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Modern analysis, cuspforms

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

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We prove that there is an orthonormal basis for square-integrable (waveform) *cuspforms* on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$.

First, we reconsider the well-known fact that the Hilbert space $L^2(\mathbb{T})$ on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ has an *orthogonal Hilbert-space basis* of exponentials e^{inx} with $n \in \mathbb{Z}$, using ideas relevant to situations lacking analogues of Dirichlet or Fejer kernels. These exponentials are *eigenfunctions* for the Laplacian $\Delta = d^2/dx^2$, so it would suffice to show that $L^2(\mathbb{T})$ has an orthogonal basis of eigenfunctions for Δ. Two technical issues must be overcome: that Δ does not map $L^2(\mathbb{T})$ to itself, and that there is no guarantee that infinite-dimensional Hilbert spaces have Hilbert-space bases of eigenfunctions for a given linear operator. [1]

About 1929, Stone, von Neumann, and Krein made sense of *unbounded* operators on Hilbert spaces, such as differential operators like Δ. In 1934, Friedrichs characterized especially good self-adjoint extensions of operators like Δ.

Unfortunately, unlike finite-dimensional self-adjoint operators, self-adjoint operators on Hilbert spaces generally do *not* give orthogonal Hilbert-space bases of eigenvectors. Fortunately, an important special class *does* have a spectral theory closely imitating finite-dimensional spectral theory: the *compact self-adjoint* operators, which *always* give an orthogonal Hilbert-space basis of eigenvectors. [2] Unbounded operators such as Δ are never *continuous*, much less *compact*, but in happy circumstances they may have *compact resolvent* $(1-\Delta)^{-1}$. Rellich's lemma and Friedrichs' construction give this compactness in precise terms and recover a good spectral theory.

The compactness of the resolvent $(1-\widetilde{\Delta})^{-1}$ of the Friedrichs self-adjoint extension $\widetilde{\Delta}$ of Δ on $L^2(\mathbb{T})$ is a *Rellich lemma*, whose proof we'll give. This will imply that $L^2(\mathbb{T})$ has an orthonormal basis of eigenfunctions for $\widetilde{\Delta}$. We also must check that $\widetilde{\Delta}$ has no further eigenfunctions than those of Δ. Here, that is demonstrably the case, by Sobolev imbedding/regularity. Thus, solving the differential equation $u'' = \lambda\cdot u$ on $\mathbb{R}$ for suitably periodic solutions u produces a Hilbert-space basis for $L^2(\mathbb{T})$.

The next example is of $L^2(\mathbb{R})$. However, here Δ does *not* have compact resolvent: the analogous decomposition, by Fourier transform and Fourier inversion, is *not* a decomposition into $L^2(\mathbb{R})$ eigenfunctions. There are no eigenfunctions of Δ in $L^2(\mathbb{R})$. To arrange a decomposition into eigenfunctions, we *perturb* the Laplace operator to a *Schrödinger* operator $S = -\Delta + x^2$, where the x^2 acts as multiplication. A

[1] Well-behaved operators on infinite-dimensional spaces may fail to have eigenvectors. For example, on $L^2[a,b]$, the multiplication operator $Tf(x) = x\cdot f(x)$ is continuous, with the symmetry $\langle Tf, g\rangle = \langle f, Tg\rangle$, but has no eigenvectors. That is, the *spectrum* of operators on infinite-dimensional Hilbert spaces typically includes more than *eigenvalues*.

[2] As we discuss further, a continuous linear operator on a Hilbert space is *compact* when it is a limit of *finite-rank* operators with respect to the usual *operator norm* $|T|_{\text{op}} = \sup_{0\neq x\in V} |Tx|_V$. *Finite-rank* continuous operators are those with finite-dimensional image. *Self-adjointness* for a *continuous* operator T is the expected $\langle Tx, y\rangle = \langle x, Ty\rangle$. The spectral theorem for self-adjoint compact operators is proven below.

Rellich lemma shows that the Friedrichs extension $\widetilde{S}$ *does* have compact self-adjoint resolvent $\widetilde{S}^{-1}$, giving an orthogonal basis of eigenfunctions. A similar Sobolev imbedding/regularity lemma proves that all eigenvectors of $\widetilde{S}$ are eigenvectors for S itself. Here, by chance, the eigenfunctions can be identified explicitly.

Finally, we consider Hilbert spaces somewhat larger than the space of cuspforms in $L^2_a(\Gamma\backslash\mathfrak{H})$ with $\Gamma = SL_2(\mathbb{Z})$, namely, those with *constant term* vanishing above height $y = a$:

$$L^2_a(\Gamma\backslash\mathfrak{H}) \;=\; \Big\{f \in L^2(\Gamma\backslash\mathfrak{H}) : c_P f(z) = 0 \text{ for } \mathrm{Im}(z) \geq a\Big\} \qquad (\text{constant term } c_P f(z) = \textstyle\int_0^1 f(x+iy)\,dx)$$

The Friedrichs extension $\widetilde{\Delta}_a$ of the restriction Δ_a of the $SL_2(\mathbb{R})$-invariant Laplacian $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})$ to $L^2_a(\Gamma\backslash\mathfrak{H})$ will be shown to have *compact resolvent*. Thus, $L^2_a(\Gamma\backslash\mathfrak{H})$ has an orthonormal basis of eigenfunctions for $\widetilde{\Delta}_a$. For $a > 0$, the space $L^2_a(\Gamma\backslash\mathfrak{H})$ does include more than cuspforms, and there are $\widetilde{\Delta}_a$-eigenfunctions which are *not* Δ_a-eigenfunctions. However, the *cuspform* eigenfunctions for $\widetilde{\Delta}_a$ are provably genuine eigenfunctions for Δ itself. The *new* eigenfunctions are certain *truncated Eisenstein series.*

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1. *Unbounded symmetric operators on Hilbert spaces*

The natural differential operator $\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$ has no sensible definition as mapping all of the Hilbert space $L^2(\mathbb{R})$ to itself, whatever else we can say. At the most cautious, it certainly does map $C^\infty_c(\mathbb{R})$ to itself, and, by integration by parts,

$$\langle \Delta f, g\rangle \;=\; \langle f, \Delta g\rangle \qquad (\text{for both } f, g \in C^\infty_c(\mathbb{R}))$$

This is *symmetry* of Δ. The possibility of thinking of Δ as differentiating L^2 functions *distributionally* does not resolve the question of L^2 behavior, unfortunately. There is substantial motivation to accommodate discontinuous (*unbounded*) linear maps on Hilbert spaces.

A not-necessarily continuous, that is, not-necessarily *bounded*, linear operator T, defined on a *dense* subspace D_T of a Hilbert space V is called an *unbounded operator on* V, even though it is likely not defined on all of V. We consider only *symmetric* unbounded operators T, meaning that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for v, w in the domain D_T of T. The Laplacian is symmetric on $C^\infty_c(\mathbb{R})$.

For unbounded operators on V, the *domain* is a potentially critical part of a description: an unbounded operator T on V is a subspace D of V and a linear map $T : D \longrightarrow V$. Nevertheless, explicit naming of the domain of an unbounded operator is often suppressed, instead writing $T_1 \subset T_2$ when T_2 is an *extension* of T_1, in the sense that the domain of T_2 contains that of T_1, and the restriction of T_2 to the domain of T_1 agrees with T_1.

Unlike self-adjoint operators on finite-dimensional spaces, and unlike self-adjoint *bounded* operators on Hilbert spaces, symmetric *unbounded* operators, even when densely defined, usually need to be *extended* in order to behave similarly to self-adjoint operators in finite-dimensional and bounded operator situations.

An operator T', D' is a *sub-adjoint* to an operator T, D when

$$\langle Tv, w\rangle \;=\; \langle v, T'w\rangle \qquad (\text{for } v \in D,\, w \in D')$$

For D *dense*, for given D' there is *at most* one T' meeting the adjointness condition.

The *adjoint* T^* is the *unique maximal* element, in terms of domain, among all sub-adjoints to T. That there is a unique maximal sub-adjoint requires proof, given below.

Paraphrasing the notion of *symmetry*: an operator T is *symmetric* when $T \subset T^*$, and *self-adjoint* when $T = T^*$. These comparisons refer to the *domains* of these not-everywhere-defined operators. In the following claim and its proof, the domain of a map S on V is incorporated in a reference to its *graph*

$$\text{graph } S \;=\; \{v \oplus Sv \;:\; v \in \text{domain } S\} \;\subset\; V \oplus V$$

The direct sum $V \oplus V$ is a Hilbert space with natural inner product

$$\langle v \oplus v', w \oplus w' \rangle \;=\; \langle v, v' \rangle + \langle w, w' \rangle$$

Define an isometry U of $V \oplus V$ by

$$U \;:\; V \oplus V \;\longrightarrow\; V \oplus V \qquad \text{by} \qquad v \oplus w \;\longrightarrow\; -w \oplus v$$

[1.0.1] Claim: Given symmetric T with *dense* domain D, there is a unique *maximal* T^*, D^* among all sub-adjoints to T, D. The adjoint T^* is *closed*, in the sense that its *graph* is closed in $V \oplus V$. In fact, the adjoint is *characterized* by its graph, which is the orthogonal complement in $V \oplus V$ to an image of the graph of T, namely,

$$\text{graph } T^* \;=\; \text{orthogonal complement of } U(\text{graph } T)$$

Proof: The adjointness condition $\langle Tv, w \rangle = \langle v, T^*w \rangle$ for given $w \in V$ is an orthogonality condition

$$\langle w \oplus T^*w,\; U(v \oplus Tv) \rangle \;=\; 0 \qquad\qquad (\text{for all } v \text{ in the domain of } T)$$

Thus, the graph of *any* sub-adjoint is a subset of

$$X \;=\; U(\text{graph } T)^\perp$$

Since T is densely-defined, for given $w \in V$ there is *at most* one possible value w' such that $w \oplus w' \in X$, so this orthogonality condition determines a well-defined function T^* on a subset of V, by

$$T^*w \;=\; w' \qquad\qquad (\text{if there exists } w' \in V \text{ such that } w \oplus w' \in X)$$

The linearity of T^* is immediate. It is maximal among sub-adjoints to T because the graph of any sub-adjoint is a subset of the graph of T^*. Orthogonal complements are closed, so T^* has a closed graph. ///

[1.0.2] Corollary: For $T_1 \subset T_2$ with dense domains, $T_2^* \subset T_1^*$, and $T_1 \subset T_1^{**}$. ///

[1.0.3] Corollary: A self-adjoint operator has a closed graph. ///

[1.0.4] Remark: The closed-ness of the graph of a self-adjoint operator is essential in proving existence of *resolvents*, below.

[1.0.5] Remark: The use of the term *symmetric* in this context is potentially misleading, but standard. The notation $T = T^*$ allows an inattentive reader to forget non-trivial assumptions on the *domains* of the operators. The equality of domains of T and T^* is understandably essential for legitimate computations.

[1.0.6] Proposition: Eigenvalues for symmetric operators T, D are *real*.

Proof: Suppose $0 \neq v \in D$ and $Tv = \lambda v$. Then

$$\lambda \langle v, v \rangle \;=\; \langle \lambda v, v \rangle \;=\; \langle Tv, v \rangle \;=\; \langle v, T^*v \rangle \qquad\qquad (\text{because } v \in D \subset D^*)$$

Further, because T^* agrees with T on D,

$$\langle v, T^*v \rangle \;=\; \langle v, \lambda v \rangle \;=\; \overline{\lambda} \overline{v}, v \rangle$$

Thus, λ is real. ///

Let $R_\lambda = (T-\lambda)^{-1}$ for $\lambda \in \mathbb{C}$ when this inverse is a continuous linear operator defined on a dense subset of V.

