On Advanced Analytic Number Theory

By

C.L. Siegel

Tata Institute of Fundamental Research
Mumbai, India

FOREWORD

"*Advanced Analytic Number Theory*" was first published by the Tata Institute of Fundamental Research in their Lecture Notes series in 1961. It is now being made available in book form with an appendix–an English translation of Siegel's paper "Berechnung von Zetafunktionen an ganzzahligen Stellen" which appeared in the *Nachrichten der Akademie der Wissenschaften in Göttingen, Math-Phy. Klasse.* (1969), pp. 87-102.

We are thankful to Professor Siegel and to the Göttingen Academy for according us permission to translate and publish this important paper. We also thank Professor S. Raghavan, who originally wrote the notes of Professor Siegel's lectures, for making available a translation of Siegel's paper.

K. G. Ramanathan

PREFACE

During the winter semester 1959/60, I delivered at the Tata Institute of Fundamental Research a series of lectures on Analytic Number Theory. It was may aim to introduce my hearers to some of the important and beautiful ideas which were developed by L. Kronecker and E. Hecke.

Mr. S. Raghavan was very careful in taking the notes of these lectures and in preparing the manuscript. I thank him for his help.

CARL LUDWIG SIEGEL

Contents

Chapter 1

Kronecker's Limit Formulas

1 The first limit formula

Let $s = \sigma + it$ be a complex variable, σ and t being real. The Riemann zeta-function $\zeta(s)$ is defined for $\sigma > 1$ by 1

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s},$$

here k^{-s} stands for $e^{-s \log k}$, with the real value of $\log k$. The series converges absolutely for $\sigma > 1$ and uniformly in every s-half-plane defined by $\sigma \geq 1 + \epsilon(\epsilon > 0)$. It follows from a theorem of Weierstrass that the sum-function $\zeta(s)$ is a regular function of s for $\sigma > 1$. Riemann proved that $\zeta(s)$ possesses an analytic continuation into the whole s-plane which is regular except for a simple pole at $s = 1$ and satisfies the well-known functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-((1-s)/2)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

$\Gamma(s)$ being Euler's gamma-function.

We shall now prove

Proposition 1. *The function $\zeta(s)$ can be continued analytically into the half-plane $\sigma > 0$ and the continuation is regular for $\sigma > 0$, except for a simple pole at $s = 1$ with residue 1. Further, at $s = 1$, $\zeta(s)$ has the expansion*

$$\zeta(s) - \frac{1}{s-1} = C + a_1(s-1) + a_2(s-1)^2 + \cdots$$

C being Euler's constant.

We first need to prove the simplest form of a summation formula due to Euler, namely.

Lemma. *If $f(x)$ is a complex-valued function having a continuous derivative $f'(x)$ in the interval $1 \leq x \leq n$, then*

$$\int_0^1 \left(\sum_{k=1}^{n-1} f'(x+k)\left(x-\frac{1}{2}\right)\right)dx + \frac{f(1)+f(n)}{2} = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx. \tag{1}$$

2

Proof. In face, if a complex-valued function $g(x)$ defined in the interval $0 \leq x \leq 1$, has a continuous derivative $g'(x)$, then we have from integration by parts,

$$\int_0^1 g'(x)\left(x=\frac{1}{2}\right)dx = \frac{1}{2}(g(0)+g(1)) - \int_0^1 g(x)dx. \tag{2}$$

(Formula (2) has a simple geometric interpretation in that the right hand side of (2) represents the area of the portion of the (x, y)-plane bounded by the curve $y = g(x)$ and the straight lines $y - g(0) = (g(1) - g(0))x$, $x = g(0)$ and $x = (g1))$. In (2), we now set $g(x) = f(x+k)$, successively for $k = 1, 2, \ldots, n-1$. We then have for $k = 1, 2, \ldots, n-1$,

$$\int_0^1 f'(x+k)\left(x-\frac{1}{2}\right)dx = \frac{1}{2}(f(k)+f(k+1)) - \int_k^{k+1} f(x)dx.$$

Adding up, we obtain formula (1). □

This is Euler's summation formula in its simplest form, with the remainder term involving only the first derivative of $f(x)$.

It is interesting to notice that if one uses the fact that the Fourier series $-\sum_n \frac{e^{2\pi inx}}{2\pi in}$ ($n = 0$, omitted) converges uniformly to $x - 1/2$ in any closed interval $(\epsilon, 1-\epsilon)(0 < \epsilon < 1)$ one obtains from (2) the following useful result, namely,

If $f(x)$ is periodic in x with period 1 and has a continuous derivative, then

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi inx} \int_0^1 f(x)e^{-2\pi inx}dx.$$

Proof of Proposition 1. Let us now specialize $f(x)$ to be the function $f(x) = x^{-s} = e^{-x\log x}$ (log x taking real values for $x > 0$). The functionx^{-s} is evidently 3
continuously differentiable in the interval $1 \leq x \leq n$; and applying (1) to the function x^{-s}, we get

$$\begin{aligned}
&-s\sum_{k=1}^{n-1}\int_0^1 (x+k)^{-s-1}\left(x-\frac{1}{2}\right)dx + \frac{1}{2}(1+n^{-s})\\
&= \sum_{k=1}^{n} k^{-s} - \int_1^n x^{-s}dx\\
&= \begin{cases} \sum_{k=1}^{n} k^{-s} - \frac{1-n^{1-s}}{s-1} (\text{if } s \neq 1)\\ \sum_{k=1}^{n} k^{-1} - \log n (\text{if } s = 1). \end{cases} \qquad (3)
\end{aligned}$$

Let us now observe that the right-hand side of (3) is an entire function of s.

We suppose now that $\sigma > 1$. When n tends to infinity $\int_1^n x^{-s}dx$ tends to $1/(s-1)$. Further, as n tends to infinity, $\sum\limits_{k=1}^{n} k^{-s}$ converges to $\zeta(s)$. Thus for $\sigma > 1$, the right hand side of (3) converges to $\zeta(s) - 1/(s-1)$ as n tends to infinity.

On the other hand, let the left-hand side of (3) be denoted by $\varphi_n(s)$. Then $\varphi_n(s)$ is an entire funciton of s and further, for $\sigma \geq \epsilon > 0$,

$$\begin{aligned}
|\varphi_n(s)| &\leq \frac{1}{2}|s|\sum_{k=1}^{n} k^{-1-\epsilon} + \frac{1+n^{-\epsilon}}{2}\\
&\leq \frac{1}{2}|s|\sum_{k=1}^{\infty} k^{-1-\epsilon} + 1.
\end{aligned}$$

Thus, as n tends to infinity, $\varphi_n(s)$ converges to a regular function of s in the half-plane $\sigma > 0$; this provides the analytic continuation of $\zeta(s) - 1/(s-1)$ for $\sigma > 0$.

Now the constant a_0 in the power-series expansion at $s = 1$ of

$$\zeta(s) - \frac{1}{s-1} = a_0 + a_1(s-1) + a_2(s-1)^2 + \cdots$$

is nothing but $\lim\limits_{n\to\infty} \varphi_n(1)$. In other words,

$$a_0 \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$$

$$= C \text{ (Euler's constant).}$$

Proposition 1 is thus proved. 4

The constant C lies between 0 and 1. It is not known whether it is rational or irrational; very probably, it is irrational.

One could determine the constants $a_1, a_2, \ldots$ also explicitly but this is more complicated.

We shall consider now an analogous problem leading to the **Kronecker limit formula** (Kroneckersche Grenzformle).

Instead of the simple linear function x, we consider a positive-definite binary quadratic form $Q(u, v) = au^2 + 2buv + cv^2$ in the *real* variables u and v and with *real* coefficients a, b, c (we then have $a > 0$ and $ac - b^2 = d > 0$). Associated with $Q(u, v)$, let us define

$$\zeta_Q(s) = \sum_{m,n=-\infty}^{\infty} (Q(m, n))^{-s}, \tag{4}$$

where $\sum'$ denotes summation over all pairs of integers (m, n), except $(0, 0)$.

$Q(u, v)$ being positive-definite, there exists a real number $\lambda > 0$, such that $Q(u, v) \geq \lambda(u^2 + v^2)$ and it is an immediate consequence that the series (4) converges absolutely for $\sigma > 1$ and uniformly in every half-plane defined by $\sigma \geq 1 + \epsilon(\epsilon > 0)$. Thus $\zeta_Q(s)$ is a regular function of s, for $\sigma > 1$.

As in the case of $\zeta(s)$ above, we shall obtain an analytic continuation of $\zeta_Q(s)$ regular in the half-plane $\sigma > 1/2$, except for a simple pole at $s = 1$. The constant a_0 in the expansion at $s = 1$ of $\zeta_Q(s)$, viz.

$$\zeta_Q(s) = \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \cdots$$

is given precisely by the Kronecker limit formula. The constant a_{-1} which is the residue of $\zeta_Q(s)$ at $s = 1$ was found by Dirichlet in his investigations on the class-number of quadratic fields.

Letus first effect some simplifications. 5

Since $Q(u, v) = Q(-u, -v)$, we see that

$$\zeta_Q(s) = 2\sum_{m=1}^{\infty}(Q(m, 0))^{-s} + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}(Q(m, n))^{-s}. \tag{5}$$

Further

$$Q(u, v) = a\left(u + \frac{b}{a}v\right)^2 + \left(c - \frac{b^2}{a}\right)v^2$$

$$= a\left(u + \frac{b}{a}v\right)^2 + \frac{d}{a}v^2 = a\left(u + \frac{b+\sqrt{-d}}{a}v\right)\left(u + \frac{b-\sqrt{-d}}{a}v\right)$$
$$= a(u + zv)(u + \bar{z}v)$$

where $\arg(\sqrt{-d}) = \pi/2$ and $z = (b + \sqrt{-d})/a = x + iy$ with $y > 0$.

