. ///

These two claims, *with their proofs*, suggest that the product $X$ *is exactly* the collection of all choices of families $\{f_\alpha : \alpha \in A\}$ of functions $f_\alpha : \{s\} \to X_\alpha$, and $\alpha^{th}$ projection given by

$$p_\alpha \;:\; \{f_\alpha : \alpha \in A\} \longrightarrow f_\alpha(s)$$

This is correct, and by uniqueness any other construction gives an isomorphic thing, but this can be simplified, since maps from a one-element set are entirely determined by their images. Thus, finally, we have been led back to the Cartesian product

$$X \;=\; \{\{x_\beta : \beta \in A\} \;:\; x_\beta \in X_\beta\}$$

and usual projections

$$p_\alpha(\{x_\beta : \beta \in A\}) \;=\; x_\alpha$$

Finally, we give the trivial proof of

**[2.0.3] Claim:** The Cartesian product and usual projections are a product for sets.

*Proof:* Given a family of maps $f_\alpha : Z \to X_\alpha$, define $f : Z \to X$ by

$$f(z) \;=\; \{f_\beta(z) : \beta \in A\}) \in X$$

This meets the defining condition, and is visibly the only map that will do so. ///

**[2.0.4] Remark:** The point here was that there is no alternative, up to (unique!) isomorphism, and that reasonable considerations based on the mapping-property definition lead to a construction.

**[2.0.5] Remark:** The little fact that the collection of set maps $\varphi$ from $S = \{s\}$ to a given set $Y$ is isomorphic to $Y$ (via $\varphi \to \varphi(s)$) is noteworthy in itself.

# 3. *Construction of product topologies*

At last, we examine the topology the mapping-property characterization requires on the Cartesian product $X$ of the underlying *sets* for non-empty topological spaces $X_\alpha$

It *is* possible to seemingly-describe non-existent (impossible) objects by mapping properties, with the impossibility not immediately clear. [4] Thus, one should respect the problem of *constructing* objects whose uniqueness (if they exist) is easy.

First, all the projections $p_\alpha : X \to X_\alpha$ defined as expected by

$$p_\alpha(\{x_\beta : \beta \in A\}) \;=\; x_\alpha$$

must be continuous. That is, for every open $U_\alpha \subset X_\alpha$, $p_\alpha^{-1}(U_\alpha)$ is open in $X$. Since

$$p_\alpha^{-1}(U_\alpha) \;=\; \prod_{\beta\in A} U_\beta \qquad\qquad (\text{where } U_\beta = X_\beta \text{ for } \beta \neq \alpha)$$

the topology on $X$ must contain *at least* these sets as opens, which entails that the topology include finite intersections of them, exactly sets of the form

$$\prod_{\beta\in A} U_\beta \qquad\qquad (\text{with } U_\beta \text{ open in } X_\beta, \text{ and } U_\beta = X_\beta \text{ for almost all } \beta)$$

Arbitrary unions of these finite intersections must be included in the topology, and so on. [5] [6] Thus, the product topology is *at least as fine as* the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$. [7]

From the other side, the condition concerning maps from another space $Z$ gives a constraint on how *coarse* the topology must be, as follows. Given a family of continuous maps $f_\alpha : Z \to X_\alpha$, the corresponding $f : Z \to X$ must be continuous, which requires $f^{-1}(U)$ be open in $Z$ for all opens $U$ in $X$. If there were *too many* opens $U$ in $X$ this condition could not be met. So the topology on $X$ must be *at least as coarse* as would be allowed by $f^{-1}(U)$ being open in $Z$ for all $f : Z \to X$. Yes, this is a less tangible constraint, since it is hard to visualize the quantification over all $f : Z \to X$.

It is reasonable to *hope* that the explicit topology on $X$ generated by the inverse images $p_\alpha^{-1}(U)$ can be *proven* to be sufficiently coarse to meet the second condition.

To prove continuity of $f : Z \to X$, thus proving that the topology on $X$ is sufficiently coarse, it suffices to prove that $f^{-1}(U)$ is open for all opens $U$ *in a sub-basis*. Happily, by $p_\alpha \circ f = f_\alpha$,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) \;=\; f_\alpha^{-1}(U_\alpha)$$

---

[4] A slightly esoteric but important example of a natural universal object *which does not exist* is a *tensor product* (as topological vector space) of two infinite-dimensional Hilbert spaces. Roughly, in contrast to happier situations in which constraints from two sides allow exactly a single solution, in this example the two constraints exclude *all* possibilities.

[5] In fact, further finite intersections or arbitrary unions produce no new sets, so all opens are already expressible as unions of finite intersections of the sets of the form $p_\alpha^{-1}(U_\alpha)$.