[1.0.7] Theorem: Let T be self-adjoint and densely defined. For $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}$, the operator R_λ is *everywhere* defined on V. For T *positive*, for $\lambda \notin [0,+\infty)$, R_λ is everywhere defined on V.

Proof: For $\lambda = x+iy$ off the real line and v in the domain of T,

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + \langle (T-x)v, iyv\rangle + \langle iyv, (T-x)v\rangle + y^2|v|^2$$

$$=\; |(T-x)v|^2 - iy\langle (T-x)v, v\rangle + iy\langle v, (T-x)v\rangle + y^2|v|^2$$

The symmetry of T, and the fact that the domain of T^* contains that of T, implies that

$$\langle v, Tv\rangle \;=\; \langle T^*v, v\rangle \;=\; \langle Tv, v\rangle$$

Thus,

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2 \;\geq\; y^2|v|^2$$

Thus, for $y \neq 0$, $(T-\lambda)v \neq 0$. We must show that $(T-\lambda)D$ is the whole Hilbert space V. If

$$0 \;=\; \langle (T-\lambda)v, w\rangle \qquad\qquad (\text{for all } v \in D)$$

then the adjoint of $T-\lambda$ can be defined on w simply as $(T-\lambda)^*w = 0$, since

$$\langle (T-\lambda)v, w\rangle \;=\; 0 \;=\; \langle v, 0\rangle \qquad\qquad (\text{for all } v \in D)$$

Thus, $T^* = T$ is defined on w, and $Tw = \overline{\lambda}w$. For λ not real, this implies $w=0$. Thus, $(T-\lambda)D$ is *dense* in V.

Since T is self-adjoint, it is *closed*, so $T-\lambda$ is closed. The equality

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2$$

gives

$$|(T-\lambda)v|^2 \;\gg_y\; |v|^2$$

Thus, for fixed $y \neq 0$, the map

$$F \;:\; v \oplus (T-\lambda)v \;\longrightarrow\; (T-\lambda)v$$

respects the metrics, in the sense that

$$|(T-\lambda)v|^2 \;\leq\; |(T-\lambda)v|^2 + |v|^2 \;\ll_y\; |(T-\lambda)v|^2 \qquad\qquad (\text{for fixed } y \neq 0)$$

The graph of $T-\lambda$ is *closed*, so is a *complete* metric subspace of $V \oplus V$. Since F respects the metrics, it preserves completeness. Thus, the metric space $(T-\lambda)D$ is *complete*, so is a closed subspace of V. Since the closed subspace $(T-\lambda)D$ is dense, it is V. Thus, for $\lambda \notin \mathbb{R}$, R_λ is everywhere-defined. Its norm is bounded by $1/|\mathrm{Im}\,\lambda|$, so it is a continuous linear operator on V.

Similarly, for T *positive*, for $\mathrm{Re}(\lambda) \leq 0$,

$$|(T-\lambda)v|^2 \;=\; |Tv|^2 - \lambda\langle Tv, v\rangle - \overline{\lambda}\langle v, Tv\rangle + |\lambda|^2\cdot|v|^2 \;=\; |Tv|^2 + 2|\mathrm{Re}\lambda|\langle Tv, v\rangle + |\lambda|^2\cdot|v|^2 \;\geq\; |\lambda|^2\cdot|v|^2$$

Then the same argument proves the existence of an everywhere-defined inverse $R_\lambda = (T-\lambda)^{-1}$, with $\|R_\lambda\| \leq 1/|\lambda|$ for $\mathrm{Re}\,\lambda \leq 0$. ///

[1.0.8] Theorem: (Hilbert) For points λ, μ off the real line, or, for T *positive* and λ, μ off $[0, +\infty)$,

$$R_\lambda - R_\mu \;=\; (\lambda - \mu) R_\lambda R_\mu$$

For the operator-norm topology, $\lambda \to R_\lambda$ is *holomorphic* at such points.

Proof: Applying R_λ to

$$1_V - (T-\lambda)R_\mu \;=\; \big((T-\mu) - (T-\lambda)\big)R_\mu \;=\; (\lambda-\mu)R_\mu$$

gives

$$R_\lambda(1_V - (T-\lambda)R_\mu) \;=\; R_\lambda\big((T-\mu) - (T-\lambda)\big)R_\mu \;=\; R_\lambda(\lambda-\mu)R_\mu$$

Then

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} \;=\; R_\lambda R_\mu$$

For holomorphy, with $\lambda \to \mu$,

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2 \;=\; R_\lambda R_\mu - R_\mu^2 \;=\; (R_\lambda - R_\mu)R_\mu \;=\; (\lambda-\mu)R_\lambda R_\mu R_\mu$$

Taking operator norm, using $\|R_\lambda\| \le 1/|\mathrm{Im}\,\lambda|$ from the previous computation,

$$\Big\|\frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2\Big\| \;\le\; \frac{|\lambda - \mu|}{|\mathrm{Im}\,\lambda| \cdot |\mathrm{Im}\,\mu|^2}$$

Thus, for $\mu \notin \mathbb{R}$, as $\lambda \to \mu$, this operator norm goes to 0, demonstrating the holomorphy.

For *positive* T, the estimate $\|R_\lambda\| \le 1/|\lambda|$ for $\mathrm{Re}\,\lambda \le 0$ yields holomorphy on the negative real axis by the same argument. ///

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2. *Friedrichs self-adjoint extensions of semi-bounded operators*

[2.1] Semi-bounded operators These are more tractable than general unbounded symmetric operators. A densely-defined symmetric operator T, D is *positive* (or *non-negative*), denoted $T \ge 0$, when

$$\langle Tv, v\rangle \;\ge\; 0 \qquad\qquad (\text{for all } v \in D)$$

All the eigenvalues of a positive operator are non-negative real. Similarly, T is *negative* when $\langle Tv, v\rangle \le 0$ for all v in the (dense) domain of T. Generally, if there is a constant $c \in \mathbb{R}$ such that $\langle Tv, v\rangle \ge c \cdot \langle v, v\rangle$ (written $T \ge c$), or $\langle Tv, v\rangle \le c \cdot \langle v, v\rangle$ (written $T \le c$), say T is *semi-bounded*.

[2.1.1] Theorem: (Friedrichs) A *positive*, densely-defined, symmetric operator T, D with D dense in Hilbert space V has a positive *self-adjoint* extension $\widetilde{T}, \widetilde{D}$, characterized by

$$\langle (1+T)v, (1+\widetilde{T})^{-1}w\rangle \;=\; \langle v, w\rangle \qquad\qquad (\text{for } v \in D \text{ and } w \in V)$$

and $\langle \widetilde{T}v, v\rangle \ge 0$ for v in the domain of $\widetilde{T}$.

Proof: [3] Define a new hermitian form $\langle , \rangle_1$ and corresponding norm $\|\cdot\|_1$ by

$$\langle v, w\rangle_1 \;=\; \langle v, w\rangle + \langle Tv, w\rangle \;=\; \langle v, (1+T)w\rangle \;=\; \langle (1+T)v, w\rangle \qquad\qquad (\text{for } v, w \in D)$$

[3] We essentially follow [Riesz-Nagy 1955], pages 329-334.

The symmetry and non-negativity of T make this positive-definite hermitian on D, and $\langle v, w\rangle_1$ has sense whenever at least one of v, w is in D.

Let V_1 be the closure in V of D with respect to the metric d_1 induced by the norm $\|\cdot\|_1$ on V. We claim that V_1 naturally continuously injects to V. Indeed, for v_i a d_1-Cauchy sequence in D, v_i is Cauchy in V in the original topology, since

$$|v_i - v_j| \;\le\; |v_i - v_j|_1$$

For two sequences v_i, w_j with the same d_1-limit v, the d_1-limit of $v_i - w_i$ is 0. Thus,

$$|v_i - w_i| \;\le\; |v_i - w_i|_1 \;\longrightarrow\; 0$$

For $h \in V$ and $v \in V_1$, the functional $\lambda_h : v \to \langle v, h\rangle$ has a bound

$$|\lambda_h v| \;\le\; |v|\cdot|h| \;\le\; |v|_1\cdot|h|$$

so the norm of the functional λ_h on V_1 is at most $|h|$. By Riesz-Fischer, there is unique Bh in the Hilbert space V_1 with $|Bh|_1 \le |h|$, such that

$$\lambda_h v \;=\; \langle v, Bh\rangle_1 \qquad\qquad (\text{for } v \in V_1)$$

Thus,

$$|Bh| \;\le\; |Bh|_1 \;\le\; |h|$$

The map $B : V \to V_1$ is verifiably linear. There is an obvious *symmetry* of B:

$$\langle Bv, w\rangle \;=\; \lambda_w Bv \;=\; \langle Bv, Bw\rangle_1 \;=\; \overline{\langle Bw, Bv\rangle_1} \;=\; \overline{\lambda_v Bw} \;=\; \overline{\langle Bw, v\rangle} \;=\; \langle v, Bw\rangle \qquad\qquad (\text{for } v, w \in V)$$

Positivity of B is similar:

$$\langle v, Bv\rangle \;=\; \lambda_v Bv \;=\; \langle Bv, Bv\rangle_1 \;\ge\; \langle Bv, Bv\rangle \ge 0$$

Finally B is *injective*: if $Bw = 0$, then for all $v \in V_1$

$$0 \;=\; \langle v, 0\rangle_1 \;=\; \langle v, Bw\rangle_1 \;=\; \lambda_w v \;=\; \langle v, w\rangle$$

Since V_1 is dense in V, $w = 0$. Similarly, if $w \in V_1$ is such that $\lambda_v w = 0$ for all $v \in V$, then $0 = \lambda_w w = \langle w, w\rangle$ gives $w = 0$. Thus, $B : V \to V_1 \subset V$ is bounded, symmetric, positive, injective, with dense image. In particular, B is self-adjoint.

Thus, B has a possibly *unbounded* positive, symmetric inverse A. Since B injects V to a dense subset V_1, necessarily A *surjects* from its domain (inside V_1) to V. We claim that A is *self-adjoint*. Let $S : V \oplus V \to V \oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A \;=\; S(\text{graph } B)$$

In computing orthogonal complements $X^\perp$, clearly

$$(S\,X)^\perp \;=\; S\big(X^\perp\big)$$

From the obvious $U \circ S = -S \circ U$, compute

$$\text{graph } A^* \;=\; (U \text{ graph } A)^\perp \;=\; (U \circ S \text{ graph } B)^\perp \;=\; (-S \circ U \text{ graph } B)^\perp$$

$$=\; -S\big((U \text{ graph } B)^\perp\big) \;=\; -\,\text{graph} A \;=\; \text{graph } A$$

since the domain of B^* is the domain of B. Thus, A is self-adjoint.

We claim that for v in the domain of A, $\langle Av, v\rangle \geq \langle v, v\rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av\rangle \;=\; \langle Bw, w\rangle \;=\; \lambda_w Bw \;=\; \langle Bw, Bw\rangle_1 \;\geq\; \langle Bw, Bw\rangle \;=\; \langle v, v\rangle$$

Similarly, with $v' = Bw'$, and $v \in V_1$,

$$\langle v, Av'\rangle \;=\; \langle v, w'\rangle \;=\; \lambda_{w'} v \;=\; \langle v, Bw'\rangle_1 \;=\; \langle v, v'\rangle_1 \qquad (v \in V_1,\ v' \text{ in the domain of } A)$$

Since B maps V to V_1, the domain of A is contained in V_1. We claim that the domain of A is dense in V_1 in the d_1-topology, not merely in the coarser subspace topology from V. Indeed, for $v \in V_1$ $\langle,\rangle_1$-orthogonal to the domain of A, for v' in the domain of A, using the previous identity,

$$0 \;=\; \langle v, v'\rangle_1 \;=\; \langle v, Av'\rangle$$

Since B injects V to V_1, A surjects from its domain to V. Thus, $v = 0$.