Also, we could, without loss of generality, suppose that $d = 1$; if $Q_1(u, v) = (1/\sqrt{d})(au^2 + 2buv + cv^2) = a_1u^2 + 2b_1uv + c_1v^2$, then $\zeta_{Q_1}(s) = d^{s/2}\zeta_Q(s)$ and $a_1c_1 - b_1^2 = 1$. The function $d^{s/2}$ (choosing a fixed branch) is a simple entire function of s and therefore, to study $\zeta_Q(s)$, it is enough to consider $\zeta_{Q_1}(s)$. Then we have $a = y^{-1}$ and $Q(u, v) = y^{-1}(u + zv)(u + \bar{z}v) = y^{-1}|u + zv|^2$. Now (5) becomes

$$\zeta_Q(s) = 2y^s\sum_{m=1}^{\infty} m^{-2s} + 2y^s\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}|m + nz|^{-2s}$$
$$= 2y^s\zeta(2s) + 2y^s\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}|m + nz|^{-2s} \tag{6}$$

We know from Proposition 1 that $\zeta(2s)$ has an analytic continuation in the half-plane $\sigma > 0$, regular except for a simple pole at $s = 1/2$. To obtain an analytic continuation for $\zeta_Q(s)$, we have, therefore, only to investigate the nature of the second term on the right hand side of (6), as a function of s. For this purpose, we need the **Poisson summation formula.**

Proposition 2. *Let* $f(x)$ *be continuous in* $(-\infty, \infty)$ *and let* $\sum_{m=-\infty}^{\infty} f(x + m)$ *be uniformly convergent in* $0 \le x \le 1$. *Then*

$$\sum_{m=-\infty}^{\infty} f(x + m) = \sum_{k=-\infty}^{\infty} e^{-2\pi ikx}\int_{-\infty}^{\infty} f(\xi)e^{2\pi ik\xi}d\xi.$$

6

Proof. The function $\varphi(x) = \sum_{m=-\infty}^{\infty} f(x + m)$ is continuous in $(-\infty, \infty)$ and periodic in x, of period 1. If $a_k = \int_0^1 \varphi(\xi)e^{2\pi ik\xi}d\xi$ then, by Féjér's Theorem, the $(C, 1)$ sum of $\sum_{k=-\infty}^{\infty} a_ke^{-2\pi ikx}$ is equal to $\varphi(x)$. In particular, if $\sum_{k=-\infty}^{\infty} |a_k|$ converges, then for x in $(-\infty, \infty)$

$$\sum_{m=-\infty}^{\infty} f(x + m) = \varphi(x) = \sum_{k=-\infty}^{\infty} e^{-2\pi ikx}\int_0^1 \varphi(\xi)e^{2\pi ik\xi}d\xi.$$

In view of the uniform convergence of $\sum\limits_{m=-\infty}^{\infty} f(x+m)$ in $0 \le x \le 1$,

$$\int_0^1 \sum_{m=-\infty}^{\infty} f(\xi+m) e^{2\pi ik\xi} d\xi = \sum_{m=-\infty}^{\infty} \int_0^1 f(\xi+m) e^{2\pi ik\xi} d\xi$$
$$= \int_{-\infty}^{\infty} f(\xi) e^{2\pi ik\xi} d\xi.$$

Thus

$$\sum_{m=-\infty}^{\infty} f(x+m) = \sum_{k=-\infty}^{\infty} e^{-2\pi ikx} \int_{-\infty}^{\infty} f(\xi) e^{2\pi ik\xi} d\xi, \tag{7}$$

which is the Poisson summation formula.

Now we set $f(x) = |x+iy|^{-2s} = |z|^{-2s}$ for x in $(-\infty, \infty)$. Then we see that the series $\sum\limits_{m=-\infty}^{\infty} |m+z|^{-2s}$ converges absolutely, uniformly in every interval $-N \le x \le N$, for $\sigma > 1/2$. For this purpose, it clearly suffices to consider the interval $0 \le x \le 1$, since the series remains unchanged when x is replaces by $x+1$. For $0 \le x \le 1$,

$$\left| \sum_{m=-\infty}^{\infty} |m+z|^{-2s} \right| \le \sum_{m=-\infty}^{\infty} |m+z|^{-2\sigma}$$
$$\le |z|^{-2\sigma} + |z-1|^{-2\sigma} + \sum_{m=1}^{\infty} (m^{-2\sigma} + m^{-2\sigma})$$
$$< 2y^{-2\sigma} + 2\sum_{m=1}^{\infty} m^{-2\sigma} \left(\sigma > \frac{1}{2}\right).$$

□ 7

Thus, by (7), for $\sigma > 1/2$,

$$\sum_{m=-\infty}^{\infty} |m+z|^{-2s} = \sum_{k=-\infty}^{\infty} e^{-2\pi ikx} \int_{-\infty}^{\infty} |\xi + iy|^{-2s} e^{2\pi ik\xi} d\xi$$
$$= \sum_{k=-\infty}^{\infty} e^{-2\pi ikx} \int_{-\infty}^{\infty} (\xi^2 + y^2)^{-s} e^{2\pi ik\xi} d\xi$$
$$= y^{1-2s} \sum_{k=-\infty}^{\infty} e^{-2\pi ikx} \int_{-\infty}^{\infty} (1+\xi^2)^{-s} e^{2\pi ik\xi} d\xi, \tag{8}$$

whenever the series (8) converges. Actually, we shall prove that it converges absolutely for $\sigma > 1/2$. For this purpose, let us consider for $k > 0$, $\int_{-\infty}^{\infty} (1 +$

$\zeta^2)^{-s}e^{2\pi ik\zeta}d\zeta$ as a line integral in the complex ζ-plane; let $\zeta = \xi + i\eta$. The function $(1+\zeta^2)^{-s}$ has a logarithmic branch-point at $\zeta = i$ and $\zeta = -i$. In the complex ζ-plane cut along the η-axis from $\eta = 1$ to $\eta = \infty$ and from $\eta = -1$ to $\eta = -\infty$, let $ABCDEFG$ denote the contour consisting of the straight line $GA(\eta = 0, |\xi| \leq R)$, the arc $AB(\zeta = Re^{i\theta}\pi \geq \theta \geq \pi/2)$, the straight line $BC(\xi = 0, 3/2 \leq \eta \leq R)$ on the left bank of the cut, the circle

$$CDE(\zeta = i + (i/2)e^{i\varphi}, 0 \leq \varphi \leq 2\pi),$$

the straight line EF on the right bank of the cut ($\xi = 0, 3/2 \leq \eta \leq R$)

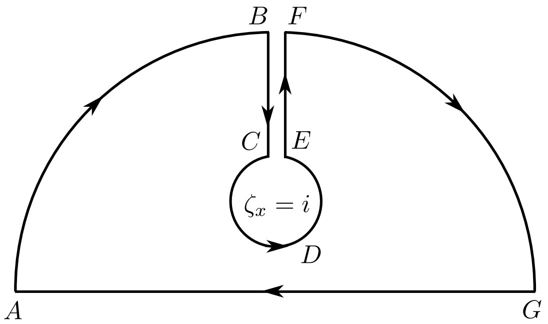


Fig. 1

andthe arc $FG(\zeta = Re^{i\theta}, \pi/2 \geq \theta \geq 0)$. If s lies in a compact set K of the **8**
s-plane with $\sigma > 0$, then on the arcs AB and FG,

$$\begin{aligned}|(1+\zeta^2)^{-s}e^{2\pi iky\zeta}| &\leq |1+\zeta^2|^{-\sigma}e^{-2\pi ky}R\sin\theta\\ &= 0(R^{-2\sigma}e^{-2\pi ky\ R\sin\theta}),\end{aligned}$$

since $-\pi \leq \arg(1+\zeta^2) \leq \pi$ on the whole contour. The constants in the O-estimate depend only on K. Thus

$$\begin{aligned}\left|\int_{AB}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta\right| &= 0\left(R^{-2\sigma}\int_0^{\pi/2}e^{-2\pi ky\ R\sin\theta}Rd\theta\right)\\ &= 0\left(R^{1-2\sigma}\int_0^{\pi/2}e^{-4kyR\theta}d\theta\right)\\ &= 0(R^{-2\sigma}),\end{aligned}$$

and hence $\int_{AB}(1+\zeta^2)^{-s}e^{2\pi ik\zeta}d\zeta$ tends to zero as R tends to infinity. Similarly $\int_{FG}(1+\zeta^2)^{-s}e^{2\pi ik\zeta}d\zeta$ also tends to zero as R tends to infinity. If Γ^+ denotes the contour $BCDEF$ in the limiting position as R tends to infinity, then for $\sigma > 0$, we have, by Cauchy's Theorem,

$$\int_{-\infty}^{\infty}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta = \int_{\Gamma^+}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta. \tag{9}$$

Let Γ^- denote the limiting position of $B'C'D'E'F'$ (which is the reflection of $BCDEF$ with respect to ξ-axis), as R tends to infinity. We can show similarly that when $k < 0$,

$$\int_{-\infty}^{\infty}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta = \int_{\Gamma^-}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta. \tag{10}$$

We shall now show that the right hand side of (**??**) defines an entire function of s. In fact, if $s = \sigma + it$ lies in a compact set K in the s-plane with $\sigma > -d(d > 0)$, then, on Γ^+ we have for a constant c_1 depending only on K, $|1+\zeta^2|^{-\sigma} \leq c_1\eta^{2d}$. Thus for $k > 0$,

$$\begin{aligned}
&\left|\int_{\Gamma^+}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta\right| \leq \int\limits_{\Gamma^+}|(1+\zeta^2)^{-s}|e^{-2\pi ky\eta}|d\zeta| \\
&\leq \int_{\Gamma^+}|1+\zeta^2|^{-\sigma}e^{\pi|t|-2\pi ky\eta}|d\zeta| \leq c_1\int_{\Gamma^+}\eta^{2d}e^{\pi|t|-2\pi ky\eta}|d\zeta| \\
&\leq c_2e^{-\pi ky}\int_{\Gamma^+}\eta^{2d}e^{-2\pi ky(\eta-\frac{1}{2})}|d\zeta| \leq c_2e^{-\pi ky}\int_{\Gamma^+}\eta^{2d}e^{-2\pi y(\eta-\frac{1}{2})}|d\zeta| \\
&\leq c_3e^{-\pi ky},
\end{aligned} \tag{11}$$

the constants c_2 and c_3 depending only on K. Hence $\int_{\Gamma^+}(1+\zeta^2)^{-s}e^{2\pi ky\zeta}d\zeta$ con- **9**
verges absolutely and uniformly in K and therefore defines an analytic function of s in any domain contained in K. Since d is an arbitrary positive number, our assertion above is proved. Similar to (11), we have also for $k < 0$,

$$\left|\int_{\Gamma^-}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta\right| \leq c_3'e^{-\pi|k|y}, \tag{12}$$

and therefore $\int_{\Gamma^-}(1+\zeta^2)^{-s}e^{2\pi iky\zeta}d\zeta$ defines for $k < 0$, an entire function of s.

In view of (**??**), (10), (11) and (12), we see that in order to prove the absolute convergence of the series (8), for $\sigma > 1/2$, we need to prove only that for $\sigma > 1/2$, $\int_{-\infty}^{\infty}(1+\zeta^2)^{-s}d\zeta$ exists. But, in fact, for $\sigma > 1/2$,

$$\int_{-\infty}^{\infty}(1+\zeta^2)^{-s}d\zeta = 2\int_0^{\infty}(1+\zeta^2)^{-s}d\zeta$$

$$= \int_0^\infty (1+\lambda)^{-s}\lambda^{-\frac{1}{2}} d\lambda \quad (\text{setting } \zeta^2 = \lambda)$$
$$= \int_0^1 \mu^{-\frac{1}{2}}(1-\mu)^{s-3/2} d\mu \left(\text{setting } \lambda = \frac{\mu}{1-\mu}\right)$$
$$= B\left(\frac{1}{2}, s-\frac{1}{2}\right),$$

where $B(a,b)$ is Euler's beta-function. Hence

$$\int_{-\infty}^{\infty}(1+\zeta^2)^{-s} d\zeta = \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

Thus, for $\sigma > 1$,

$$\sum_{m=-\infty}^{\infty} |m+z|^{-2s} = \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + y^{1-2s} \sideset{}{'}\sum_{m=-\infty}^{\infty} e^{-2\pi imx} \int_{-\infty}^{\infty}(1+\zeta^2)^{-s} e^{2\pi imy\zeta} d\zeta,$$

where the accent on Σ indicates the omission of $m = 0$. Substituting this in (6), **10**
we have for $\sigma > 1$,

$$\zeta_Q(s) = 2y^s\zeta(2s) +$$
$$+ 2y^s \sum_{n=1}^{\infty} \left\{ \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)} n^{1-2s}y^{1-2s} + n^{1-2s}y^{1-2s} \times \right.$$
$$\left. \times \sideset{}{'}\sum_{m=-\infty}^{\infty} e^{-2\pi imnx} \int_{-\infty}^{\infty}(1+\zeta^2)^{-s} e^{2\pi imny\zeta} d\zeta \right\}$$