[6] We could say that the product topology (assuming that it exists) is *generated by* these sets $p^{-1}(U_\alpha)$, but it is more usual (and less explanatory) to say that these sets are a *sub-basis* for the topology. Recall that a *basis* for a topology is a set of opens such that *any* open set is a union of some of the given opens. A *sub-basis* for a topology is a set of opens so that any open is a union of *finite intersections* of the given opens.

[7] Recall that one topology on a set $X$ is *finer* than a second topology on the same set when it contains all the opens of the second topology, and possibly more. The second topology is *coarser* than the first.

which is open by the continuity of $f_\alpha$. So $f$ is continuous, which is to say that the topology on $X$ is coarse enough (not too fine), so succeeds in being suitable for a product. ///

**[3.0.1] Remark:** Thus, the product topology must be *fine enough* so that inverse images $p_\alpha(U_\alpha)$ in $X$ of projections $p_\alpha : X \to X_\alpha$ are open, and *coarse enough* so that inverse images $f^{-1}(U)$ in $Z$ of induced maps $f : Z \to X$ are open.

**[3.0.2] Remark:** The *main* issue here is *existence* of any topology at all that will work as a product. By contrast, the *uniqueness* of products (if they exist) was proven earlier, in a standard (even clichéd) mapping-property fashion. That is, the uniqueness *up to unique isomorphism* asserts that if $X$ and $Y$ were two products, they must be (uniquely) homeomorphic.

**[3.0.3] Remark:** To repeat: the question of *existence* of a product topology can be viewed as being the question of whether or not the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$ might accidentally be *too fine* for all induced maps $Z \to X$ to be continuous. That is, the continuity of the projections and the continuity of the induced maps $Z \to X$ are *opposing* constraints. The for-general-reasons uniqueness tells us *a priori* that there is *at most one* simultaneous solution, so the question is whether or not there is *any*. In this case, it turns out that these conflicting constraints *do* allow a common solution. Still, it does happen in other circumstances that two opposing conditions allow *no* simultaneous satisfaction.

---

# 4. *Why not something else, instead?*

For context, we should ask what happens if some or all of the arrows in the diagram defining the product are *reversed*. On the face of it, this might seem an idle question, but still deserves an answer. The fact that products are characterized by mapping properties was *not* the final goal. This was an exercise illustrating a larger point.

As mentioned earlier, when *all* the arrows are reversed, the object defined is a *coproduct*. That is, given a family $\{X_\alpha : \alpha \in A\}$, a *coproduct* of the $X_\alpha$ is an $X$ with maps [8] $i_\alpha : X_\alpha \to X$ such that, for all $Z$ and maps $f_\alpha : X_\alpha \to Z$, there is a unique $f : X \to Z$ such that every $f_\alpha$ factors through $f$, that is, such that $f_\alpha = f \circ i_\alpha$ for all $\alpha$. In diagrams this asserts that there exists a unique $f : X \to Z$ such that all triangles commute in

$$\begin{tikzcd}
& Z & & & \\
& & & X \arrow[ull, dashed, "f"'] & \\
\cdots & X_\alpha \arrow[uu, "f_\alpha"] \arrow[urr, "i_\alpha"'] & \cdots & X_\beta \arrow[uull, "f_\beta"] \arrow[u, "i_\beta"'] & \cdots
\end{tikzcd}$$

**[4.0.1] Remark:** We are scrupulously avoiding saying what kind of objects $X_\alpha$ are involved, and what kind of maps are involved. The objects might be sets and the maps set maps, or the objects topological spaces and the maps continuous, or the objects abelian groups and the maps group homomorphisms, and so on.

**[4.0.2] Claim:** For sets and set maps, coproducts are *disjoint unions*. That is, given sets $X_\alpha$ for $\alpha \in A$, the

[8] These maps $i_\alpha$ would usually be called *inclusions* or *imbeddings*, but use of these terms might prejudice the reader. They will indeed *turn out* to be provably injections.

coproduct of the $X_\alpha$ is the *disjoint* union [9] $X = \bigsqcup_{\alpha \in A} X_\alpha$ with literal inclusions $i_\alpha : X_\alpha \to X$.

We leave the discussion or proof of this to the reader. In any case, disjoint unions are of some interest, though perhaps not as much as products. For contrast, also consider

**[4.0.3] Claim:** For abelian groups and group homomorphisms, coproducts are *direct sums*. That is, given abelian groups $X_\alpha$ for $\alpha \in A$, the coproduct of the $X_\alpha$ is the direct sum

$$X \;=\; \bigoplus_{\alpha \in A} X_\alpha \;=\; \{\{x_\beta : \beta \in A\} \;:\; \text{all but finitely-many } x_\beta\text{'s are } 0 \in X_\alpha\}$$

with inclusions $i_\alpha : X_\alpha \to X$ by

$$i_\alpha(x_\alpha) \;=\; \{y_\beta : y_\beta = 0 \in X_\beta \text{ for } \beta \neq \alpha, \;\; \text{and } y_\alpha = x_\alpha\}$$

We also leave the proof or discussion of this second claim to the interested reader.