Last, prove that A is an extension of $S = 1 + T$. On one hand, as above,

$$\langle v, Sw\rangle \;=\; \lambda_{Sw} v \;=\; \langle v, BSw\rangle_1 \qquad (\text{for } v, w \in D)$$

On the other hand, by definition of $\langle,\rangle_1$,

$$\langle v, Sw\rangle \;=\; \langle v, w\rangle_1 \qquad (\text{for } v, w \in D)$$

Thus,

$$\langle v, w - BSw\rangle_1 \;=\; 0 \qquad (\text{for all } v, w \in D)$$

Since D is d_1-dense in V_1, $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of B, so is in the domain of A, and

$$Aw \;=\; A(BSw) \;=\; Sw$$

Thus, the domain of A contains that of S and extends S. ///

[2.2] Compactness of resolvents

Friedrichs' self-adjoint extensions have extra features not always shared by the other self-adjoint extensions of a given symmetric operator. For example,

[2.2.1] Claim: When the inclusion $V_1 \to V$ is *compact*, the resolvent $(1+\widetilde{T})^{-1} : V \to V$ is compact.

Proof: The proof above actually shows that $B = (1+\widetilde{T})^{-1}$ maps $V \to V_1$ continuously even with the finer $\langle,\rangle_1$-topology on V_1: the relation

$$\langle v, Bw\rangle_1 \;=\; \langle v, w\rangle \qquad (\text{for } v \in V_1)$$

with $v = Bw$, gives

$$|Bw|_1^2 \;=\; \langle Bw, Bw\rangle_1 \;=\; \langle Bw, w\rangle \;\leq\; |Bw| \cdot |w| \;\leq\; |Bw|_1 \cdot |w|$$

The resultant $|Bw|_1 \leq |w|$ gives continuity in the finer topology. Thus, we may view $B : V \to V_1 \to V$ as the composition of this continuous map with the injection $V_1 \to V$ where V_1 has the finer topology. The composition of a continuous linear map with a compact operator is compact,[4] so compactness of $V_1 \to V$ with the finer topology on V_1 suffices to prove compactness of the resolvent. ///

[4] Use the characterization of compactness of $C : Y \to Z$ that the image of the unit ball of Y is pre-compact. Let $B : X \to Y$ be continuous and $C : Y \to Z$ compact. Replacing B by a scalar multiple does not affect the compactness-or-not of $C \circ B$, so without loss of generality the operator norm of B can be 1. Then B maps the unit ball of X to a subset of the unit ball in Y. Pre-compactness is inherited by subsets, so, as the image under C of the unit ball in Y is pre-compact, the image of the unit ball in X under $C \circ B$ is pre-compact.

3. *Examples of incommensurable self-adjoint extensions*

The differential operator $T = \frac{d^2}{dx^2}$ on $L^2[a,b]$ or $L^2(\mathbb{R})$ is a prototypical natural *unbounded operator*. It is undeniably *not continuous* in the L^2 topology: on $L^2[0,1]$ the norm of $f(x) = x^n$ is $1/\sqrt{2n+1}$, and the second derivative of x^n is $n(n-1)x^{n-2}$, so

$$\text{operator norm } \frac{d^2}{dx^2} \text{ on } L^2[0,1] \quad \geq \quad \sup_{n\geq 1} \frac{n(n-1)\cdot \frac{1}{\sqrt{2n-3}}}{\frac{1}{\sqrt{2n+1}}} \quad = \quad +\infty$$

That is, $\frac{d^2}{dx^2}$ is not a L^2-bounded operator on *polynomials* on $[0,1]$, so has no bounded *extension*[5] to $L^2[0,1]$.

As the simple example here illustrates, it is unreasonable to expect naturally-occurring positive, symmetric operators to have *unique* self-adjoint extensions.

In brief, for unbounded operators arising from differential operators, imposition of varying *boundary conditions* often gives rise to mutually incomparable self-adjoint extensions.

[3.1] Non-symmetric adjoints of symmetric operators Just below, *many different* positive, symmetric extensions of a natural, positive, symmetric, densely-defined operator T are exhibited, with *no two having a common symmetric extension.* This is not obviously possible. In that situation, the *graph-closure* $\overline{T} = T^{**}$ is *not* self-adjoint. Equivalently, T^* is *not* symmetric, proven as follows.

Suppose positive, symmetric, densely-defined T has positive, symmetric extensions A, B *admitting no common symmetric extension.* Let $\overline{A} = A^{**}, \overline{B} = B^{**}$ be the graph-closures of A, B. Friedrichs' construction $T \to \widetilde{T}$ applies to T, A, B. The inclusion-reversing property of $S \to S^*$ gives a diagram of extensions, where ascending lines indicate extensions:

$$\begin{tikzcd}
& T^* \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
A^* \arrow[d, dash] & & B^* \arrow[d, dash] \\
\widetilde{A} = \widetilde{A}^* \arrow[r, dotted, dash, "?"] \arrow[d, dash] & \widetilde{T} = \widetilde{T}^* \arrow[r, dotted, dash, "?"] \arrow[dd, dash] & \widetilde{B} = \widetilde{B}^* \arrow[d, dash] \\
\overline{A} \arrow[d, dash] \arrow[dr, dash] & & \overline{B} \arrow[d, dash] \arrow[dl, dash] \\
A \arrow[dr, dash] & \overline{T} \arrow[d, dash] & B \arrow[dl, dash] \\
& T &
\end{tikzcd}$$

Since T^* *is* a common extension of A, B, but A, B have no common *symmetric* extension, T^* cannot be symmetric. Thus, any such situation gives an example of *non-symmetric adjoints* of symmetric operators. Equivalently, $\overline{T}$ cannot be self-adjoint, because its adjoint is T^*, which cannot be *symmetric.*

---

[5] Whether or not the Axiom of Choice is used to artificially extend $\frac{d^2}{dx^2}$ to $L^2[0,1]$, that extension is not continuous, because the restriction to polynomials is already not continuous. The unboundedness/non-continuity is inescapable.

Further, although the graph closures $\overline{A}$ and $\overline{B}$ *are* (not necessarily *proper*) extensions of $\overline{T}$, neither of their Friedrichs extensions can be directly comparable to that of $\overline{T}$ without being *equal* to it, since *comparable* self-adjoint densely-defined operators are necessarily equal. [6] By hypothesis, A, B have no common symmetric extension, so it cannot be that *both* equalities hold.

[3.2] Example: symmetric extensions lacking a common symmetric extension

Let $V = L^2[a,b]$, $T = -d^2/dx^2$, with domain

$$D_T \;=\; \{f \in C_c^\infty[a,b] \;:\; f \text{ vanishes to infinite order at } a, b\}$$

The sign on the second derivative makes T *positive*: using the boundary conditions, integrating by parts,

$$\langle Tv, v\rangle \;=\; -\langle v'', v\rangle \;=\; -v'(b)\overline{v}(b) + v'(a)\overline{v}(a) + \langle v', v'\rangle \;=\; \langle v', v'\rangle \;\geq\; 0 \qquad\qquad (\text{for } v \in D_T)$$

Integration by parts *twice* proves *symmetry*:

$$\langle Tv, w\rangle \;=\; -\langle v'', w\rangle \;=\; -v'(b)\overline{w}(b) + v'(a)\overline{w}(a) + \langle v', w'\rangle \;=\; \langle v', w'\rangle$$

$$=\; v(b)\overline{w}'(b) - v(a)\overline{w}'(a) - \langle v, w''\rangle \;=\; \langle v, Tw\rangle \qquad\qquad (\text{for } v, w \in D_T)$$

For each pair α, β of complex numbers, an extension $T_{\alpha,\beta} = -d^2/dx^2$ of T is defined by taking a larger domain, namely, by relaxing the boundary conditions in various ways:

$$D_{\alpha,\beta} \;=\; \{f \in C_c^\infty[a,b] \;:\; f(a) = \alpha \cdot f(b),\; f'(a) = \beta \cdot f'(b)\}$$

Integration by parts gives

$$\langle T_{\alpha,\beta} v, w\rangle \;=\; -v'(b)\overline{w}(b) \cdot (1 - \beta\overline{\alpha}) v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}) \;+\; \langle v, T_{\alpha,\beta} w\rangle \qquad\qquad (\text{for } v, w \in D_{\alpha,\beta})$$

The values $v'(b)$, $v(b)$, $w(b)$, and $w'(b)$ can be arbitrary, so the extension $T_{\alpha,\beta}$ is *symmetric* if and only if $\alpha\overline{\beta} = 1$, and in that case T is *positive*, since again

$$\langle T_{\alpha,\beta} v, v\rangle \;=\; -\langle v'', v\rangle \;=\; \langle v', v'\rangle \;\geq\; 0 \qquad\qquad (\text{for } \alpha\overline{\beta} = 1 \text{ and } v \in D_{\alpha,\beta})$$

For two values α, α', taking $\beta = 1/\overline{\alpha}$ and $\beta' = 1/\overline{\alpha}'$, for the symmetric extensions $T_{\alpha,\beta}$ and $T_{\alpha',\beta'}$ to have a *common symmetric extension* $\widetilde{T}$ requires that the domain of $\widetilde{T}$ include both $D_{\alpha,\beta} \cup D_{\alpha',\beta'}$. The integration by parts computation gives

$$\langle \widetilde{T} v, w\rangle \;=\; -v'(b)\overline{w}(b) \cdot (1 - \beta\overline{\alpha}) + v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}) \;+\; \langle v, T_{\alpha,\beta} w\rangle$$

$$=\; -v'(b)\overline{w}(b)(1 - \beta\overline{\alpha}') + v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}') \;+\; \langle v, \widetilde{T} w\rangle \qquad\qquad (\text{for } v \in D_{\alpha,\beta},\; w \in D_{\alpha',\beta'})$$

Thus, the required symmetry $\langle \widetilde{T} v, w\rangle = \langle v, \widetilde{T} w\rangle$ holds only for $\alpha = \alpha'$ and $\beta = \beta'$. That is, the original operator T has a continuum of distinct symmetric extensions, no two of which admit a common symmetric extension.

In particular, no two of these symmetric extensions can have a common *self-adjoint* extension. Yet, each does have at least the *Friedrichs* positive, self-adjoint extension. Thus, T has infinitely-many distinct positive, self-adjoint extensions.

---

[6] A densely-defined self-adjoint operator cannot be a proper extensions of another such: for $S \subset T$ with $S = S^*$ and $T = T^*$, the inclusion-reversing property gives $T = T^* \subset S^* = S$.

For example, the two similar boundary-value problems on $L^2[0,2\pi]$

$$\begin{cases} u'' = \lambda \cdot u \quad \text{and} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi) \\ \\ u'' = \lambda \cdot u \quad \text{and} \quad u(0) = 0 = u(2\pi) \end{cases}$$

(provably) have eigenfunctions and eigenvalues

$$\begin{cases} 1,\, \sin(nx),\, \cos(nx) \quad n = 1,2,3,\ldots \quad \text{eigenvalues} \quad 0,\, 1,\, 1,\, 4,\, 4,\, 9,\, 9,\ldots \\ \\ \qquad \sin(\frac{nx}{2}) \qquad\quad n = 1,2,3,\ldots \quad \text{eigenvalues} \quad \frac{1}{4},\, 1,\, \frac{9}{4},\, 4,\, \frac{25}{4},\, 9,\, \frac{49}{4},\ldots \end{cases}$$

That is, *half* the eigenfunctions and eigenvalues are common, while the other half of eigenvalues of the first situation are shifted upward for the second situation. Both collections of eigenfunctions give orthogonal bases for $L^2[0,2\pi]$.

The expressions of the unshared eigenfunctions of one in terms of those of the other are not trivial.