By (**??**) and (10), therefore, for $\sigma > 1$,

$$\zeta_Q(s) = 2y^s\zeta(2s) + 2y^{1-s}\frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) +$$
$$+ 2y^{1-s}\sum_{n=1}^{\infty} n^{1-2s} \sum_{m=-\infty}^{\infty} e^{-2\pi imnx} \int_{-\infty}^{\infty}(1+\zeta^2)^{-s} e^{2\pi imny\zeta} d\zeta;$$

i.e.

$$\zeta_Q(s) = 2y^s\zeta(2s) + 2y^{1-s}\frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) + 2y^{1-s}\sum_{n=1}^{\infty} n^{1-2s} \times$$
$$\times \left\{ \sum_{m=1}^{\infty} e^{-2\pi imnx} \int_{\Gamma^+}(1+\zeta^2)^{-s} e^{2\pi imny\zeta} d\zeta + \right.$$

$$\sum_{m=1}^{\infty} e^{2\pi imnx} \int_{\Gamma^-} (1+\zeta^2)^{-s} e^{-2\pi imny\zeta} d\zeta \Bigg\}. \tag{13}$$

The double series in (13) converges absolutely, uniformly in every compact subset of the s-plane, in view of the fact that by (11) and (12), it is majorised by

$$2y^{1-\sigma} \sum_{n=1}^{\infty} n^{1-2\sigma} \left(c_3 \sum_{m=1}^{\infty} e^{-\pi mny} + c_3' \sum_{m=1}^{\infty} e^{-\pi mny} \right),$$

whichclearly converges. Since we have seen earlier that the terms of the double series define entire functions of s, it follows by the theorem of Weierstrass that the double series in (13) defines an entire function of s. **11**

Formula (13) gives us an analytic continuation of $\zeta_Q(s)$ for $\sigma > 1/2$. For, $\zeta(2s)$ is regular for $\sigma > 1/2$ and from the fact that $\zeta(s)$ can be analytically continued into the half-plane $\sigma > 0$ such that $\zeta(s) - 1/(s-1)$ is regular for $\sigma > 0$, we see on replacing s by $2s-1$ that $\zeta(2s-1)$ can be continued analytically in the half-plane $\sigma > 1/2$ such that $\zeta(2s-1) - 1/2(s-1)$ regular for $\sigma > 1/2 : \Gamma(s-\frac{1}{2})/\Gamma(s)$ is regular for $\sigma > 1/2$ and non-zero at $s=1$. Hence from (13), we see that $\zeta_Q(s)$ can be continued analytically in the half-plane $\sigma > 1/2$ and its only singularity in this half-plane is a simple pole at $s=1$ with residue equal to $2 \cdot \pi^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \cdot 1/2 = \pi$.

$\zeta_Q(s) - \pi/(s-1)$ has therefore a convergent power-series expansion $a_0 + a_1(s-1) + \cdots$ at $s=1$. We shall now determine the constant a_0. Clearly, from (13),

$$\begin{aligned} a_0 = 2y\zeta(2) &+ \lim_{s\to 1} \left(2y^{1-s} \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1) - \frac{\pi}{s-1} \right) + \\ &+ 2\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} e^{-2\pi imnx} \int_{\Gamma^+} (1+\zeta^2)^{-1} e^{2\pi imny\zeta} d\zeta + \right. \\ &\left. + \sum_{m=1}^{\infty} e^{2\pi imnx} \int_{\Gamma^-} (1+\zeta^2)^{-1} e^{-2\pi imny\zeta} d\zeta \right) \end{aligned}$$

Now from

$$\zeta(s) = \frac{1}{s-1} + C + a_1(s-1) + \cdots$$

we have

$$\zeta(2s-1) = \frac{1}{2(s-1)} + C + 2a_1(s-1) + \cdots.$$

Also replacing s by $s - 1/2$ in the formula

$$\Gamma\left(s+\frac{1}{2}\right)\Gamma(s) = \sqrt{\pi}\cdot 2^{1-2s}\Gamma(2s).$$

wehave 12

$$\Gamma\left(s-\frac{1}{2}\right)\Gamma(s) = \sqrt{\pi}2^{2-2s}\Gamma(2s-1).$$

At $s = 1$, $\Gamma(s)$ has the expansion

$$\Gamma(s) = 1 + a(s-1) + \cdots$$

and hence

$$\Gamma^2(s) = 1 + 2a(s-1) + \cdots$$

Further at $s = 1$,

$$\Gamma(2s-1) = 1 + 2a(s-1) + \cdots$$

Hence we have at $s = 1$, by Cauchy multiplication of power-series,

$$\Gamma(2s-1)\Gamma^{-2}(s) = 1 + b(s-1)^2 + \cdots$$

and thus

$$\begin{aligned} 2y^{1-s}\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) &= 2\pi(2\sqrt{y})^{2-2s}\frac{\Gamma(2s-1)}{\Gamma^2(s)}\zeta(2s-1) \\ &= 2\pi(1 + b(s-1)^2 + \cdots)\times \\ &\quad \times (1 - 2\log(2\sqrt{y})(s-1) + \cdots)\times \\ &\quad \times \left(\frac{1}{2(s-1)} + C + \cdots\right). \end{aligned}$$

In other words,

$$\lim_{s\to 1}\left(2y^{1-s}\pi^{\frac{1}{2}}\frac{\Gamma(S-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) - \frac{\pi}{s-1}\right) = 2\pi(C - \log(2\sqrt{y}))/$$

Now (see Fig. 1), for $mn > 0$, since $(1+\zeta^2)^{-1}$ has no branch-point at $\zeta = i$ or $\zeta = -i$,

$$\begin{aligned} \int_{\Gamma^+}(1+\zeta^2)^{-1}e^{2\pi imny\zeta}d\zeta &= \int_{CDE}(1+\zeta^2)^{-1}e^{2\pi imny\zeta}d\zeta \\ &= 2\pi i\left(\text{Residue of } \frac{e^{2\pi imny\zeta}}{1+\zeta^2} \text{ at } \zeta = i\right) \end{aligned}$$

$$= \pi e^{-2\pi mny}.$$

Similarly, for $mn > 0$, **13**

$$\int_{\Gamma^-} (1+\zeta^2)^{-1} e^{-2\pi imny\zeta} d\zeta = -2\pi i \left(\text{Residue of } \frac{e^{-2\pi imny\zeta}}{1+\zeta^2} \text{ at } \zeta = -i \right)$$
$$= \pi e^{-2\pi mny}$$

Hence

$$a_0 = \frac{\pi^2}{3} y + 2\pi(C - \log(2\sqrt{y})) + 2\pi \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} e^{-2\pi imnx - 2\pi mny} +$$
$$+ 2\pi \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} e^{2\pi imnx - 2\pi mny}$$

These series converge absolutely and it is practical to sum over n first. Then

$$a_0 = \frac{\pi^2}{3} y + 2\pi(C - \log 2\sqrt{y}) + 2\pi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi imn\bar{z}} +$$
$$+ 2\pi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi imnz}$$
$$= \frac{\pi^2}{3} y + 2\pi(C - \log(2\sqrt{y})) - 2\pi \sum_{m=1}^{\infty} \log(1 - e^{-2\pi im\bar{z}}) -$$
$$- 2\pi \sum_{m=1}^{\infty} \log(1 - e^{2\pi imz})$$
$$= 2\pi(C - \log(2\sqrt{y})) - 2\pi(\log e^{(\pi i/12)(z-\bar{z})} + \log \prod_{m=1}^{\infty} (1 - e^{2\pi imz}) +$$
$$+ \log \prod_{m=1}^{\infty} (1 - e^{-2\pi im\bar{z}})).$$

For complex $z = x + iy$ with $y > 0$, Dedekind defined the η-function,

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}).$$

Inthe notation of Dedekind, then, **14**

$$\begin{aligned} a_0 &= 2\pi(C - \log 2\sqrt{y}) - 2\pi \log \eta(z)\eta(-z)) \\ &= 2\pi(C - \log 2 - \log(\sqrt{y}|\eta(z)|^2)). \end{aligned}$$

Thus we have proved

Theorem 1. *Let* $z = x + iy$, $y > 0$ *and let* $Q(u, v) = y^{-1}(u + vz) \cdot (u + v\bar{z})$. *Then the zeta function* $\zeta_Q(s)$ *associated with* $Q(u, v)$ *and defined by* $\zeta_{Q(s)} = \sum_{m,n}'(Q(m, n))^{-s}$, $s = \sigma + it$, $\sigma > 1$, *can be continued analytically into a function of s regular for* $\sigma > 1/2$ *except for a simple pole at* $s = 1$ *with residue* π *and at* $s = 1$, $\zeta_Q(s)$ *has the expansion*

$$\zeta_Q(s) - \frac{\pi}{s-1} = 2\pi(C - \log 2 - \log(\sqrt{y}|\eta(z)|^2)) + a_1(s-1) + \cdots$$

This leads us to the interesting **first limit formula of Kronecker**, viz.

$$\lim_{s \to 1}\left(\zeta_Q(s) - \frac{\pi}{s-1}\right) = 2\pi(C - \log 2 - \log \sqrt{y}|\eta(z)|^2).$$

We make a few remarks. It is remarkable that the residue of $\zeta_Q(s)$ at $s = 1$ does not involve a, b, c and this was utilized by Dirichlet in determining the class-number of positive binary quadratic forms. Of course, we had supposed that $d = ac - b^2 = 1$; in the general case, the residue of $\zeta_Q(s)$ at $s = 1$ would be $\pi/\sqrt{d}$.

This limit formula has several applications; Kronecker himself gave one, namely that of finding solutions of Pell's (diophantine) equation $x^2 - dy^2 = 4$, by means of elliptic functions. It has sereval other applications and it can be generalized in many ways. In the next section, we shall show that as an application of this formula, the transformation-theory of $\eta(z)$ under the elliptic modular group can be developed.

2 The Dedekind η-function

Let $\mathfrak{H}$ denote the complex upper halt-plane, namely the set of $z = x + iy$ with $y > 0$. For $z \in \mathfrak{H}$, Dedekind defined the η-function

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty}(1 - e^{2\pi imz}).$$

15

This infinite product converges absolutely, uniformly in every compact subset of $\mathfrak{H}$. Thus, as a function of z, $\eta(z)$ is regular in $\mathfrak{H}$. Since none of the factors of the convergent infinite product is zero in $\mathfrak{H}$, it follows that $\eta(z) \neq 0$, for z in $\mathfrak{H}$. If g_2 and g_3 are the usual constants occurring in Weierstrass' theory of elliptic functions with the period-pair $(1, z)$, then $(2\pi)^{12}\eta^{24}(z) = g_2^3 - 27g_3^2$. The function $\eta(z)$ is a "modular form of dimension $-1/2$".