**[4.0.4] Remark:** The difference between these two coproducts is in contrast to the analogues for *products*. That is, the underlying *set* of a product of groups *is* the set-product of the underlying sets. By contrast, the underlying set of a coproduct of abelian groups is *not* the set-coproduct of the underlying sets.

*Further variations on the arrows turn out not to be interesting*:

For example, if we take the characterizing diagram for a product of $X_\alpha$'s, but only reverse the dotted arrow, we require that there is $X$ and maps $p_\alpha : X \to X_\alpha$ such that for all families $f_\alpha : Z \to X_\alpha$ there is a unique $f : X \to Z$ (this is the arrow reversal) such that $p_\alpha = f_\alpha \circ f$. This family of conditions is the only possible thing to require of a similar sort, given the directions of the arrows. That is, we require the commutativity of all triangles in diagrams

$$\begin{tikzcd}
Z \arrow[ddr, "f_\alpha"'] \arrow[ddrrr, "f_\beta"] & & & & \\
& & X \arrow[ull, dashed, "f"'] \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & & \\
\dots & X_\alpha & \dots & X_\beta & \dots
\end{tikzcd}$$

**[4.0.5] Claim:** For non-empty sets $X_\alpha$, if any one has at least two elements, then there is no such set $X$ (with set maps $p_\alpha$).

*Proof:* As in the discussion of products of sets, we may take $Z = \{z\}$, a set with a single element $z$, and for a selection of elements $x_\alpha \in X_\alpha$ for all $\alpha \in A$, let $f_\alpha : Z \to X_\alpha$ be $f_\alpha(z) = x_\alpha$. For non-empty set $X$, this implies that

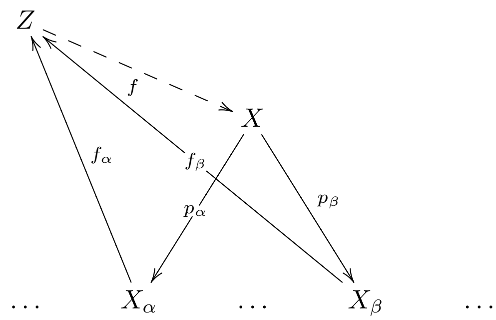
$$p_\alpha(X) \;=\; (f_\alpha \circ f)(X) \subset f_\alpha(Z) \;=\; \{x_\alpha\}$$

for all $\alpha$. But when $X_\alpha$ has two or more elements, this is impossible, since $p_\alpha$ is surjective. ///

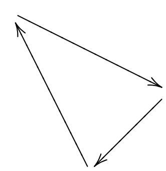
---

[9] The notion of *disjoint union* is that, when by chance the sets $X_\alpha$ as they naturally occur have non-trivial intersections, we add artifical labels or play other tricks to make *isomorphic* sets which are now pairwise *disjoint*, and *then* take the union.

As another uninteresting outcome, take the characterizing diagram for a product of $X_\alpha$'s, and reverse the $f_\alpha$'s, that is, consider diagrams




There is a different failure, namely that all the patterns of arrows in triangles are of the *cyclic* form



which suggests no obvious commutativity condition. Other choices of conditions are left to the bored or whimsical reader.

In summary, mapping properties characterize *products* and *coproducts*. This characterization depicts the interactions of these objects with other objects, which is of greater import than their internal construction.

---

## *Bibliography*

E. Noether had promoted *abstract algebra* for at least two decades prior to the overt introduction of what is nowadays recognizable as *category theory* in [Eilenberg-MacLane 1942] and [Eilenberg-MacLane 1945], motivated by issues in algebraic topology and group cohomology. In the intervening years, the pervasiveness of categorical notions has become well-known. [MacLane 1971] gives a relatively early account, while [Lawvere-Rosebrugh 2003] and [Lawvere-Schanuel 1997] make a good case for the utility of category theory at all levels of mathematics. This is a very small sample of relevant literature.

[Eilenberg-MacLane 1942] S. Eilenberg, S. MacLane, *Natural isomorphism in group theory*, Proc. Nat. Acad. Sci. USA **28** (1942, 537-543.

[Eilenberg-MacLane 1945] S. Eilenberg, S. MacLane, *General theory of natural equivalences*, Trans. AMS **58** (1945), 231-294.

[Lawvere-Rosebrugh 2003] F. Lawvere, R. Rosebrugh, *Sets for mathematics*, Cambridge Univ. Press, 2003.

[Lawvere-Schanuel 1997] F. Lawvere, S. Schanuel, *Conceptual mathematics: a first introduction to categories*, Cambridge Univ. Press, 1997.

[MacLane 1971] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Math., Springer-Verlag, 1971.

---