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4. *Discrete spectrum of* Δ *on* $L^2(\mathbb{T})$

On the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$, there are no boundary terms in integration by parts, Δ has the *symmetry*

$$\langle \Delta f, g\rangle \;=\; \langle f, \Delta g\rangle \qquad \text{(with the usual } \langle f,g\rangle = \textstyle\int_{\mathbb{T}} f \cdot \overline{g}, \text{ for } f,g \in C^\infty(\mathbb{T}))$$

Friedrichs' *self-adjoint extension* $\widetilde{\Delta}$ of Δ is essentially described by the relation

$$\langle (1-\widetilde{\Delta})^{-1}x,\, (1-\Delta)y\rangle \;=\; \langle x,y\rangle \qquad \text{(for } x \in L^2(\mathbb{T}) \text{ and } y \in C^\infty(\mathbb{T}))$$

The *compactness* of the *resolvent* $(\widetilde{\Delta}-z)^{-1}$ enables the *spectral theorem* for compact, self-adjoint operators to yield an orthogonal Hilbert-space basis for $L^2(\mathbb{T})$ consisting of eigenfunctions for every $(\widetilde{\Delta}-z_o)^{-1}$ for z_o not in the spectrum. These eigenfunctions will be eigenfunctions for $\widetilde{\Delta}$ itself, and then provably for Δ.

The compactness of the resolvent will follow from Friedrichs' construction of $\widetilde{\Delta}$ via the continuous linear map $(1-\widetilde{\Delta})^{-1}$, itself a continuous linear map $L^2(\mathbb{T}) \longrightarrow H^1(\mathbb{T})$, where the Sobolev space $H^1(\mathbb{T})$ is the completion of $C^\infty(\mathbb{T})$ with respect to the Sobolev norm

$$|f|_{H^1(\mathbb{T})} \;=\; \Big(|f|^2_{L^2(\mathbb{T})} + |f'|^2_{L^2(\mathbb{T})}\Big)^{\frac{1}{2}} \;=\; \langle (1-\Delta)f, f\rangle^{\frac{1}{2}}$$

To prove compactness of the resolvent, we will prove *Rellich's lemma:* the inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$ is *compact.* [7] The map $(1-\widetilde{\Delta})^{-1}$ of $L^2(\mathbb{T})$ to itself is compact because it is the composition of the continuous map $(1-\widetilde{\Delta})^{-1} : L^2(\mathbb{T}) \to H^1(\mathbb{T})$ and the compact inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$.

The eigenfunctions for the extension $\widetilde{\Delta}$ certainly include the Δ-eigenfunctions e^{inx} with $n \in \mathbb{Z}$, but we must show that there are no *more* eigenfunctions for $\widetilde{\Delta}$ than for Δ. That is, while we can solve the differential equation $\Delta u = \lambda \cdot u$ on $\mathbb{R}$ and identify λ having $2\pi\mathbb{Z}$-periodic solutions, more must be done to assure that there are no *other* $\widetilde{\Delta}$-eigenfunctions in the orthogonal basis promised by the spectral theorem. We *hope* that the natural heuristic, of straightforward solution of the differential equation $\Delta u = \lambda \cdot u$, gives the whole orthogonal basis, but this is exactly the issue: details about the extension $\widetilde{\Delta}$. [8] Sobolev regularity will

[7] Rellich proved an L^2 version, as here, and Kondrachev proved an L^p version.

[8] We will see that there is a *unique* self-adjoint extension $\widetilde{\Delta}$ of Δ on $\mathbb{T}$. A *symmetric*, densely-defined operator with a *unique* self-adjoint extension is called *essentially* self-adjoint. In that case, the self-adjoint extension is the graph-closure of the original. However, in equally innocuous situations, such as Δ on $L^2[0,2\pi]$, there is a *continuum* of *boundary conditions* giving mutually incomparable self-adjoint extensions $\widetilde{\Delta}$, each with compact resolvent.

show that *all* eigenfunctions u for $\widetilde{\Delta}$ are C^∞ functions, so the extension $\widetilde{\Delta}$ is evaluated on u by evaluating the original Δ, so are obtained by solving the differential equation $u'' = \lambda \cdot u$ in classical terms.

The k^{th} Sobolev space $H^k(\mathbb{T})$ is the Hilbert space completion of $C^\infty(\mathbb{T})$ with respect to k^{th} Sobolev norm

$$|f|_{H^1(\mathbb{T})} \;=\; \Big(|f|^2_{L^2(\mathbb{T})} + |f'|^2_{L^2(\mathbb{T})} + \ldots + |f^{(k)}|^2\Big)^{\frac{1}{2}} \;\asymp\; \langle (1-\Delta)^k f, f\rangle^{\frac{1}{2}} \qquad (\text{for } 0 \le k \in \mathbb{Z})$$

Precise comparison of constants between the two versions of the Sobolev norms is irrelevant. The imbedding theorem asserts that $H^{k+1}(\mathbb{T})$ is inside $C^k(\mathbb{T})$, the latter a Banach space with natural norm

$$|f|_{C^k(\mathbb{T})} \;=\; \sup_{0\le i\le k}\,\sup_{x\in\mathbb{T}} |f^{(i)}(x)|$$

It is convenient that Hilbert spaces capture relevant information, since our geometric intuition is much more accurate for Hilbert spaces than for Banach spaces. [9] The spaces $C^k(\mathbb{T})$ are *not* Hilbert spaces, but Sobolev's imbedding theorem shows they are naturally *interlaced* with Hilbert spaces $H^r(\mathbb{T})$, in the sense that the norms satisfy the dominance relation

$$\big|f\big|_{H^k(\mathbb{T})} \;\ll\; \big|f\big|_{C^k(\mathbb{T})} \;\ll\; \big|f\big|_{H^{k+1}(\mathbb{T})} \qquad \text{giving} \qquad H^{k+1}(\mathbb{T}) \;\subset\; C^k(\mathbb{T}) \;\subset\; H^k(\mathbb{T})$$

The inclusions $C^k(\mathbb{T}) \subset H^k(\mathbb{T})$ follow from the density of $C^\infty(\mathbb{T})$ in every $C^k(\mathbb{T})$. Letting $H^\infty(\mathbb{T}) = \cap_k H^k(\mathbb{T})$, the intersection $C^\infty(T)$ of Banach spaces $C^k(\mathbb{T})$ is an intersection of Hilbert spaces [10]

$$H^\infty(\mathbb{T}) \;=\; \bigcap_k H^k(\mathbb{T}) \;=\; \bigcap_k H^{k+1}(\mathbb{T}) \;\subset\; \bigcap_k C^k(\mathbb{T}) \;=\; C^\infty(\mathbb{T}) \;\subset\; H^\infty(\mathbb{T})$$

The smoothness of $\widetilde{\Delta}$-eigenfunctions can be seen as follows. *By construction*, the *domain* of $\widetilde{\Delta}$ is inside $H^1(\mathbb{T})$. Indeed, $(1-\widetilde{\Delta})^{-k}L^2(\mathbb{T}) \subset H^k(\mathbb{T})$. To say u is a $\widetilde{\Delta}$-eigenfunction requires that u be in the domain of $\widetilde{\Delta}$. The eigenfunction property $\widetilde{\Delta}u = \lambda \cdot u$ gives $(1-\widetilde{\Delta})u = (1-\lambda)u$, and

$$u \;=\; (1-\widetilde{\Delta})^{-1}(1-\lambda)u \;\subset\; (1-\widetilde{\Delta})^{-1}H^1(\mathbb{T}) \;\subset\; H^2(\mathbb{T})$$

By induction, $u \in H^\infty(\mathbb{T}) = C^\infty(\mathbb{T})$. In particular, the extension $\widetilde{\Delta}$ of Δ is just Δ on the original domain $C^\infty(\mathbb{T})$ of Δ, so

$$\lambda \cdot u \;=\; \widetilde{\Delta}u \;=\; \Delta u \;=\; u''$$

This differential equation is easily solved on $\mathbb{R}$, and uniqueness of solutions proven: for $\lambda \neq 0$, the solutions are linear combinations of $u(x) = e^{\pm\sqrt{\lambda}\cdot x}$; for $\lambda = 0$, the solutions are linear combinations of $u(x) = 1$ and $u(x) = x$. The $2\pi\mathbb{Z}$-periodicity is equivalent to $\lambda \in i\mathbb{Z}$ in the former case, and eliminates $u(x) = x$ in the latter. As usual, uniqueness is proven via the mean-value theorem.

This proves that the exponentials $\{e^{inx} : n \in \mathbb{Z}\}$ are a Hilbert-space basis for $L^2(\mathbb{T})$.

Now we give the proofs of Sobolev inequalities/imbedding and Rellich compactness.

[9] The *minimum principle* in a topological vector space asserts that, given a point x and in a non-empty convex set C there is a unique point in C closest to x. This holds in Hilbert spaces, giving *orthogonal projections* and other analogues of finite-dimensional Euclidean geometry. In Banach spaces, the minimum principle can fail by having *no* minimizing point, or by having *infinitely many*. It was not until 1906 that B. Levi pointed out that the Dirichlet Principle is only reliable in a Hilbert space.

[10] The canonical topologies on the nested intersections $C^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ are *projective limits*.

[4.1] Sobolev imbedding on $\mathbb{T}$ This is just an application of the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality.

[4.1.1] Theorem: $H^{k+1}(\mathbb{T}) \subset C^k(\mathbb{T})$.

Proof: The case $k=0$ illustrates the causality: prove that the H^1 norm dominates the C^o norm, namely, sup-norm, on $C_c^\infty(\mathbb{T}) = C^\infty(\mathbb{T})$. Use coordinates on the real line. For $0 \le x \le y \le 1$, the difference between maximum and minimum values of $f \in C^\infty[0,1]$ is constrained:

$$|f(y) - f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_0^1 |f'(t)|\,dt \;\le\; \Big(\int_0^1 |f'(t)|^2\,dt\Big)^{1/2} \cdot \Big(\int_x^y 1\,dt\Big)^{1/2} \;=\; |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$$

Let $y \in [0,1]$ be such that $|f(y)| = \min_x |f(x)|$. Using the previous inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x) - f(y)| \;\le\; \int_0^1 |f(t)|\,dt + |f(x) - f(y)|$$

$$\le\; \int_0^1 |f| \cdot 1 + |f'|_{L^2} \cdot 1 \;\le\; |f|_{L^2} + |f'|_{L^2} \;\ll\; 2\big(|f|^2 + |f'|^2\big)^{1/2} \;=\; 2|f|_{H^1}$$

Thus, on $C_c^\infty(\mathbb{T})$ the H^1 norm dominates the sup-norm. Thus, this comparison holds on the H^1 completion $H^1(\mathbb{T})$, and $H^1(\mathbb{T}) \subset C^o(\mathbb{T})$. ///

The space $C^\infty(\mathbb{T})$ of smooth functions on the circle $\mathbb{T}$ is the nested intersection of the spaces $C^k(\mathbb{T})$, which is an instance of a *(projective) limit* of Banach spaces:

$$C^\infty(\mathbb{T}) \;=\; \bigcap_{k=0}^{\infty} C^k(\mathbb{T}) \;=\; \lim_k C^k(\mathbb{T})$$

so it has a uniquely-determined topology, which is in fact a *Fréchet* space. Similarly,

$$H^\infty(\mathbb{T}) \;=\; \bigcap_{k=0}^{\infty} H^k(\mathbb{T}) \;=\; \lim_k H^k(\mathbb{T})$$

[4.1.2] Corollary: $C^\infty(\mathbb{T}) = H^\infty(\mathbb{T})$.