Let $\alpha, \beta, \gamma, \delta$ be rational integers such that $\alpha\delta - \beta\gamma = 1$. The transformation $z \to z^* = (\alpha z + \beta)(\gamma z + \delta)^{-1}$ takes $\mathfrak{H}$ onto itself; for, if $z^* = x^* + iy^*$, then

$$\begin{aligned} y^* = \frac{1}{2i}(z^* - \overline{z}^*) &= \frac{1}{2i}\left(\frac{\alpha z + \beta}{\gamma^z + \delta} - \frac{\alpha\overline{z} + \beta}{\gamma\overline{z} + \delta}\right) \\ &= \frac{1}{2i}(z - \overline{z})|\gamma z + \delta|^{-2} = y|\gamma z + \delta|^{-2} > 0, \end{aligned} \tag{14}$$

and clearly $z = (\delta z^* - \beta)(-\gamma z^* + \alpha)^{-1}$. These transformations are called "modular transformations" and they form a group called the "elliptic modular group". It is known that the elliptic modular group is generated by the simple modular transformations, $z \to z + 1$ and $z \to -1/z$. In other words, any general modular transformation can be obtained by iterating the transformations $z \to z \pm 1$ and $z \to -1/z$.

We shall give, in this section, two proofs of the transformation-formula for the behaviour of $\eta(z)$ under the modular transformation $z \to -1/z$. The first proof is a consequence of the Kronecker limit formula proved in § 1.

With $z = x + iy \in \mathfrak{H}$, we associate the positive-definite binary quadratic form $Q(u, v) = y^{-1}(u + vz)(u + v\overline{z})$. By the Kronecker limit formula for $\zeta_Q(s) = \sum_{m,n=-\infty}^{\infty}{}' (Q(m, n))^{-s}$

$$\lim_{s\to 1}\left(\zeta_Q(s) - \frac{\pi}{s-1}\right) = 2\pi(C - \log 2 - \log(\sqrt{y}|\eta(z)|^2)). \tag{15}$$

Let $z^* = (\alpha z + \beta)(\gamma z + \delta)^{-1} = x^* + iy^*$ be the image of z under a modular transformation. Then with z^*, let us associate the positive-definite binary quadratic form

$$Q^*(u, v) = y^{*-1}(u + vz^*)(u + v\overline{z}^*).$$

Again, by the Kronecker limit formula for **16**

$$\zeta_{Q^*}(s) = \sum_{m,n=-\infty}^{\infty}{}' (Q^*(m, n))^{-s},$$

$$\lim_{s\to 1}\left(\zeta_{Q^*}(s) - \frac{\pi}{s-1}\right) = 2\pi(C - \log 2 - \log \sqrt{y^*}|\eta(z^*)|^2). \tag{16}$$

Now, by (**??**),

$$y^{*-1}|m + nz^*|^2 = y^{-1}|\gamma z + \delta|^2|m + n(\alpha z + \beta)(\gamma z + \delta)^{-1}|^2$$

$$= y^{-1}|m(\gamma z+\delta)+n(\alpha z+\beta)|^2$$
$$= y^{-1}|(m\delta+n\beta)+(m\gamma+n\alpha)z|^2.$$

Since α, β, γ, δ are integral and $\alpha\delta-\beta\gamma=1$, when m, n run over all pairs of rational integers except $(0,0)$, so do $(m\delta+n\beta, m\gamma+n\alpha)$. Thus, in view of absolute convergence of the series, for $\sigma>1$,

$$\begin{aligned}\zeta_{Q^*}(s) &= \sum_{m,n}{}' (y^{*-1}|m+nz^*|^2)^{-s}\\ &= \sum_{m,n}{}' (y^{-1}|(m\delta+n\beta)+(m\gamma+n\alpha)z|^2)^{-s}\\ &= \sum_{m,n}{}' (y^{-1}|m+nz|^2)^{-s};\end{aligned}$$

i.e. for $\sigma>1$,

$$\delta_Q(s)=\zeta_{Q^*}(s)$$

($\zeta_Q(s)$ is what is called a "non-analytic modular function of z").

Since $\zeta_Q(s)$ and $\zeta_{Q^*}(s)$ can be analytically continued in the half-plane $\sigma>1/2$, $\zeta_Q(s)=\zeta_{Q^*}(s)$ even for $\sigma>1/2$. Hence, from (15) and (16),

$$\log(\sqrt{y}|\eta(z)|^2)=\log(\sqrt{y^*}|\eta(z^*)|^2);$$

i.e.

$$|\eta(z)|y^{\frac{1}{4}}=|\eta(z^*)|y^{\frac{1}{4}}.$$

By(**??**), then, we have **17**

$$\left|\eta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\right| = |\eta(z)||\sqrt{\gamma z+\delta}|. \tag{17}$$

Let us consider the function

$$f(z)=\frac{\eta((\alpha z+\beta)(\gamma z+\delta)^{-1})}{\sqrt{\gamma z+\delta\eta(z)}}$$

where the branch of $\sqrt{\gamma z+\delta}$ is chosen as follows; namely, if $\gamma=0$, then $\alpha=\delta=\pm1$ and assuming without loss of generality that $\alpha=\delta=1$, we choose $\sqrt{\gamma z+\delta}=1$. If $\gamma\neq0$, we might suppose that $\gamma>0$ and then $\sqrt{\gamma z+\delta}$ is that branch whose argument always lies between 0 and π for $z\in\mathfrak{H}$. Since $\eta(z)$ and $\sqrt{\gamma z+\delta}$ never vanish in $\mathfrak{H}$, (17) means that the funciton $f(z)$ which is regular in $\mathfrak{H}$ is of obsolute value 1. By the maximum modulus principle, $f(z)=\epsilon$, a constant of absolute value 1. We have thus

Proposition 3. *The Dedekind η-function*

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}), z \in \mathfrak{H}$$

satisfies, under a modular transformation $z \to (\alpha z + \beta) \cdot (\gamma z + \delta)^{-1}$ the transformation formula

$$\eta((\alpha z + \beta)(\gamma z + \delta)^{-1}) = \epsilon \sqrt{\gamma z + \delta}\eta(z)$$

with $\epsilon = \epsilon(\alpha, \beta, \gamma, \delta)$ and $|\epsilon| = 1$.

We shall determine ϵ for the special modular transformations, $z \to z + 1$ and $z \to -1/z$.

From the very definition of $\eta(z)$,

$$\eta(z + 1) = e^{\pi i/12}\eta(z) \tag{18}$$

and so here $\epsilon = e^{\pi i/12}$.

Also if we set $z = i$ in the formula $\eta(-1/z) = \epsilon \sqrt{z}\eta(z)$, we have $\eta(i) = \epsilon \sqrt{i}\eta(i)$. Since $\eta(i) \neq 0$, $\epsilon = 1/\sqrt{i} = e^{-\pi i/4}$.

Thus$\eta(-1/z) = e^{-\pi i/4} \sqrt{z}\eta(z)$. We rewrite this as **18**

$$\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z), \tag{19}$$

$\sqrt{z/i}$ being that branch taking the value 1 at $z = i$.

To determine ϵ in the general case, we only observe that any modular transformation is obtained by iteration of the transformations $z \to z \pm 1$ and $z \to -1/z$. Since every time we apply these transformations we get, in view of (18) and (19), a factor which is $e^{\pm\pi i/12}$ or $e^{-\pi i/4}$, we see that ϵ is a 24^{th} root of unity; ϵ can be determined this way, by a process of "reduction".

The question arises as to whether there exists an explicit expression for ϵ, in terms of α, β, γ and δ. This problem was considered and solved by Dedekind who for this purpose, had to investigate the behaviour of $\eta(z)$ as z approaches the 'rational points' on the real line. This problem has also been considered recently by Rademacher in connection with the so-called 'Dedekind sums'.

We now give a very simple *alternative proof* of formula (19).

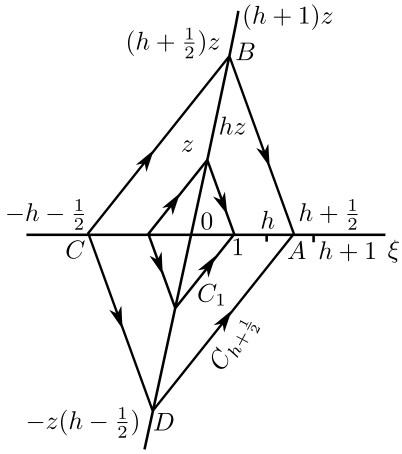


Fig. 2

Letus consider the integral 19

$$\frac{1}{8}\int_{C_t} \cot \pi\zeta \cot \frac{\pi\zeta}{z}\frac{d\zeta}{\zeta}$$

where C_t is the contour of the parallelogram in the ζ-plane with vertices at $\zeta = t$, tz, $-t$ and $-tz$ (t positive and not integral).

The integrand is a meromorphic function of ζ. It has simple poles at $\zeta = \pm k$ and $\zeta = \pm kz (k = 1, 2, 3, \ldots)$ with residues $\dfrac{1}{\pi k}\cot\dfrac{\pi k}{z}$ and $\dfrac{1}{\pi k}\cot \pi kz$ respectively, as one can readily notice from the expansion

$$\cot \pi u = \frac{1}{\pi u} - \frac{\pi u}{3} + \cdots$$

valid for $0 < |u| < 1$. Also, the integrand has a pole of the third order at $\zeta = 0$, with residue $-\frac{1}{3}(z + 1/z)$.

Let $t = h + 1/2$, h being a positive integer. Then there are no poles of the integrand on C_t and inside C_t there are poles at $\zeta = 0$, $\pm k$ and $\pm kz(k = 1, 2, \ldots, h)$. By Cauchy's theorem on residues,

$$\frac{1}{8}\int_{C_{h+\frac{1}{2}}} \cot \pi\zeta \cot \pi\frac{\zeta}{z}\frac{d\zeta}{\zeta} = \frac{2\pi i}{8}\left\{\sum_{k=-h}^{h}{}' \frac{1}{\pi k}\cot\frac{\pi k}{z} + \sum_{k=-h}^{h}{}' \frac{1}{\pi k}\cot \pi kz - \frac{1}{3}\left(z + \frac{1}{z}\right)\right\},$$

the accent on Σ indicating the omission of $k = 0$ from the summation. In other words,

$$\frac{1}{8}\int_{C_{h+\frac{1}{2}}}\cot\pi\zeta\cot\pi\frac{\zeta}{z}\frac{d\zeta}{12}\left(z+\frac{1}{z}\right)=\frac{i}{2}\left\{\sum_{k=1}^{h}\frac{1}{k}\left(\cot\pi kz+\cot\frac{\pi k}{z}\right)\right\}. \tag{20}$$

We wish to consider the limit of the relation (20) as h tends to infinity through the sequence of natural numbers.

For this purpose, let us first observe that if $\zeta=\xi+i\eta$, then

$$\cot\pi\zeta=i\frac{e^{\pi i\zeta}+e^{-\pi i\zeta}}{e^{\pi i\zeta}-e^{-\pi i\zeta}}$$

tendsto $-i$ if η tends to ∞, and tends to $+i$, if η tends to $-\infty$. Similarly $\cot\pi(\zeta/z)$ **20**
tends to $-i$, when the imaginary part of ζ/z tends to ∞ and to i, when the latter tends to $-\infty$. If O is the origin and ζ tends to infinity along a ray OP with P on one of the open segments AB, BC, CD, DA of $C_{h+\frac{1}{2}}$ (see figure), then $\cot\pi\zeta\cdot\pi(\zeta/z)$ tends to the values $+1$, -1, $+1$, -1 respectively. Moreover, this process of tending to the respective values is uniform, when the ray OP lies between two fixed rays from O, which lie in one of the sectors AOB, BOC, COD or DOA and neither of which coincides with the lines $\eta=0$ or $\zeta=\lambda z$ (λ real). This means that the sequence of functions $\{\cot\pi(h+1/2)\xi\cot\pi(h+1/2)(\zeta/z)\}$ $(h=1,2,\ldots)$ tends to the limits $+1$ or -1 uniformly on any proper closed subsegment of a side of C_1.