Proof: From the interlacing property $C^{k+1}(\mathbb{T}) \subset H^{k+1}(\mathbb{T}) \subset C^{k+1}(\mathbb{T})$, both the spaces $C^k(\mathbb{T})$ and the spaces $H^k(\mathbb{T})$ are *cofinal* in the larger projective system that includes *both*, so all three projective limits are the same. ///

[4.2] Rellich's lemma on $\mathbb{T}$ This uses some finer details from the discussion just above, namely, the Lipschitz property $|f(x) - f(y)| \ll |x-y|^{\frac{1}{2}}$ for $|f|_{H^1} \le 1$, and the related fact that the map $H^1(T) \to C^o(\mathbb{T})$ has operator norm at most 2. [11]

[11] It would be contrary to our purpose here to use the spectral description of Sobolev spaces, but this description is important. Namely,

$$H^k(\mathbb{T}) \;=\; \{\sum_n c_n\, e^{2\pi i n x} \;:\; \sum_n |c_n|^2 \cdot (1+n^2)^k < \infty\}$$

From this viewpoint, Rellich's lemma on $\mathbb{T}$ is very easy, since the exponentials give an orthonormal basis e_i of H^{k+1} and f_i of H^k such that $e_i \to f_i/(1+n^2)^{\frac{1}{2}}$. Such a map $e_i \to \lambda_i \cdot f_i$ is *compact* exactly when $\lambda_i \to 0$, which is manifest here.

[4.2.1] Theorem: The inclusion $H^{k+1}(\mathbb{T}) \to H^k(\mathbb{T})$ is *compact.*

Proof: The principle is adequately illustrated by showing that the unit ball in $H^1(\mathbb{T})$ is *totally bounded* in $L^2(\mathbb{T})$. Approximate $f \in H^1(\mathbb{T})$ in $L^2(\mathbb{T})$ by piecewise-constant functions

$$F(x) \;=\; \begin{cases} c_1 & \text{for } 0 \le x < \frac{1}{n} \\ c_2 & \text{for } \frac{1}{n} \le x < \frac{2}{n} \\ \ldots & \\ c_n & \text{for } \frac{n-1}{n} \le x \le 1 \end{cases}$$

The sup norm of $|f|_{H^1} \le 1$ is bounded by 2, so we only need c_i in the range $|c_i| \le 2$.

Given $\varepsilon > 0$, take N large enough such that the disk of radius 2 in $\mathbb{C}$ is covered by N disks of radius less than ε, with centers C. Given $f \in H^1(\mathbb{T})$ with $|f|_1 \le 1$, choose constants $c_k \in C$ such that $|f(k/n) - c_k| < \varepsilon$. Then

$$|f(x) - c_k| \;\le\; |f\Big(\frac{k}{n}\Big) - c_k| + \Big|f(x) - f\Big(\frac{k}{n}\Big)\Big| \;<\; \varepsilon + \Big|x - \frac{k}{n}\Big|^{\frac{1}{2}} \;\le\; \varepsilon + \frac{1}{\sqrt{n}} \qquad (\text{for } \tfrac{k}{n} \le x \le \tfrac{k+1}{n})$$

Then

$$\int_0^1 |f - F|^2 \;\le\; \sum_{k=1}^n \int_{k/n}^{(k+1)/n} \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2 \;\le\; n \cdot \frac{1}{n} \cdot \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2 \;=\; \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2$$

For ε small and n large, this is small. Thus, the image in $L^2(\mathbb{T})$ of the unit ball in $H^1(\mathbb{T})$ is *totally bounded*, so has compact closure. This proves that the inclusion $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is *compact.* ///

[4.3] Sobolev imbedding and Rellich compactness on $\mathbb{T}^n$ Either by reducing to the case of a single circle $\mathbb{T}$, or by repeating analogous arguments directly on $\mathbb{T}^n$, one proves[12]

[4.3.1] Theorem: *(Sobolev inequality/imbedding)* For $\ell > k + \frac{n}{2}$, there is a continuous inclusion $H^\ell(\mathbb{T}^n) \subset C^k(\mathbb{T}^n)$. ///

[4.3.2] Theorem: *(Rellich compactness)* $H^{k+1}(\mathbb{T}^n) \subset H^k(\mathbb{T}^n)$ is *compact.* ///

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5. *Discrete spectrum of* $\Delta - x^2$ *on* $L^2(\mathbb{R})$

In contrast to Δ on $\mathbb{T}$, there are *no* square-integrable Δ-eigenfunctions on $\mathbb{R}$, so there is *no* orthogonal basis for $L^2(\mathbb{R})$ consisting of Δ-eigenfunctions. Perturb the Laplacian Δ to a *Schrödinger operator*[13]

$$S \;=\; -\Delta + x^2 \;=\; -\frac{d^2}{dx^2} + x^2$$

[12] In fact, the general discussion of compactness of the resolvent of the Laplace-Beltrami operator on *compact* Riemannian manifolds reduces to the case of $\mathbb{T}^n$, by smooth partitions of unity.

[13] This particular Schrödinger operator is also known as a *Hamiltonian*, and arose in 20th-century physics as the operator expressing *total energy* of the *quantum harmonic oscillator*, whatever that is taken to mean. Despite this later-acquired significance, Mehler had determined many spectral properties of this operator in 1866.

by adding the *confining potential* x^2, where x^2 is construed as a *multiplication operator*:

$$Sf(x) \;=\; -f''(x) + x^2 \cdot f(x)$$

We will see that the resolvent $\widetilde{S}^{-1}$ of the Friedrichs extension $\widetilde{S}$ of S *is* compact, so $\widetilde{S}$ has entirely discrete spectrum.

[5.1] Eigenfunctions of the Schrödinger operator

In contrast to Δ itself, whose eigenfunctions are well-known and easy to obtain from solving the constant-coefficient equation $u'' = \lambda u$, the eigenfunctions for S are less well-known, and solution of $Su = \lambda u$ is less immediate. The standard device to obtain eigenfunctions is as follows.

With Dirac operator [14]

$$\mathbb{D} \;=\; i\frac{\partial}{\partial x} \qquad \text{so that} \qquad \mathbb{D}^2 \;=\; -\Delta$$

the factorization

$$-\Delta + x^2 \;=\; (\mathbb{D} - ix)(\mathbb{D} + ix) + [ix, \mathbb{D}] \;=\; (\mathbb{D} - ix)(\mathbb{D} + ix) + 1 \qquad (\text{with } [ix, \mathbb{D}] = ix \circ \mathbb{D} - \mathbb{D} \circ ix)$$

allows determination of many S-eigenfunctions, although proof that *all* are produced requires some effort.

Rather than attempting a direct solution of the differential equation $Su = \lambda u$, fortunate special features are exploited. First, a smooth function u annihilated by $\mathbb{D} + ix$ will be an eigenfunction for S with eigenvalue 1:

$$Su \;=\; \Big((\mathbb{D} - ix)(\mathbb{D} + ix) + 1\Big)u \;=\; (\mathbb{D} - ix)\, 0 + u \;=\; 1 \cdot u \qquad (\text{for } (\mathbb{D} + ix)u = 0)$$

Dividing through by i, the equation $(\mathbb{D} + ix)u = 0$ is

$$\Big(\frac{\partial}{\partial x} + x\Big)u \;=\; 0$$

That is, $u' = -xu$ or $u'/u = -x$, so $\log u = -x^2/2 + C$ for arbitrary constant C. With $C = 1$

$$u(x) \;=\; e^{-x^2/2}$$

Conveniently, this is in $L^2(\mathbb{R})$, and in fact is in the Schwartz space on $\mathbb{R}$. The alternative factorization

$$S \;=\; -\Delta + x^2 \;=\; (\mathbb{D} + ix)(\mathbb{D} - ix) - [ix, \mathbb{D}] \;=\; (\mathbb{D} + ix)(\mathbb{D} - ix) - 1$$

does also lead to an eigenfunction $u(x) = e^{x^2/2}$, but this grows too fast for present purposes.

It is unreasonable to expect this more generally, but *here* the *raising* and *lowering* operators

$$R \;=\; \text{raising} \;=\; \mathbb{D} - ix \qquad\qquad L \;=\; \text{lowering} \;=\; \mathbb{D} + ix$$

map S-eigenfunctions to other eigenfunctions: for $Su = \lambda u$, noting that $S = RL + 1 = LR - 1$,

$$\begin{aligned} S(Ru) \;&=\; (RL + 1)(Ru) \;=\; RLRu + Ru \;=\; R(LR)u + Ru \;=\; R(LR - 1)u + 2Ru \\ &=\; RSu + 2Ru \;=\; R\lambda u + 2Ru \;=\; (\lambda + 2) \cdot Ru \qquad (\text{for } Su = \lambda u) \end{aligned}$$

[14] Conveniently, the Dirac operator in this situation has *complex* coefficients. In two dimensions, Dirac operators have *Hamiltonian quaternion* coefficients. The two-dimensional case is a special case of the general situation, that Dirac operators have coefficients in *Clifford algebras*.

Similarly, $S(Lu) = (\lambda - 2) \cdot Lu$. Many eigenfunctions are produced by application of R^n to $u_1(x) = e^{-x^2/2}$:

$$R^n e^{-x^2/2} \;=\; (2n+1) - \text{eigenfunction for } -\Delta + x^2$$

Repeated application of R to $e^{-x^2/2}$ produces polynomial multiples of $e^{-x^2/2}$ [15]

$$R^n e^{-x^2/2} \;=\; H_n(x) \cdot e^{-x^2/2} \qquad\qquad \text{(with polynomial } H_n(x) \text{ of degree } n)$$

The commutation relation shows that application of $LR^n u$ is just a multiple of $R^{n-1}u$, so application of L to the eigenfunctions $R^n u$ produces nothing new.

We can almost prove that the functions $R^n u$ are *all* the square-integrable eigenfunctions. The integration-by-parts symmetry

$$\langle (\mathbb{D} - ix)f,\, g\rangle \;=\; \langle f, (\mathbb{D} + ix)g\rangle \qquad\qquad (\text{for } f, g \in \mathscr{S}(\mathbb{R}))$$

gives

$$\langle (-\Delta + x^2)f,\, f\rangle \;=\; \langle (\mathbb{D} + ix)f,\, (\mathbb{D} + ix)f\rangle + \langle f, f\rangle \;=\; |(\mathbb{D} + ix)f|^2_{L^2} + |f|^2_{L^2} \;\geq\; |f|^2_{L^2}$$

In particular, an L^2 eigenfunction has *real* eigenvalue $\lambda \geq 1$. *Granting* that repeated application of L to a λ-eigenfunction u stays in $L^2(\mathbb{R})$, the function $L^n u$ has eigenvalue $\lambda - 2n$, and the requirement $\lambda - 2n \geq 1$ on $L^2(\mathbb{R})$ implies that $L^n u = 0$ for some n. Then $L(L^{n-1}u) = 0$, but we already have shown that the only $L^2(\mathbb{R})$-function in the kernel of L is $u_1(x) = e^{-x^2/2}$.

[5.2] Sobolev norms associated to the Schrödinger operator

A genuinely-self-adjoint Friedrichs extension $\widetilde{S}$ requires specification of a *domain* for S. The space $C_c^\infty(\mathbb{R})$ of test functions is universally reasonable, but we have already seen many not-compactly-supported eigenfunctions for the differential operator S. Happily, those eigenfunctions are in the Schwartz space $\mathscr{S}(\mathbb{R})$. This suggests specifying $\mathscr{S}(\mathbb{R})$ as the domain of an unbounded operator S.

There is a hierarchy of Sobolev-like norms

$$|f|_{\mathfrak{B}^\ell} \;=\; \Big\langle (-\Delta + x^2)^\ell f,\, f\Big\rangle^{\frac{1}{2}}_{L^2(\mathbb{R})} \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{R}))$$

with corresponding Hilbert-space completions

$$\mathfrak{B}^\ell \;=\; \text{completion of } \mathscr{S}(\mathbb{R}) \text{ with respect to } |f|_{\mathfrak{B}^\ell}$$

and $\mathfrak{B}^0 = L^2(\mathbb{R})$. The Friedrichs extension $\widetilde{S}$ is characterized via its *resolvent* $\widetilde{S}^{-1}$, the resolvent characterized by

$$\langle \widetilde{S}^{-1} f,\, Sg\rangle \;=\; \langle f, g\rangle \qquad\qquad (\text{for } f \in L^2(\mathbb{R}) \text{ and } g \in \mathscr{S}(\mathbb{R}))$$

and $\widetilde{S}^{-1}$ maps $L^2(\mathbb{R})$ continuously to $\mathfrak{B}^1$. Thus, an eigenfunction u for $\widetilde{S}$ is in $\mathfrak{B}^\infty = \bigcap_\ell \mathfrak{B}^\ell = \lim_\ell \mathfrak{B}^\ell$. We will see that

$$\mathfrak{B}^\infty \;=\; \mathscr{S}(\mathbb{R})$$

In particular, $\widetilde{S}$-eigenfunctions are in the natural domain of S, so evaluation of $\widetilde{S}$ on them is evaluation of S. Thus, $\widetilde{S}$-eigenfunctions are S-eigenfunctions. Further, repeated application of the lowering operator stabilizes $\mathscr{S}(\mathbb{R})$, so the *near-proof* above becomes a *proof* that all eigenfunctions in $L^2(\mathbb{R})$ are of the form $R^n\, e^{-x^2/2}$.

To prove that these eigenfunctions are a Hilbert space basis for $L^2(\mathbb{R})$, we will prove a Rellich-like *compactness* result: the injection $\mathfrak{B}^{\ell+1} \to \mathfrak{B}^\ell$ is *compact*. The map $\widetilde{S}^{-1}$ of $L^2(\mathbb{R})$ to itself is compact because it is the

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[15] The polynomials H_n are the *Hermite polynomials*, but everything needed about them can be proven from this spectral viewpoint.

composition of the continuous map $\widetilde{S}^{-1} : L^2(\mathbb{R}) \to \mathfrak{B}^1$ and the compact inclusion $\mathfrak{B}^1 \to \mathfrak{B}^0 = L^2(\mathbb{R})$. Thus, the eigenfunctions for the resolvent form an orthogonal Hilbert-space basis, and these are eigenfunctions for $\widetilde{S}$ itself, and then for S.

That is, *there is an orthogonal basis for $L^2(\mathbb{R})$ consisting of S-eigenfunctions, all obtained as*

$$(2n+1) - \text{eigenfunction} \;=\; R^n e^{-x^2/2} \;=\; \Big(i\frac{\partial}{\partial x} - ix\Big)^n e^{-x^2/2}$$

[5.3] Rellich compactness

On $\mathbb{R}$, such a compactness result depends on *both* smoothness and decay properties of the functions, in contrast to $\mathbb{T}$, where smoothness was the only issue.

[5.3.1] Proposition: The Friedrichs extension $\widetilde{S}$ of $S = -\frac{d^2}{dx^2} + x^2$ has compact resolvent.

Proof: The Friedrichs extension $\widetilde{S}$ of S is defined via its resolvent $\widetilde{S}^{-1} : L^2(\mathbb{R}) \to \mathfrak{B}^1$, the resolvent itself characterized by

$$\langle \widetilde{S}^{-1} v, w\rangle_1 \;=\; \langle v, w\rangle \qquad\qquad (\text{for } v \in L^2(\mathbb{R}) \text{ and } w \in \mathscr{S}(\mathbb{R}))$$

The resolvent $\widetilde{S}^{-1}$ is continuous with respect to the $|\cdot|_{\mathfrak{B}^1}$-topology on $\mathfrak{B}^1$. Thus, to prove that the resolvent is *compact* as a map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, factoring through the injection $\mathfrak{B}^1 \to L^2(\mathbb{R})$, it suffices to show that the latter injection is compact.

Show compactness of $\mathfrak{B}^1 \to L^2(\mathbb{R})$ by showing *total boundedness* of the image of the unit ball. Let φ be a smooth cut-off function, with

$$\varphi_N(x) \;=\; \begin{cases} 1 & (\text{for } |x| \le N) \\ \text{smooth, between 0 and 1} & (\text{for } N \le |x| \le N+1) \\ 0 & (\text{for } |x| \ge N+1) \end{cases}$$

The derivatives of φ_N in $N \le |x| \le N+1$ can easily be arranged to be independent of N. For $|f|_1 \le 1$, write $f = f_1 + f_2$ with

$$f_1 \;=\; \varphi_N \cdot f \qquad\qquad f_2 \;=\; (1-\varphi_N)\cdot f$$

The function f_1 on $[-N-1, N+1]$ can be considered as a function on a circle $\mathbb{T}$, by sticking $\pm(N+1)$ together. Then the usual Rellich-Kondrachev compactness lemma on $\mathbb{T}$ shows that the image of the unit ball from $\mathfrak{B}^1$ is totally bounded in $L^2(\mathbb{T})$, which we can identify with $L^2[-N-1, N+1]$. The L^2 norm of the function f_2 is directly estimated

$$|f_2|^2_{L^2(\mathbb{R})} \;=\; \int_{|x|\ge N} \varphi_N^2(x)\cdot |f_2(x)|^2\,dx \;\le\; \frac{1}{N^2}\int_{|x|\ge N} |f_2(x)\cdot x|^2\,dx$$

$$\le\; \frac{1}{N^2}\int_{\mathbb{R}} x^2 f(x)\cdot \overline{f}(x)\,dx \;\le\; \frac{1}{N^2}\int_{\mathbb{R}} (-\frac{d^2}{dx^2} + x^2) f(x)\cdot \overline{f}(x)\,dx \;=\; \frac{1}{N^2}|f|_1^2 \;\le\; \frac{1}{N^2}$$

Thus, given $\varepsilon > 0$, for N large the *tail* f_2 lies within a single ε-ball in $L^2(\mathbb{R})$. This proves total boundedness of the image of the unit ball, and compactness. ///

[5.3.2] Corollary: The spectrum of $S = -\frac{d^2}{dx^2} + x^2$ is entirely *discrete*. There is an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions for S, all of which lie in $\mathscr{S}(\mathbb{R})$.

Proof: Self-adjoint compact operators have discrete spectrum with finite multiplicities for non-zero eigenvalues. These eigenfunctions are also eigenfunctions for the Friedrichs extension $\widetilde{S}$. Since these eigenfunctions give an orthogonal Hilbert-space basis, $\widetilde{S}$ has no further spectrum. Since the eigenfunctions are in $\mathscr{S}(\mathbb{R})$, it is legitimate to say that they are eigenfunctions of S itself, rather than of an extension. ///

[5.4] Sobolev imbedding

We prove that $\mathfrak{B}^{k+1} \subset C^k(\mathbb{R})$ by reducing to the case of the circle $\mathbb{T}$.

[... iou ...]

[5.5] $\mathfrak{B}^\infty = \mathscr{S}$

The projective limit $\mathfrak{B}^\infty$ is the Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R})$. This will be proven by comparison of seminorms. As a corollary, $\mathscr{S}$ is *nuclear Fréchet*.

The *Weyl algebra* $A = A_1$ of operators, generated over $\mathbb{C}$ by the multiplication x and derivative $\partial = d/dx$, is also generated by $R = i\partial - ix$ and $L = i\partial + ix$. The Weyl algebra is *filtered* by degree in R and L: let $A^{\le n}$ be the C-subspace of A spanned by all non-commuting monomials in R, L of total degree at most n, with $A^{\le 0} = \mathbb{C}$. Note that R and L *commute* modulo $A^{\le 0}$: as operators, $\partial \circ x = 1 + x \circ \partial$, and the commutation relation is obtained again, by

$$[R, L] \;=\; RL - LR \;=\; (i\partial - ix)(i\partial + ix) - (i\partial + ix)(i\partial - ix) \;=\; -(\partial - x)(\partial + x) + (\partial + x)(\partial - x)$$

$$=\; -(\partial^2 - x\partial + \partial x - x^2) + (\partial^2 + x\partial - \partial x - x^2) \;=\; 2(x\partial - \partial x) \;=\; -2$$

[5.5.1] Claim: For a monomial w_{2n} in R and L of degree $2n$,

$$|\langle w_{2n} \cdot f, f\rangle_{L^2(\mathbb{R})}| \;\ll_n\; |f|^2_{\mathfrak{B}^n} \qquad\qquad (\text{for } f \in C_c^\infty(\mathbb{R}))$$

Proof: Induction. First,

$$\langle RLf, f\rangle \;=\; \langle (RL+1)f, f\rangle - \langle f, f\rangle \;\le\; \langle (RL+1)f, f\rangle \;=\; \langle Sf, f\rangle \;=\; |f|^2_{\mathfrak{B}^1}$$

A similar argument applies to LR. For the length-two word L^2,

$$|\langle L^2 f, f\rangle| \;=\; |\langle Lf, Rf\rangle| \;\le\; |Lf| \cdot |Rf| \;=\; \langle Lf, Lf\rangle^{\frac{1}{2}} \cdot \langle Rf, Rf\rangle^{\frac{1}{2}}$$

$$=\; \langle RLf, f\rangle^{\frac{1}{2}} \cdot \langle LRf, f\rangle^{\frac{1}{2}} \;\le\; \langle Sf, f\rangle \;=\; |f|^2_{\mathfrak{B}^1}$$

A similar argument applies to R^2, completing the argument for $n = 1$.

For the induction step, any word w_{2n} of length $2n$ is equal to $R^a L^b$ mod $A^{\le 2n-2}$ for some $a + b = 2n$, so, by induction,

$$|\langle w_{2n} f, f\rangle| \;=\; |\langle R^a L^b f, f\rangle| + |f|^2_{\mathfrak{B}^{n-1}}$$

In the case that $a \ge 1$ and $b \ge 1$, by induction

$$|\langle R^a L^b f, f\rangle| \;=\; |\langle R^{a-1} L^{b-1}(Lf),\, Lf\rangle| \;\ll_n\; =\; |Lf|^2_{\mathfrak{B}^{n-1}} \;=\; \langle S^{n-1} Lf, Lf\rangle \;=\; \langle RS^{n-1} Lf, f\rangle$$

Since $RS^{n-1}L$ is S^n mod $A^{\le 2n-2}$, by induction

$$\langle RS^{n-1} Lf, f\rangle \; ll_n \; \langle S^n f, f\rangle + |f|^2_{\mathfrak{B}^{n-1}} \;=\; |f|^2_{\mathfrak{B}^n} + |f|^2_{\mathfrak{B}^{n-1}} \;\ll\; |f|^2_{\mathfrak{B}^n}$$

In the extreme case $a = 0$,

$$\langle L^{2n} f, f\rangle \;=\; \langle L^n f, R^n f\rangle \;\le\; |L^n f| \cdot |R^n f| \;=\; \langle L^n f, L^n f\rangle^{\frac{1}{2}} \cdot \langle R^n f, R^n f\rangle^{\frac{1}{2}} \;=\; \langle R^n L^n f, f\rangle^{\frac{1}{2}} \cdot \langle L^n R^n f, f\rangle^{\frac{1}{2}}$$

which brings us back to the previous case. The extreme case $b = 0$ is similar. ///

[5.5.2] Corollary: For a monomial w_n in R and L of degree n,

$$|\langle w_n \cdot f, f\rangle_{L^2(\mathbb{R})}| \ll_n |f|_{\mathfrak{B}^n} \cdot |f|_{L^2} \qquad (\text{for } f \in C_c^\infty(\mathbb{R}))$$

Proof: By Cauchy-Schwarz-Bunyakowsky and the claim,

$$|\langle w_n \cdot f, f\rangle_{L^2}| \le |w_n f|_{L^2} \cdot |f|_{L^2} = \langle w_n^* w_n f, f\rangle^{\frac{1}{2}} \cdot |f|_{L^2} \le |f|_{\mathfrak{B}^n} \cdot |f|_{L^2}$$

as claimed. ///

[5.5.3] Claim: The seminorms $\mu_w(f) = |\langle wf, f\rangle|^{\frac{1}{2}}$ on test functions dominate (collectively) the usual Schwartz seminorms

$$\nu_{m,n}(f) = \sup_{x\in\mathbb{R}} \left|(1+x^2)^m f^{(n)}(x)\right|$$

Proof:

[... iou ...]