Moreover, this sequence of functins is uniformly bounded on C_1. This can be seen as follows. Let K_j denote the discs $|\zeta-j|<1/4$, $j=0,\pm1,\pm2,\ldots$. In view of the periodicity of $\cot\pi\zeta$, let us confine ourselves to the strip, $0\le\xi\le2$. We then readily see that in the complement of the set-union of the discs K_0, K_1 and K_2 with respect to this vertical strip, the function $\cot\pi\zeta$ is bounded; for $\cot\pi\zeta$ tends to πi as η tends to $\pm\infty$ and is bounded away from its poles at $\zeta=0$, 1 and 2. Indeed therefore, $\cot\pi\zeta$ is bounded in the complement of the set union of the discs K_j, $j=0,\pm1,\ldots$ with respect to the entire plane. Since the contours $C_{h+\frac{1}{2}}$, for $h=1,2,\ldots$ lie in the complement, we have $|\cot\pi\zeta|\le\alpha$ for $\zeta\in C_{h+\frac{1}{2}}$ and α independent of h and ζ. By a similar argument for $\cot(\pi\zeta/z)$ concerning its poles at $\pm kz$, we see that for $\zeta\in C_{h+\frac{1}{2}}$, $h=1,2,\ldots|\cot(\pi\zeta/z)|\le\beta$, β independent of h and ζ. In other words, the sequence of functions $\{\cot\pi(h+\frac{1}{2})\zeta\cot\pi(h+\frac{1}{2})(\zeta/z)\}$ is uniformly bounded on c_1. Now

$$\int_{C_{h+\frac{1}{2}}}\cot\pi\zeta\cot\frac{\pi\zeta}{z}\frac{d\zeta}{\zeta}=\int_{C_1}\cot\pi\left(h+\frac{1}{2}\right)\zeta\cot\pi\left(h+\frac{1}{2}\right)\frac{\zeta}{z}\frac{d\zeta}{\zeta}, \tag{21}$$

and in view of the foregoing considerations, we can interchange the passage to the limit (as h tends to infinity) and the integration, on the right side of (21). Thus, letting h tend to infinity, we obtain from (20),

$$\frac{\pi i}{12}\left(z+\frac{1}{z}\right)+\frac{1}{8}\left(\int_1^z-\int_z^{-1}+\int_{-1}^{-z}-\int_{-z}^{1}\right)\frac{d\zeta}{\zeta}$$
$$=\frac{i}{2}\left\{\sum_{k=1}^{\infty}\frac{1}{k}\left(\cot\pi kz+\cot\frac{\pi k}{z}\right)\right\}. \tag{22}$$

Now 21

$$\frac{1}{8}\left(\int_1^z-\int_z^{-1}+\int_{-1}^{-z}-\int_{-z}^{1}\right)\frac{d\zeta}{\zeta}=\frac{1}{4}\left(\int_1^z+\int_{-1}^{z}\right)\frac{d\zeta}{\zeta}.$$

For z in the ζ-plane cut along the negative ξ-axis, let $\log z$ denote the branch that is real on the positive ξ-axis. Then we see

$$\int_1^z\frac{d\zeta}{\zeta}=\log z \quad\text{and}\quad \int_{-1}^{z}\frac{d\zeta}{\zeta}=\log z-\pi i.$$

Thus from (22),

$$\frac{\pi i}{12}\left(z+\frac{1}{z}\right)+\frac{1}{2}\left(\log z-\frac{\pi i}{2}\right)=\frac{i}{2}\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(\cot\pi kz+\cot\frac{\pi k}{z}\right)\right\}. \tag{23}$$

We insert in (23), the expansions

$$\cot\pi kz=-i\left(1+2\frac{e^{2\pi ikz}}{1-e^{2\pi ikz}}\right)=-i\left(1+2\sum_{m=1}^{\infty}e^{2\pi imkz}\right)$$

and

$$\cot\frac{\pi k}{z}=i\left(1+2\frac{e^{-2\pi ik/z}}{1-e^{-2\pi ik/z}}\right)=i\left(1+2\sum_{m=1}^{\infty}e^{-2im\pi k/z}\right);$$

then we see that the resulting double series is absolutely convergent. We are therefore justified in summing over k first; we see as a result that the series (23) goes over into

$$-\sum_{m=1}^{\infty}\log\frac{(1-e^{2\pi imz})}{(1-e^{-2\pi im/z})}=\log\frac{\eta\left(-\frac{1}{z}\right)}{\eta(z)}+\frac{\pi i}{12}\left(z+\frac{1}{z}\right).$$

Finally

$$\log \frac{\eta\left(-\frac{1}{z}\right)}{\eta(z)} = \frac{1}{2}\left(\log z - \frac{\pi i}{2}\right) = \log \sqrt{\frac{z}{i}},$$

where $\sqrt{z/i}$ is that branch which takes the value 1 at $z = i$ i.e. $\eta(-1/z) =$ **22**
$\sqrt{(z/i)}\eta(z)$.

We wish to remark that it was only for the sake of simplicity that we took a parallelogram C_1 and considered 'dilatations' $C_{h+\frac{1}{2}}$ of C_1. It is clear that instead of the parallelogram, we could have taken any other closed contour passing through 1, z, -1, $-z$ and through no other poles of the integrand and worked with corresponding 'dilatations' of the same.

3 The second limit formula of Kronecker

We Shall consider now some problems more general than the ones concerning the analytic continuation of the function $\zeta_Q(s)$ in the half-plane $\sigma > 1/2$ and the Kronecker limit formula for $\zeta_Q(s)$.

In the first place, we ask whether it is possible to continue $\zeta_Q(s)$ analytically in a larger half-plane; this question shall be postponed for the present. We shall show later (§ 5) that $\zeta_Q(s)$ has an analytic continuation in the entire plane which is regular except for the simple pole at $s = 1$ and satisfies a functional equation similar to that of the Riemann ζ-function.

Instead of $\zeta_Q(s) = \sum\limits_{m,n=-\infty}^{\infty}{}' (Q(m,n))^{-s}$, one might also consider

$$\sum_{m,n=-\infty}^{\infty} (Q(m+\mu, n+\nu))^{-s}$$

where μ and ν are non-zero real numbers which are not both integral and where m and n run independently from $-\infty$ to $+\infty$. Or, instead of the positive-definite binary quadratic form $Q(u,v)$, we might take a positive-definite quadratic form in more than two variables. Let, in fact, $S = (s_{ij})$, $1 \leq i, j \leq p$ be the matrix of a positive-definite quadratic form $\sum\limits_{1 \leq i,j \leq p} s_{ij} x_i x_j$ in p variables. Let us consider

$$\zeta_S(S) = \sum_{x_1,\ldots,x_p=-\infty}^{\infty} \left(\sum_{i,j} S_{ij} x_i x_j\right)$$

$x_1, \ldots, x_p$ running over all p-tuples of integers, except $(0, \ldots, 0)$. The function$\zeta_S(S)$, known as the Epstein zeta-function, is regular for $\sigma ? p/2$. It was **23** shown by Epstein that $\zeta_S(S)$ can be continued analytically in the entire s-plane, into a function regular except for a simple pole at $S = p/2$ and satisfying the functional equation

$$\pi^{-s}\Gamma(S)\zeta_S(S) = |S|^{-\frac{1}{2}}\pi^{-(p/2-s)}\Gamma\left(\frac{p}{2}-s\right)\zeta_{S^{-1}}\left(\frac{p}{2}-s\right)$$

Epstein also obtained for $\zeta_S(s)$, an analogue of the first limit formula of Kronecker but this naturally involves more complicated functions than $\eta(z)$.

We shall now consider a generalization of $\zeta_Q(s)$ which shall lead us to the second limit formula of Kronecker. namely, let u and v be independent real parameters. Let, as before, $Q(m,n) = am^2 + 2bmn + cn^2$ be a positive-definite binary quadratic form and let $ac - b^2 = 1$, without loss of generality. We then define, for $\sigma > 1$,

$$\zeta_Q(s,u,v) = \sum_{m,n}{}' e^{2\pi i(mu+nv)}(Q(m,n))^{-s} \tag{24}$$

m, n running over all ordered pairs of integers except $(0,0)$. The series converges absolutely for $\sigma > 1$ and converges uniformly in every half-plane $\sigma \geq 1 + \epsilon(\epsilon > 0)$. Thus $\zeta_Q(s,u,v)$ is a regular function of s for $\sigma > 1$. If u and v are both integers, then $\zeta_Q(s,u,v) = \zeta_Q(s)$ which has been already considered. We shall, suppose, in the following, that

$$u \text{ and } v \text{ are not both integers.} \tag{$*$}$$

Furthermore, on account of the periodicity of $\zeta_Q(s,u,v)$ in u and v, we can clearly suppose that $0 \leq u, v < 1$. The condition $(*)$ then means that at least one of the two conditions $0 < u < 1$, $0 < v < 1$ is satisfied. We might, without loss of generality, suppose that $0 < u < 1$. For, otherwise, if $u = 0$, then necessarily $0 < v < 1$. In this case, let us take the positive-definite binary quadratic form $Q_1(m,n) = cm^2 + 2bmn + an^2$; we see then that, for $\sigma > 1$,

$$\zeta_Q(s,u,v) = \sum_{m,n}{}' e^{2\pi i(mv+nu)}(Q_1(m,n))^{-s}.$$

And here, since $0 < v < 1$, v shall play the role of u in (24).

Weshall prove now that $\zeta_Q(s,u,v)$ can be continued analytically into a func- **24**
tion regular in the half-plane $\sigma > 1/2$ and then determine its value at $s = 1$; this will lead us to the second limit formula of Kronecker.

For $\sigma > 1$, by our earlier notation,

$$\zeta_Q(s,u,v) = y^s \sum_{m=-\infty}^{\infty} e^{2\pi imu}|m|^{-2s} + y^s \sum_{n=-\infty}^{\infty} e^{2\pi inv} \sum_{m=-\infty}^{\infty} e^{2\pi imu}|m+nz|^{-2s}. \tag{25}$$

The first series in (25) converges absolutely for $\sigma > 1/2$ and uniformly in every compact subset of this half-plane. Hence it defines a regular function of s for $\sigma > 1/2$. The double series in (25) again defines a regular function of s in the half-plane $\sigma > 1$. To obtain its analytic continuation in a larger half-plane, we shall carry out the summation of the double series in the following particular way. Namely, we consider the finite partial sums

$$\sum_{n_1 \le n \le n_2}{}' \sum_{m_1 \le m \le m_2} e^{2\pi i(mu+nv)} |m + nz|^{-2s}.$$

We shall show that when m_1 and n_1 tend to $-\infty$ independently and m_2 and n_2 tend to ∞ independently, then these partial sums which are entire functions of s converge uniformly in every compact subset of the half-plane $\sigma > 1/2$. The proof is based on the method of partial summation in Abel's Theorem.