///

[5.6] Hermite polynomials

Up to a constant, the n^{th} *Hermite polynomial* $H_n(x)$ is characterized by

$$H_n(x) \cdot e^{-x^2/2} = R^n e^{-x^2/2} = (i\frac{\partial}{\partial x} - ix)^n e^{-x^2/2}$$

The above discussion shows that $H_0, H_1, H_2, \ldots$ are orthogonal on $\mathbb{R}$ with respect to the weight e^{-x^2}, and give an orthogonal basis for the weighted L^2-space

$$\{f : \int_{\mathbb{R}} |f(x)|^2 \cdot e^{-x^2} dx < \infty\}$$

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6. *Discrete spectrum of Δ on $L^2_a(\Gamma\backslash\mathfrak{H})$*

The invariant Laplacian Δ on $\Gamma\backslash\mathfrak{H}$ with $\Gamma = SL_2(\mathbb{Z})$ definitely *does* have some continuous spectrum, namely, pseudo-Eisenstein series, which we have shown are decomposable as integrals of Eisenstein series E_s. We prove that the orthogonal complement to all pseudo-Eisenstein series in $L^2(\Gamma\backslash\mathfrak{H})$, which was shown to be the space of L^2 *cuspforms*, has an orthonormal basis of Δ-eigenfunctions.

The original arguments were cast in terms of *integral operators*, as in [Selberg 1956], [Roelcke 1956]. The proofs here still do rely upon the spectral theory of compact, self-adjoint operators, but via resolvents of restrictions of the Laplacian. Compare [Faddeev 1967], [Faddeev-Pavlov 1972], [Lax-Phillips 1976], and [Venkov 1979].

This approach illustrates some interesting idiosyncracies of Friedrichs extensions of restrictions, as exploited in [ColinDeVerdière 1981/2/3].

The H^1-norm is

$$|f|_1 = \Big(\langle f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})} + \langle -\Delta f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})}\Big)^{\frac{1}{2}} \qquad (\text{for } f \in C_c^\infty(\Gamma\backslash\mathfrak{H}))$$

For $a \geq 0$, let

$$L^2_a(\Gamma\backslash\mathfrak{H}) \quad = \quad \{f \in L^2(\Gamma\backslash\mathfrak{H}) \;:\; c_P f(iy) = 0 \text{ for } y \geq a\}$$

$$H^1_a(\Gamma\backslash\mathfrak{H}) \quad = \quad H^1\text{-norm completion of } C^\infty_c(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$$

[6.1] A seeming paradox The space $L^2(\Gamma\backslash\mathfrak{H})$ contains the space $L^2(\Gamma\backslash\mathfrak{H})$ of L^2 cuspforms, for every $a \geq 0$. For $a \gg 1$, it is properly larger, so contains various parts of the continuous spectrum for Δ, namely, appropriate integrals of Eisenstein series. Yet the Friedrichs extension $\widetilde{\Delta}_a$ of a restriction Δ_a of Δ, discussed below, has entirely discrete spectrum. Evidently, some part of the continuous spectrum of Δ becomes discrete for $\widetilde{\Delta}_a$. That is, some integrals of Eisenstein series E_s become L^2 eigenfunctions for $\widetilde{\Delta}_a$. In particular, for $a \geq 1$, *certain truncated Eisenstein series* become eigenfunctions. The truncation is necessary for L^2-ness, but creates some non-smoothness. The non-smoothness *can* be understood in terms of the behavior of Friedrichs extensions, but that discussion is not strictly necessary here.

[6.2] Compactness of $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})_a$ As earlier, to prove that a Friedrichs extension has compact resolvent, it suffices to prove that a corresponding inclusion of Sobolev H^1 into L^2 is *compact*.

[6.2.1] Theorem: $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})_a$ is *compact.*

Proof: We roughly follow [Lax-Phillips 1976], adding some details. The *total boundedness* criterion for relative compactness requires that, given $\varepsilon > 0$, the image of the unit ball B in H^1_a in $L^2(\Gamma\backslash\mathfrak{H})_a$ can be covered by finitely-many balls of radius ϵ.

The idea is that the usual Rellich lemma on $\mathbb{T}^n$ reduces the issue to an estimate on the *tail*, which follows from the H^1_a condition.

The usual Rellich compactness lemma asserts the compactness of proper inclusions of Sobolev spaces on products of circles. Given $c \geq a$, cover the image Y_o of $\frac{\sqrt{3}}{2} \leq y \leq c+1$ in $\Gamma\backslash\mathfrak{H}$ by small coordinate patches U_i, and one large open U_∞ covering the image Y_∞ of $y \geq c$. Invoke compactness of Y_o to obtain a finite sub-cover of Y_o. Choose a smooth partition of unity $\{\varphi_i\}$ subordinate to the finite subcover along with U_∞, letting φ_∞ be a smooth function that is identically 1 for $y \geq c$. A function f in the Sobolev $+1$-space on Y_o is a finite sum of functions $\varphi_i \cdot f$. The latter can be viewed as having compact support on small opens in $\mathbb{R}^2$, thus identified with functions on products of circles, and lying in Sobolev $+1$-spaces there. Apply the Rellich compactness lemma to each of the finitely-many inclusion maps of Sobolev $+1$-spaces on product of circles. Thus, certainly, $\varphi_i \cdot B$ is totally bounded in $L^2(\Gamma\backslash\mathfrak{H})$.

Thus, to prove compactness of the global inclusion, it suffices to prove that, given $\varepsilon > 0$, the cut-off c can be made sufficiently large so that $\varphi_\infty \cdot B$ lies in a single ball of radius ε inside $L^2(\Gamma\backslash\mathfrak{H})$. That is, it suffices to show that

$$\lim_{c\to\infty} \int_{y>c} |f(z)|^2 \, \frac{dx\,dy}{y^2} \;\longrightarrow\; 0 \qquad\qquad (\text{uniformly for } |f|_1 \leq 1)$$

We note prove a reassuring, if unsurprising, lemma asserting that the H^1-norms of systematically specified families of smooth *tails* are dominated by the H^1-norms of the original functions.

Let ψ be a smooth real-valued function on $(0, +\infty)$ with

$$\begin{cases} \psi(y) = 0 & (\text{for } 0 < y \leq 1) \\[2ex] 0 \leq \psi(y) \leq 1 & (\text{for } 1 < y < 2) \\[2ex] \psi(y) = 1 & (\text{for } 1 \leq y) \end{cases}$$

[6.2.2] Claim: For fixed ψ, for $t \ge 1$, the smoothly cut-off tail $f^{[t]}(x+iy) = \psi\Big(\dfrac{y}{t}\Big) \cdot f(x+iy)$ has H^1-norm dominated by that of f itself:

$$|f^{[t]}|_{H^1} \ll_\psi |f|_{H^1} \qquad\qquad \text{(implied constant independent of } f \text{ and } t \ge 1)$$

Proof: *(of claim)* Since $|a+bi|^2 = a^2+b^2$ and Δ has real coefficients, it suffices to treat real-valued f. Since $0 \le \psi \le 1$, certainly $|\psi f|_{L^2} \le |f|_{L^2}$. For the other part of the H^1-norm,

$$\langle -\Delta f^{[t]}, f^{[t]} \rangle \;=\; -\int_{S^1}\int_{y\ge t} \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f^{[t]} \cdot f^{[t]} \, dx\,dy$$

$$= -\int_{S^1}\int_{y\ge t} \psi\Big(\frac{y}{t}\Big)^2 f_{xx} f + \frac{1}{t^2}\psi''\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\, f^2 + \frac{2}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\, f_y f + \psi\Big(\frac{y}{t}\Big)^2 f_{yy} f \;\; dx\,dy$$

Using $0 \le \psi \le 1$, the first and fourth summands are

$$\int_{S^1}\int_{y\ge t} -\psi\Big(\frac{y}{t}\Big)^2 f_{xx} f - \psi\Big(\frac{y}{t}\Big)^2 f_{yy} f \;\; dx\,dy \;\le\; \int_{S^1}\int_{y\ge t} -f_{xx} f - f_{yy} f \;\; dx\,dy \;\le\; \langle -\Delta f, f\rangle \;\le\; |f|^2_{H^1}$$

Using the fact that ψ' and ψ'' are supported on $[1,2]$, the second summand is

$$\int_{S^1}\int_{y\ge t} +\Big|\frac{1}{t^2}\psi''\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\, f^2\Big| \;\; dx\,dy \;\ll_\psi\; \int_{S^1}\int_{t\le y\le 2t} \frac{f^2}{t^2}\, dx\,dy$$

$$\le \int_{S^1}\int_{t\le y\le 2t} \frac{(2t)^2\, f^2}{t^2}\;\frac{dx\,dy}{y^2} \;\le\; 4|f|^2_{L^2} \;\ll\; |f|^2_{H^1}$$

The remaining term is usefully transformed by an integration by parts:

$$\int_{S^1}\int_{y\ge t} \frac{2}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\, f_y f \;\; dx\,dy \;=\; \int_{S^1}\int_{t\le y\le 2t} \frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\cdot \frac{\partial}{\partial y}(f^2) \;\; dx\,dy$$

$$= \int_{S^1}\int_{t\le y\le 2t} \frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\Big)\cdot f^2 \;\; dx\,dy$$

and then is dominated by

$$\int_{S^1}\int_{t\le y\le 2t} \Big|\frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\Big)\Big| \cdot f^2 \;\; dx\,dy \;\le\; \int_{S^1}\int_{t\le y\le 2t} \Big|\frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\Big)\Big| \cdot f^2 \cdot (2t)^2 \;\; \frac{dx\,dy}{y^2}$$

$$= 4\int_{S^1}\int_{t\le y\le 2t} \Big|\psi''\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big) + \psi'\Big(\frac{y}{t}\Big)^2\Big| \cdot f^2 \;\; \frac{dx\,dy}{y^2} \;\ll_\psi\; |f|^2_{L^2}$$

with implied constant independent of f and $t \ge 1$. ///

[6.2.3] Remark: To legitimize the following computation, recall that we proved above that $f \in H^1(\Gamma\backslash\mathfrak{H})$ has *square-integrable* first derivatives, so this differentiation is necessarily in an L^2 sense.

Let the Fourier coefficients of f be $\widehat{f}(n)$. Take $c > a$ so that the 0^{th} Fourier coefficient $\widehat{f}(0)$ vanishes identically. By Plancherel for the Fourier expansion in x, and then elementary inequalities: integrating over the part of Y_∞ above $y = c$, letting $\mathbb{F}$ be Fourier transform in x,

$$\int\int_{y>c} |f|^2 \,\frac{dx\,dy}{y^2} \;\le\; \frac{1}{c^2}\int\int_{y>c} |f|^2\,dx\,dy \;=\; \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c} |\widehat{f}(n)|^2\,dy$$

$$\le\; \frac{1}{c^2}\sum_{n\neq 0}(2\pi n)^2\int_{y>c}|\widehat{f}(n)|^2\,dy \;=\; \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c}\Big|\mathbb{F}\frac{\partial f}{\partial x}(n)\Big|^2\,dy \;=\; \frac{1}{c^2}\int\int_{y>c}\Big|\frac{\partial f}{\partial x}\Big|^2\,dx\,dy$$

$$=\; \frac{1}{c^2}\int\int_{y>c} -\frac{\partial^2 f}{\partial x^2}\cdot\overline{f}\,(x)\,dx\,dy \;\le\; \frac{1}{c^2}\int\int_{y>c} -\frac{\partial^2 f}{\partial x^2}\cdot\overline{f}\,(x) - \frac{\partial^2 f}{\partial y^2}\cdot\overline{f}\,(x)\,dx\,dy$$

$$=\; \frac{1}{c^2}\int\int_{y>c} -\Delta f\cdot\overline{f}\,\frac{dx\,dy}{y^2} \;\le\; \frac{1}{c^2}\int\int_{\Gamma\backslash\mathfrak{H}} -\Delta f\cdot\overline{f}\,\frac{dx\,dy}{y^2} \;=\; \frac{1}{c^2}|f|_1^2 \;\le\; \frac{1}{c^2}$$

This uniform bound completes the proof that the image of the unit ball in $H^1_a(\Gamma\backslash\mathfrak{H})$ in $L^2(\Gamma\backslash\mathfrak{H})_a$ is *totally bounded*. Thus, the inclusion is a compact map. ///

[6.2.4] Corollary: For λ off a *discrete* set of points in $\mathbb{C}$, $\widetilde{\Delta}_a$ has *compact resolvent* $(\widetilde{\Delta}_a - \lambda)^{-1}$, and the parametrized family of compact operators

$$(\widetilde{\Delta}_a - \lambda)^{-1} \;:\; L^2(\Gamma\backslash\mathfrak{H})_a \;\longrightarrow\; L^2(\Gamma\backslash\mathfrak{H})_a$$

is *meromorphic* in $\lambda \in \mathbb{C}$.

Proof: Friedrichs' construction shows that $(\widetilde{\Delta}_a - \lambda)^{-1} : L^2(\Gamma\backslash\mathfrak{H})_a \to H^1_a(\Gamma\backslash\mathfrak{H})$ is continuous even with the stronger topology of $H^1_a(\Gamma\backslash\mathfrak{H})$. Thus, the composition

$$L^2(\Gamma\backslash\mathfrak{H})_a \longrightarrow H^1_a(\Gamma\backslash\mathfrak{H}) \subset L^2(\Gamma\backslash\mathfrak{H})_a \qquad \text{by} \qquad f \longrightarrow (\widetilde{\Delta}_a - \lambda)^{-1}f \longrightarrow (\widetilde{\Delta}_a - \lambda)^{-1}f$$

is the composition of a continuous operator with a compact operator, so is compact. Thus,

$$(\widetilde{\Delta}_a - \lambda)^{-1} \;:\; L^2(\Gamma\backslash\mathfrak{H})_a \longrightarrow L^2(\Gamma\backslash\mathfrak{H})_a \qquad \text{is a compact operator}$$

We claim that, for a (not necessarily bounded) normal operator T, if T^{-1} exists and is *compact*, then $(T-\lambda)^{-1}$ exists and is a compact operator for λ off a *discrete* set in $\mathbb{C}$, and is *meromorphic* in λ.[16] To prove the claim, first recall from the spectral theory of normal compact operators, the non-zero spectrum of compact T^{-1} is all *point spectrum*. We claim that the spectrum of T and non-zero spectrum of T^{-1} are in the bijection $\lambda \leftrightarrow \lambda^{-1}$. From the algebraic identities

$$T^{-1} - \lambda^{-1} \;=\; T^{-1}(\lambda - T)\lambda^{-1} \qquad\qquad T - \lambda \;=\; T(\lambda^{-1} - T^{-1})\lambda$$

failure of either $T - \lambda$ or $T^{-1} - \lambda^{-1}$ to be *injective* forces the failure of the other, so the point spectra are identical. For (non-zero) λ^{-1} not an eigenvalue of *compact* T^{-1}, $T^{-1} - \lambda^{-1}$ is injective *and* has a continuous, everywhere-defined inverse.[17] For such λ, inverting the relation $T - \lambda = T(\lambda^{-1} - T^{-1})\lambda$ gives

$$(T-\lambda)^{-1} \;=\; \lambda^{-1}(\lambda^{-1} - T^{-1})^{-1}T^{-1}$$

[16] This assertion and its proof are standard. For a similar version in a standard source, see [Kato 1966], p. 187 and preceding. The same compactness and meromorphy assertion plays a role in the (somewhat apocryphal) Selberg-Bernstein treatment of the meromorphic continuation of Eisenstein series.

[17] That $S - \lambda$ is *surjective* for compact self-adjoint S and $\lambda \neq 0$ not an eigenvalue is a corollary of the spectral theory of self-adjoint compact operators, which says that all the spectrum consists of eigenvalues. This is the easiest initial part of *Fredholm theory*.

from which $(T-\lambda)^{-1}$ is continuous and everywhere-defined. That is, λ is *not* in the spectrum of T. Finally, $\lambda = 0$ is not in the spectrum of T, because T^{-1} exists and is continuous. This establishes the bijection.

Thus, when T^{-1} is compact, the spectrum of T is *countable*, with no accumulation point in $\mathbb{C}$. Letting $R_\lambda = (T-\lambda)^{-1}$, the resolvent relation

$$R_\lambda \;=\; (R_\lambda - R_0) + R_0 \;=\; (\lambda - 0)R_\lambda R_0 + R_0 \;=\; (\lambda R_\lambda + 1)\circ R_0$$

shows R_λ is the composite of a continuous and a compact operator, proving compactness. ///

[6.3] Discreteness of cuspforms We claim that the space $L^2_{\mathrm{cfm}}(\Gamma\backslash\mathfrak{H})$ has a Hilbert space basis of eigenfunctions for Δ.

The compactness of the inclusion $j_a : H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})_a \subset L^2(\Gamma\backslash\mathfrak{H})$, is the bulk of the proof.

[... iou ...]

[6.3.1] Claim: $(\widetilde{\Delta}_a - \lambda)^{-1}$ stabilizes $L^2_{\mathrm{cfm}}(\Gamma\backslash\mathfrak{H})$

This stability property would imply that $(\widetilde{\Delta}_a - \lambda)^{-1}$ restricted to $L^2_{\mathrm{cfm}}(\Gamma\backslash\mathfrak{H})$ is a compact operator.

Proof: The space of L^2 cuspforms can be characterized as the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to the space of pseudo-Eisenstein series Ψ_φ with arbitrary data $\varphi \in C^\infty_c(0,+\infty)$. However, the relation

$$\langle(\widetilde{\Delta}_a - \lambda)^{-1} f, \Psi_\varphi\rangle \;=\; \langle f, (\widetilde{\Delta}_a - \overline{\lambda})^{-1}\Psi_\varphi\rangle \;=\; \langle f, (\Delta - \overline{\lambda})^{-1}\Psi_\varphi\rangle \;=\; \langle f, \Psi_{(\Delta-\overline{\lambda})^{-1}\varphi}\rangle$$

suggests considering a class of data φ closed under solution of the corresponding differential equation. Letting $y = e^x$ and $\varphi(e^x) = v(x)$, as above, the differential equation is

$$u'' - u' - \lambda u \;=\; v$$

Taking Fourier transform,

$$\widehat{u} \;=\; \frac{-\widehat{v}}{x^2 - ix + \lambda}$$

With $\lambda = s$, the zeros of the denominator are at is and $i(1-s)$. Taking s large positive real moves these poles as far away from the real line as desired. Thus, from Paley-Wiener-type considerations, if $\widehat{v}$ were holomorphic on the strip $|\mathrm{Im}\,(\xi)| \le N$, and integrable and square-integrable on horizontal lines inside that strip, certainly the same will be true of $\widehat{u}$. The inverse Fourier transform will have a bound $e^{-N|x|}$.

The corresponding function u on $(0,+\infty)$ will be bounded by y^N as $y \to 0^+$, and by y^{-N} as $y \to +\infty$. A soft argument (for example, via *gauges*) proves good convergence of the associated pseudo-Eisenstein series.

Thus, we can redescribe the space of cuspforms to make visible the stability under $(\widetilde{\Delta}_a - \lambda)^{-1}$. This completes the proof that $L^2_{\mathrm{cfm}}(\Gamma\backslash\mathfrak{H})$ decomposes discretely, that is, has an orthonormal Hilbert space basis of $\widetilde{\Delta}_a$-eigenvectors. ///

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7. *Appendix: spectrum of T versus T^{-1} versus $(T-\lambda)^{-1}$*

Following [Kato 1966], p. 187 and preceding, we show that, for a (not necessarily bounded) self-adjoint operator T, if T^{-1} exists and is *compact*, then $(T-\lambda)^{-1}$ exists and is a compact operator for λ off a *discrete* set in $\mathbb{C}$, and is *meromorphic* in λ.

Further, we show that the spectrum of T and non-zero spectrum of T^{-1} are in the bijection $\lambda \leftrightarrow \lambda^{-1}$.

First, from the spectral theory of self-adjoint compact operators, the non-zero spectrum of T^{-1} is all *point spectrum*. From the algebraic identities

$$T^{-1} - \lambda^{-1} \;=\; T^{-1}(\lambda - T)\lambda^{-1} \qquad\qquad T - \lambda \;=\; T(\lambda^{-1} - T^{-1})\lambda$$

failure of either $T - \lambda$ or $T^{-1} - \lambda^{-1}$ to be *injective* forces the failure of the other, so the point spectra are identical. For (non-zero) λ^{-1} not an eigenvalue of *compact* T^{-1}, $T^{-1} - \lambda^{-1}$ is injective *and* has a continuous, everywhere-defined inverse. That $S - \lambda$ is *surjective* for compact self-adjoint S and $\lambda \neq 0$ not an eigenvalue is a consequence of the spectral theorm for self-adjoint compact operators[18] For such λ, inverting the relation $T - \lambda = T(\lambda^{-1} - T^{-1})\lambda$ gives

$$(T - \lambda)^{-1} \;=\; \lambda^{-1}(\lambda^{-1} - T^{-1})^{-1}T^{-1}$$

from which $(T - \lambda)^{-1}$ is continuous and everywhere-defined. That is, λ is *not* in the spectrum of T. Finally, $\lambda = 0$ is not in the spectrum of T, because T^{-1} exists and is continuous. This establishes the bijection.

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8. *Appendix: total boundedness and pre-compactness*

For us, *pre-compact* means *has compact closure.* A subset of E a metric space is *totally bounded* when, for every $\varepsilon > 0$, the set E has a finite cover of open balls of radius ε.

[8.0.1] Claim: A subset of a complete metric space is pre-compact if and only if it is totally bounded.

Proof: If a set has compact closure then it admits a finite covering by open balls of arbitrarily small radius.

On the other hand, suppose that a set E is totally bounded in a complete metric space X. To show that E has compact closure it suffices to show that any sequence $\{x_i\}$ in E has a Cauchy subsequence.

Choose such a subsequence as follows. Cover E by finitely-many open balls of radius 1. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball B_1, and let i_1 be the smallest index so that x_{i_1} lies in this ball.

The set $E \cap B_1$ is totally bounded, and contains infinitely-many elements from the sequence. Cover it by finitely-many open balls of radius $1/2$, and choose a ball B_2 with infinitely-many elements of the sequence lying in $E \cap B_1 \cap B_2$. Choose i_2 to be the smallest so that both $i_2 > i_1$ and so that x_{i_2} lies inside $E \cap B_1 \cap B_2$.

Inductively, suppose that indices $i_1 < \ldots < i_n$ have been chosen, and balls B_i of radius $1/i$, so that

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it B_{n+1}, containing infinitely-many elements of the sequence. Let i_{n+1} be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \;\in\; E \cap B_1 \cap \ldots \cap B_{n+1}$$

For $m < n$, $d(x_{i_m}, x_{i_n}) \leq \frac{1}{m}$, so this subsequence is Cauchy. ///

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