Let us define for any integer k, $c_k = \dfrac{e^{2\pi iku}}{e^{2\pi iu} - 1}$. (Recall that $0 < u < 1$). Then $e^{2\pi imu} = c_{m+1} - c_m$. Moreover,

$$|c_m| \le |1 - e^{2\pi iu}|^{-1} = \alpha,$$

where α is independent of m. Now

$$\begin{aligned} &\sum_{n_1 \le n \le n_2}{}' \sum_{m_1 \le m \le m_2} e^{2\pi i(mu+nv)} |m + nz|^{-2s} \\ &= \sum_{n=n_1}^{n_2} e^{2\pi inv} \sum_{m=m_1}^{m_2} \frac{c_{m+1} - c_m}{|m + nz|^{2s}} \\ &= \sum_{n=n_1}^{n_2} e^{2\pi inv} \left(\sum_{m=m_1+1}^{m_2} c_m (|m - 1 + nz|^{-2s} - |m + nz|^{-2s}) + \right. \\ &\qquad\qquad \left. + c_{m_2+1} |m_2 + nz|^{-2s} - c_{m_1} |m_1 + nz|^{-2s} \right). \end{aligned} \tag{26}$$

Let us also observe that 25

$$\left| c_{m_2+1} |m_2 + nz|^{-2s} \right| \le \alpha |n|^{-2\sigma} y^{-2\sigma}$$

and

$$\left| c_{m_1} |m_1 + nz|^{-2s} \right| \le \alpha |n|^{-2\sigma} y^{-2\sigma}.$$

Further

$$\left| |m - 1 + nz|^{-2s} - |m + nz|^{-2s} \right|$$

$$\left|-2s\int_{m-1}^{m}((\mu+nx)^2+n^2y^2)^{-s-1}(\mu+nx)d\mu\right| \leq 2|s|\int_{m-1}^{m}((\mu+nx)^2+n^2y^2)^{-(\sigma+\frac{1}{2})}du.$$

Thus (26) is majorised by

$$2|s|\alpha\sum_{n=n_1}^{n_2}{}'\sum_{m=m_1+1}^{m_2}\int_{m-1}^{m}((\mu+nx)^2+n^2y^2)^{-(\sigma+\frac{1}{2})}d\mu+2\alpha y^{-2\sigma}\sum_{n=n_1}^{n_2}{}'|n|^{-2\sigma}. \quad (27)$$

In (27), we sum m from $-\infty$ to $+\infty$ instead of summing from m_1+1 to m_2 and then we see that (26) has the majorant

$$2\alpha|s|\sum_{n=n_1}^{n_2}{}'\int_{-\infty}^{\infty}((\mu+nx)^2+n^2y^2)^{-(\sigma+\frac{1}{2})}d\mu+2\alpha y^{-2\sigma}\sum_{n=n_1}^{n_2}|n|^{-2\sigma}.$$

Now **26**

$$\int_{-\infty}^{\infty}((\mu+nx)^2+n^2y^2)^{-(\sigma+\frac{1}{2})}d\mu=\int_{-\infty}^{\infty}(\mu^2+n^2y^2)^{-(\sigma+\frac{1}{2})}d\mu$$
$$=|n|^{-2\sigma}y^{-2\sigma}\int_{-\infty}^{\infty}(1+\mu^2)^{-(\sigma+\frac{1}{2})}d\mu. \quad (28)$$

Since the integral in (28) converges for $\sigma > 1/2$, we see that (26) is majorised by $\beta\sum\limits_{n=n_1}^{n_2}|n|^{-2\sigma}$, β depending only on the compact set in which s lies in the half-plane $\sigma > 1/2$ and on α and y.

If, in (26), we now carry out the passage of m_1 and n_1 to $-\infty$ and of m_2 and n_2 to $+\infty$ independently, then we see that the double series in (25) summed in this particular manner, is majorised by the series $2\beta\sum\limits_{n=1}^{\infty}n^{-2\sigma}$. Hence it converges uniformly when s lies in a compact set in the half-plane $\sigma > 1/2$. Since each partial sum (26) is an entire function of s, we see, by Weierstrass' Theorem that the double series in (25) converges uniformly to a function which is regular for $\sigma > 1/2$ and which provides the necessary analytic continuation in this half-plane.

Thus under the assumption that u and v are not both integers, $\zeta_Q(s,u,v)$ has an analytic continuation which is regular for $\sigma > 1/2$. We shall now determine its value at $s=1$. We wish to make it explicit that hereafter we shall make only the assumption $(*)$ and not the specific assumption $0 < u < 1$.

From the convergence proved above, it follows that

$$\zeta_Q(1,u,v)=\frac{z-\bar{z}}{2i}\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{|m|^2}+$$

$$+\frac{z-\bar{z}}{2i}\sum_{n=-\infty}^{\infty}{}' e^{2\pi inv}\sum_{m=-\infty}^{\infty}\frac{e^{2\pi imu}}{(m+nz)(m+n\bar{z})} \tag{29}$$

where the accents on Σ have the usual meaning.

In order to sum $\sum_{m=-\infty}^{\infty}\frac{e^{2\pi imu}}{|m|^2}$, we observe that since the series $\sum_{m=-\infty}^{\infty}\frac{e^{2\pi imu}}{m}$ 27
converges uniformly to $2\pi(1/2-u)$ in any closed interval. $0<p\le u\le q<1$, we have

$$\int_0^u\left(\sum_{m=-\infty}^{\infty}\frac{e^{2\pi imu}}{m}\right)du=\lim_{\epsilon\to 0}\int_\epsilon^u\left(\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m}\right)du$$
$$=\lim_{\epsilon\to 0}\sum_{m=-\infty}^{\infty}{}'\int_\epsilon^u\frac{e^{2\pi imu}}{m}du,$$

i.e.

$$2\pi i(u-u^2)=\frac{1}{2\pi i}\lim_{\epsilon\to 0}\left(\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^2}-\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^2}\right),$$

i.e.

$$2\pi^2(u^2-u)=\sum_{m=-\infty}^{\infty}{}'-\frac{e^{2\pi imu}}{m^2}-\frac{\pi^2}{3},$$

since $\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^2}$ is continuous in u. In other words,

$$\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^2}=2\pi^2\left(u^2-u+\frac{1}{6}\right).$$

We shall sum the double series in (29) more generally with $\bar{z}$ replaced by r, where $-r\in\mathfrak{H}$. Leter, we shall set $r=\bar{z}$. We remark that with slight changes, our arguments concerning the series $\sum' e^{2\pi i(mu+nv)}|m+nz|^{-2s}$ above will also go through for the double series $\sum' e^{2\pi i(mu+nv)}((m+nz)\cdot(m+nr))^{-s}$. Now for summing the series

$$\frac{z-r}{2i}\sum_{n=-\infty}^{\infty}{}' e^{2\pi inv}\sum_{m=-\infty}^{\infty}\frac{e^{2\pi imu}}{(m+nz)(m+nr)}$$
$$=\frac{1}{2i}\sum_{n=-\infty}^{\infty}{}'\frac{e^{2\pi inv}}{n}\sum_{m=-\infty}^{\infty}e^{2\pi imu}\left(\frac{1}{m+nr}-\frac{1}{m+nz}\right). \tag{30}$$

we could apply the Poisson summation formula with respect to m to the inner sum on the left side of (30), but we shall adopt a different method here.

Letus define, for complex $z = x + iy$, **28**

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} + \sum_{m=1}^{\infty} \left(\frac{e^{2\pi imu}}{z+m} + \frac{e^{-2\pi imu}}{z-m} \right)$$

and consider $f(z)$ in the strip $-1/2 \le x \le 1/2$. It is not hard to see that whenever z lies in a compact set not containing $z = 0$ in this strip, the series converges uniformly and hence $f(z)$ is regular in this strip except at $z = 0$, where it has a simple pole with residue 1. Moreover, one can show that $f(z) - 1/z$ is bounded in this strip. Further

$$f(z+1) = e^{-2\pi iu} f(z).$$

Consider now the function

$$g(z) = 2\pi i \frac{e^{-2\pi iuz}}{1 - e^{-2\pi iz}};$$

$g(z)$ is again regular in this strip except at $z = 0$ where it has a simple pole with residue 1. Moreover $g(z) = 1/z$ is bounded in this strip; as a matter of fact, if $0 < u < 1$, $g(z)$ tends to 0 if y tends to $+\infty$ or $-\infty$ and if $u = 0$, then $g(z)$ tends to 0 or $2\pi i$ according as y tends to $+\infty$ or $-\infty$. Further $g(z+1) = e^{-2\pi iu} g(z)$.

As a consequence, the function $f(z) - g(z)$ is regular and bounded in this strip and since $f(z+1) - g(z+1) = e^{-2\pi iu}(f(z) - g(z))$, it is regular in the whole z-plane and bounded, too. By Liouville's theorem, $f(z) - g(z) = c$, a constant. It can be seen that, if $u \ne 0$, $c = 0$ and if $u = 0$, $c = \pi i$, by using the series expansion for the function $\pi \cot \pi z$. In any case,

$$f(nr) - f(nz) = g(nr) - g(nz). \tag{31}$$

In view of the convergence-process of the series (29) described above, we can rewrite the double series (30) as

$$\frac{1}{2i} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi inv}}{n} \left\{ \frac{1}{nr} - \frac{1}{nz} + \sum_{m=1}^{\infty} e^{2\pi imu} \left(\frac{1}{m+nr} + \frac{1}{-m+nr} - \frac{1}{m+nz} - \frac{1}{-m+hz} \right) \right\}$$
$$= \frac{1}{2i} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi inv}}{n} (f(nr) - f(nz)) = \pi \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi inv}}{n} \left(\frac{e^{-2\pi iunr}}{1 - e^{-2\pi inr}} - \frac{e^{-2\pi iunz}}{1 - e^{-2\pi inz}} \right), \tag{32}$$

by (31). Now, we insert in (32), the expansions **29**

$$(1 - e^{-2\pi inz})^{-1} = \begin{cases} -\sum_{m=1}^{\infty} e^{2\pi imnz}, & (n > 0) \\ \sum_{m=0}^{\infty} e^{-2\pi mnz}, & (n < 0) \end{cases}$$

$$(1-e^{-2\pi inr})^{-1} = \begin{cases} \sum_{m=0}^{\infty} e^{-2\pi imnr}, & (n>0) \\ -\sum_{m=1}^{\infty} e^{2\pi imnr}, & (n<0) \end{cases}$$

valid for $z, -r \in \mathfrak{H}$ and then the series (32) goes over into

$$\pi \sum_{n=1}^{\infty} \frac{1}{n} \Bigg\{ e^{2\pi in(v-ur)} + e^{-2\pi in(v-uz)} + \sum_{m=1}^{\infty} (e^{-2\pi in(v-uz-mz)} + \\ + e^{2\pi in(v-ur-mr)} + e^{2\pi in(v-uz+m'z)} + e^{-2\pi in(v-ur+mr)}) \Bigg\} \tag{33}$$

If $0 < u < 1$, then this series converges absolutely, since $z, -r \in \mathscr{H}$. Carrying out the summation over n first, we see that this double series is equal to

$$-\pi \log \Bigg\{ \prod_{m=0}^{\infty} (1-e^{-2\pi i(v-uz-mz)})(1-e^{2\pi i(v-ur-mr)}) \times \\ \times \prod_{m=1}^{\infty} (1-e^{2\pi i(v-uz+mz)})(1-e^{-2\pi i(v-ur+mr)}) \Bigg\}. \tag{34}$$

If $u = 0$, then necessarily $0 < v < 1$ and in this case,

$$\pi \sum_{n=1}^{\infty} \frac{1}{n} (e^{2\pi inv} + e^{-2\pi inv}) = -\pi \log(1-e^{2\pi iv})(1-e^{-2\pi iv}).$$

Therest of the series (33) is absolutely convergent and summing over n first, **30**
we see that the value of the series (33) is given by the expression (34).

Let us now set $v - uz = w$, $v - ur = q$. Then replacing $\bar{z}$ by r in (29), we have finally, for $0 \le u, v < 1$ and u and v not simultaneously zero,

$$\frac{z-r}{2i} \sum_{m,n=-\infty}^{\infty} \frac{e^{2\pi i(mu+nv)}}{(m+nz)(m+nr)} \\ = -\pi^2 i(z-r)\left(u^2 - u + \frac{1}{6}\right) - \pi \log \Bigg\{ \prod_{m=0}^{\infty} (1-e^{-2\pi i(w-mz)}) \times \\ \times (1-e^{2\pi i(q-mr)}) \times \prod_{m=1}^{\infty} (1-e^{2\pi i(w+mz)})(1-e^{-2\pi i(q+mr)}) \Bigg\} \tag{35}$$

For $z \in \mathfrak{H}$ and arbitrary complex w, the elliptic theta-function $\vartheta_1(w, z)$ is defined by the infinite series,

$$\vartheta_1(w, z) = \sum_{n=-\infty}^{\infty} e^{\pi i(n+\frac{1}{2})^2 z + 2\pi i(n+\frac{1}{2})(w-\frac{1}{2})}.$$

It is clear that this series converges absolutely, uniformly when z lies in a compact set in $\mathfrak{H}$ and w in a compact set of the w-plane. Hence $\vartheta_1(w, z)$ is an analytic function of z and w for $z \in \mathfrak{H}$ and complex w. It is proved in the theory of the elliptic theta-functions that $\vartheta_1(w, z)$ can also be expressed as an absolutely convergent infinite product, namely,

$$\begin{aligned}\vartheta_1(w, z) = -ie^{\pi i(z/4)}(e^{\pi iw} - e^{-\pi iw})\times \\ \times \prod_{m=1}^{\infty}(1 - e^{2\pi i(w+mz)})(1 - e^{-2\pi i(w-mz)})(1 - e^{2\pi imz}). \end{aligned} \tag{36}$$

We shall identify these two forms of $\vartheta_1(w, z)$ later on. For the present, we shall consider $\vartheta_1(w, z)$ as defined by the infinite product.

Now a simple rearrangement of the expression (35) gives

$$\begin{aligned}&\frac{z-r}{2i}\sum_{m,n=-\infty}^{\infty}{}' \frac{e^{2\pi i(mu+nv)}}{(m+nz)(m+nr)} \\ &= -\pi^2 i(z-r)\left(u^2 - u + \frac{1}{6}\right) + \pi^2 i\left((w-q) + \frac{1}{6}(z-r)\right) - \\ &\quad - \pi \log \frac{\vartheta_1(w, q)\vartheta_1(q, -r)}{\eta(z)\eta(-r)} \\ &= -\pi^2 i\frac{(w-q)^2}{z-r} - \pi \log \frac{\vartheta_1(w, q)\vartheta_1(q, -r)}{\eta(z)\eta(-r)}\end{aligned}$$

$$\frac{z-r}{-2\pi i}\sum_{m,n}{}' \frac{e^{2\pi i(mu+nv)}}{(m+nz)(m+nr)} = \log \frac{\vartheta_1(w, z)\vartheta_1(q, -r)}{\eta(z)\eta(-r)} e^{\pi i((w-q)^2/(z-r))}. \tag{37}$$

Setting $r = \overline{z}$ again, we have $q = \overline{w}$ and then 31

$$\zeta_Q(1, u, v) = -\pi^2 i\frac{(w-\overline{w})^2}{z-\overline{z}} - \pi \log \frac{\vartheta_1(w, z)\vartheta_1(\overline{w}, -\overline{z})}{|\eta(z)|^2}$$

One can see easily from (36) that $\overline{\vartheta_1(w, z)} = \vartheta_1(\overline{w}, -\overline{z})$. Hence

$$\zeta_Q(1, u, v) = -\pi^2 iu^2(z-r) - 2\pi \log \left|\frac{\vartheta_1(w, z)}{\eta(z)}\right|, \tag{38}$$

for $0 \leq u, v < 1$ and u and v not both zero.

We contend that formula (38) is valid for all u and v not both integral. Since $\zeta_Q(1, u+1, v) = \zeta_Q(1, u, v+1) = \zeta_Q(1, u, v)$ and w goes over into $w + 1$ and

$w - z$ respectively under the transformations $v \to v + 1$ and $u \to u + 1$, it would suffice for this purpose to prove that

$$\zeta_Q(1, u, v) = -\pi^2 i u^2 (z - r) - 2\pi \log \left| \frac{\vartheta_1(w + 1, z)}{\eta(z)} \right|$$

and

$$\zeta_Q(1, u, v) = -\pi^2 i (u + 1)^2 (z - r) - 2\pi \log \left| \frac{\vartheta_1(w - z, z)}{\eta(z)} \right|.$$

The first assertion is an immediate consequence of the fact that $\vartheta_1(w + 1, z) = -\vartheta_1(w, z)$. To prove the second, we observe that from (36), we have

$$\begin{aligned} \frac{\vartheta_1(w - z, z)}{\vartheta_1(w, z)} &= \frac{(e^{\pi i(w-z)} - e^{-\pi i(w-z)})(1 - e^{2\pi i w})}{(e^{\pi i w} - e^{-\pi i w})(1 - e^{-2\pi i(w-z)})} \\ &= e^{+2\pi i w - \pi i z} \end{aligned}$$

and hence 32

$$\begin{aligned} 2\pi \left(\log \left| \frac{\vartheta_1(w - z, z)}{\eta(z)} \right| - \log \left| \frac{\vartheta_1(w, z)}{\eta(z)} \right| \right) &= +\pi^2 i \{(2(w - \overline{w}) - (z - \overline{z})\} \\ &= -\pi^2 i (z - \overline{z})((u + 1)^2 - u^2). \end{aligned}$$

Thus the second assertion is proved and we have

Theorem 2. *If u and v are real and not both integral, then the Epstein zeta-function $\zeta_Q(s, u, v)$ defined by* (24) *for $\sigma > 1$, can be continued analytically into a function regular for $\sigma > 1/2$ and its value at $s = 1$ is given by* (38).

We are finally led to the following formula: viz. *for all u and v not both integral*,

$$\frac{z - \overline{z}}{2i} \sum_{m,n}{}' \frac{e^{2\pi i(mu+nv)}}{|m + nz|^2} = -\pi^2 i \frac{(w - \overline{w})^2}{z - \overline{z}} - 2\pi \log \left| \frac{\vartheta_1(w, z)}{\eta(z)} \right|.$$

This is the second limit formula of Kronecker. We may rewrite it as

$$\frac{z - \overline{z}}{-2\pi i} \sum{}' \frac{e^{2\pi i(mu+nv)}}{|m + nz|^2} = \log \left| \frac{\vartheta_1(v - uz, z)}{\eta(z)} e^{\pi i z u^2} \right|^2. \qquad (39)$$

If u and v are both integral, then $v - uz$ is a zero of $\vartheta_1(w, z)$ and then both sides are infinite.

4 The elliptic theta-function $\vartheta_1(w, z)$

For $z = x + iy \in \mathfrak{H}$ and arbitrary complex w, the elliptic theta-function $\vartheta_1(w, z)$ was defined in § 3, by the infinite product (36). $\vartheta_1(w, z)$ is a regular function of z and w, for z in $\mathfrak{H}$ and arbitrary complex w. We shall now develop the transformation-theory of $\vartheta_1(w, z)$ under modular transformations, as a consequence of the second limit formula of Kronecker.

Let$u$ and v be arbitrary real numbers which are not simultaneously integral. **33**
Let $z \to z^* = (\alpha z + \beta)(\gamma z + \delta)^{-1} = x^* + iy^*$ be a modular transformation. Then $u^* = \delta u + \gamma v$ and $v^* = \beta u + \alpha v$ are again real numbers, which are not both integral. For $\sigma > 1$, we have

$$y^{*s} \sum_{m,n=-\infty}^{\infty} \frac{e^{2\pi i(mu^*+nv^*)}}{|m+nz^*|^{2s}} = y^s {\sum_{m,n}}' \frac{e^{2\pi i((m\delta+n\beta)u+(m\gamma+n\alpha)v)}}{|(m\delta+n\beta)+(m\gamma+n\alpha)z|^{2s}}$$
$$= y^s {\sum_{m,n}}' \frac{e^{2\pi i(mu+nv)}}{|m+nz|^{2s}}.$$

From § 3, we know that both sides, as functions of s, have analytic continuations regula in the half-plane $\sigma > 1/2$ and hence the equality of the two sides is valid even for $\sigma > 1/2$. In particular, for $s = 1$,

$$y^* {\sum_{m,n}}' \frac{e^{2\pi i(mu^*+nv^*)}}{|m+nz^*|^2} = y {\sum_{m,n}}' \frac{e^{2\pi i(mu+nv)}}{|m+nz|^2}. \tag{40}$$

By Kronecker's second limit formula,

$$-\pi^2 i \frac{(w^* - \overline{w}^*)^2}{z^* - \overline{z}^*} - 2\pi \log\left|\frac{\vartheta_1(w^*, z^*)}{\eta(z^*)}\right| = -\pi^* i \frac{(w - \overline{w})^2}{z - \overline{z}} - 2\pi \log\left|\frac{\vartheta_1(w, z)}{\eta(z)}\right|$$

where $w = v - uz$ and $w^* = v^* - u^* z^* = w/(\gamma z + \delta)$. Therefore,

$$\log\left|\frac{\vartheta_1(w^*, z^*)}{\eta(z^*)}\right| - \log\left|\frac{\vartheta_1(w, z)}{\eta(z)}\right| = \frac{\pi i}{2}\frac{(w\overline{w})^2}{z - \overline{z}} - \frac{\pi i}{2}\frac{(w^* - \overline{w}^*)^2}{z^* - \overline{z}^*}$$
$$= \frac{\pi i \gamma}{2}\left(\frac{w^2}{\gamma z + \delta} - \frac{\overline{w}^2}{\gamma \bar{z} + \delta}\right) = \log\left|e^{(\pi i \gamma w^2)/(\gamma z + \delta)}\right|$$

Thus

$$\frac{\vartheta_1\left(\frac{w}{\gamma z+\delta}, \frac{\alpha z+\beta}{\gamma z+\delta}\right)}{\eta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)} = \frac{\vartheta_1(w, z)}{\eta(z)} e^{(\pi i \gamma w^2)/(\gamma z+\delta)} \cdot \omega, \tag{41}$$

where $|\omega| = 1$ and ω might depend on every one of $w, z, \alpha, \beta, \gamma$ and δ. Butfrom **34**

(41), we observe that ω is a regular function of z in $\mathfrak{H}$ and w, except possibly when w is of the form $m + nz$ with integral m and n. But since $|\omega| = 1$, we see, by applying the maximum modulus principle to ω as a function of w and of z, that ω is independent of w and z. Hence $\omega = \omega(\alpha, \beta, \gamma, \delta)$.

Now from the fact that

$$\vartheta_1(w, z) = 2\pi\eta^3(z)w + \cdots$$

we note that $\vartheta_1'(0, z)$, the value of the derivative of $\vartheta_1(w, z)$ with respect to w at $w = 0$, is equal to $2\pi\eta^3(z)$. Differentiating the relation (41) with respect to w at $w = 0$, we get

$$\frac{1}{\gamma z + \delta} \frac{\vartheta_1'\left(0, \frac{\alpha z+\beta}{\gamma z+\delta}\right)}{\eta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)} = \frac{\vartheta_1'(0, z)}{\eta(z)}\omega,$$

i.e.

$$(\gamma z + \delta)^{-1}\eta^2\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \omega\eta^2(z).$$

We have thus rediscovered the formula

$$\eta\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \epsilon\sqrt{\gamma z + \delta}\,\eta(z)$$

with ϵ being a 24^{th} root of unity depending only on α, β, γ and δ and $\omega = \epsilon^2$. Rewriting (41), we obtain

Proposition 4. *The elliptic theta-function has the transformation formula*

$$\vartheta_1\left(\frac{w}{\gamma z + \delta}, \frac{\alpha z + \beta}{\gamma z + \delta}\right) = \epsilon^3\sqrt{\gamma z + \delta}\,e^{(\pi i\gamma w^2)/(\gamma z+\delta)}\vartheta_1(w, z)$$

under a modular substitution $z \to (\alpha z + \beta)(\gamma z + \delta)^{-1}$, ϵ^3 *being an* 8^{th} *root of unity depending only on* α, β, γ *and* δ.

From the transformation-theory of $\eta(z)$ we know that corresponding to the modular transformations $z \to z+1$ and $z \to -1/z$, ϵ^3 has respectively the values $e^{\pi i/4}$ and $e^{-3\pi i/4}$. Hence

$$\left.\begin{aligned} \vartheta_1(w, z + 1) &= e^{\pi i/4}\vartheta_1(w, z) \\ \vartheta_1\left(\frac{w}{z}, -\frac{1}{z}\right) &= e^{\pi i w^2/z}\frac{1}{i}\sqrt{\frac{z}{i}}\,\vartheta_1(w, z). \end{aligned}\right\} \tag{42}$$

Once again using the fact that the modular transformations $z \to z + 1$ and 35

$z \to -1/z$ generate the elliptic modular group, we can determine ϵ^3, with the help of (42), by a process of 'reduction'. Hermite found an explicit expression for ϵ^3, involving the well-known Gaussian sums.

We now wish to make another remark which is of function-theoretic significance and which is again, a consequence of Kronecker's second limit formula.

Let us define for $z \in \mathfrak{H}$ and real u and v (not both integral), the function

$$f(z,u,v) = \log\left(\frac{\vartheta_1(v-uz,z)}{\eta(z)} e^{\pi izu^2}\right)$$

choosing a fixed branch of the logarithm in $\mathfrak{H}$. Then from (40) and (39),

$$\log\left|\frac{\vartheta_1(v^*-u^*z^*,z^*)}{\eta(z^*)} e^{\pi iz^*u^{*2}}\right| = \log\left|\frac{\vartheta_1(v-uz,z)}{\eta(z)} e^{\pi izu^2}\right|.$$

Hence

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \delta u+\gamma v, \beta u+\alpha v\right) = f(z,u,v) + i\lambda,$$

where λ is real and might depend on z, u, v, α, β, γ and δ. But since λ is a regular function of z, whose imaginary part is zero in $\mathfrak{H}$, λ is independent of z. Thus

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \delta u+\gamma v, \beta u+\alpha v\right) = f(z,u,v) + i\lambda(u,v,\alpha,\beta,\gamma,\delta). \tag{43}$$

Let now $q > 1$, be a fixed integer and let u and v be proper rational fractions with reduced denominator q i.e. $u = a/q$, $v = a/q$, $(a,b,q) = 1$ and $0 \leq u$, $v < 1$. Since $\alpha\delta - \beta\gamma = 1$, we have again.

$$(\delta a+\gamma b, \beta a+\alpha b, q) = 1.$$

Nowfrom (39), we see that $\log\left|\frac{\vartheta_1(v-uz,z)}{\eta(z)} e^{\pi izu^2}\right|$ is invariant under the **36**
transformations $u \to u+1$ and $v \to v+1$. Hence $f(z,u,v)$ picks up a purely imaginary additive constant under these transformations. If now, $(\delta a+\gamma b)/q \equiv (a^*/q)(\mod 1)$ and $(\beta a+\alpha b)/q \equiv (b^*/q)(\mod 1)$ where a^*/q, b^*/q are proper rational fractions with reduced denominator q, then

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \frac{\delta a+\gamma b}{q}, \frac{\beta a+\gamma b}{q}\right) = f\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \frac{a^*}{q}, \frac{b^*}{q}\right) + i\lambda'$$

where λ' is real and depends on a, b, α, β, γ and δ. Writing $f(z,a/q,b/q)$ as $f_{a,b}(z)$, we get from (43),

$$f_{a^*,b^*}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = f_{a,b}(z) + i\lambda^*(a,b,\alpha,\beta,\gamma,\delta) \tag{44}$$

$\lambda^*(a, b, \alpha, \beta, \gamma, \delta)$ being a real constant depending on a, b, α, β, γ and δ.

The number of reduced fractions a/q, b/q with reduced denominator q is given by $v(q) = q^2 \prod_{p|q}(1-1/p^2)p$ running through all prime divisors of q. Corresponding to each pair a/q, b/q, we have a function $f_{a,b}(z)$ and the relation (44) asserts that when we apply a modular transformation on z, these functions are permuted among themselves, except for a purely imaginary additive constant.

The modular transformations $z\, to(\alpha z + \beta)(\gamma z + \delta)^{-1}$ for which $\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \equiv \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}($ mod $q)$, form a subgroup $\mathscr{M}(q)$ of the elliptic modular group called the **principal congruence subgroup of level** (stufe) q.

The group $\mathscr{M}(q)$ has index $qv(q)$ in the elliptic modular group and acts discontinuously on $\mathfrak{H}$. We can construct in the usual way, a fundamental domain for $\mathscr{M}(q)$ in $\mathfrak{H}$. This can be made into a compact Riemann surface by identifying the points on the boundary which are 'equivalent' under $\mathscr{M}(q)$ and 'adding the cusps'. Complex-valued functions $f(z)$ which are meromorphic in $\mathfrak{H}$ and invariant under the modular transformations in $\mathscr{M}(q)$ and have at most a pole in the 'local uniformizer' at the 'cusps' of the fundamental domain are called **modular functions of level q.** They form an algebraic function field of one variable (over the field of complex numbers) which coincides with the field of meromorphic functions on the Riemann surface.

Inthe case when the transformation $z \to (\alpha z + \beta)(\gamma z + \delta)^{-1}$ is in $\mathscr{M}(q)$, then **37**
referring to (44), we have $a^* = a$, $b^* = b$ and

$$f_{a,b}\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = f_{a,b}(z) + i\lambda^*(a, b, \alpha, \beta, \gamma, \delta). \tag{45}$$

The function $f_{a,b}(z)$ is regular everywhere in $\mathfrak{H}$. To examine its singularities (in the local uniformizers) at the 'cusps' on the Riemann surface, it suffices in virtue of (44) to consider the singularity of $f_{a,b}(z)$ at the 'cusp at infinity'. It can be seen that at the 'cusp at infinity', $f_{a,b}(z)$ has the singularity given by $\log e^{\pi i z(a^2 q^{-2} - a q^{-1} + 1/6)}$. In view of (45), then $f_{a,b}(z)$ is an **abelian integral** on the Riemann surface associated with $\mathscr{M}(q)$ and the quantities $\lambda^*(a, b, \alpha, \beta, \gamma, \delta)$ are 'periods'. Since the expression $u^2 - u + 1/6$ does not vanish for rational u, $f_{a,b}(z)$ is an abelian integral *strictly of the third kind.* It might be an interesting problem to determine the 'periods', $\lambda^*(a, b, \alpha, \beta, \gamma, \delta)$.

We had defined the function $\vartheta_1(w, z)$ by the infinite product (36); now, we shall identify it with the infinite series expansion by which it is usually defined. For this purpose, we define for $z \in \mathfrak{H}$ and complex w,

$$f(w, z) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+\frac{1}{2})^2 z + 2\pi i (n+\frac{1}{2})(w-\frac{1}{2})}. \tag{46}$$

Clearly $f(w,z)$ is a regular function of w and z. Moreover, it is easy to see that

$$f(w+1,z) = -f(w,z). \tag{47}$$

Also replacing n by $n+1$ on the right-hand side of (46), we have

$$\begin{aligned} f(w,z) &= \sum_{n=-\infty}^{\infty} e^{\pi i z(n+1+\frac{1}{2})^2+2\pi i(n+1+\frac{1}{2})(w-\frac{1}{2})} \\ &= -e^{2\pi i w+\pi i z} f(w+z,z), \end{aligned}$$

i.e.

$$f(w+z,z) = -e^{-2\pi i w-\pi i z} f(w,z). \tag{48}$$

We know already that

$$\vartheta_1(w+1,z) = -\vartheta_1(w,z)$$

and

$$\vartheta_1(w+z,z) = -e^{-2\pi i w-\pi i z}\vartheta_1(w,z).$$

Hence,for fixed $z \in \mathfrak{H}$, the function $h(w,z) = \dfrac{f(w,z)}{\vartheta_1(w,z)}$ is a doubly periodic 38
function of w with independent periods 1 and z and is regular in w except possibly when $w = m + nz$ with integral m and n, since $\vartheta_1(w,z)$ has simple zeros at these points. But $f(w,z)$ also has zeros at these points, since, replacing n by $-n-1$ on the right-hand side of (46), we have $f(-w,z) = -f(w,z)$ and hence $f(0,z) = 0$ and from (47) and (48), $f(m+nz,z) = 0$ for integral m and n. Thus $h(w,z)$ is a doubly periodic entire function of w and by Liouville's theorem, it is independent of w, i.e. $h(w,z) = h(z)$.

The function $h(z)$ is a regular function of z in $\mathfrak{H}$ and to determine its behaviour under the modular transformations, it suffices to consider its behaviour under the transformations $z \to z+1$ and $z \to -z^{-1}$. We can show easily that $h(z+1) = h(z)$. Moreover, by (42),

$$\vartheta_1\left(\frac{w}{z}, -\frac{1}{z}\right) = e^{\pi i w^2/z}\frac{1}{i}\sqrt{\frac{z}{i}}\vartheta_1(w,z).$$

Let us assume, for the present, that we have proved the formula

$$f\left(\frac{w}{z}, -\frac{1}{z}\right) = e^{\pi i w^2 . z}\frac{1}{i}\sqrt{\frac{z}{i}} f(w,z). \tag{49}$$

It will then follow that $h(-1/z) = h(z)$. Hence $h(z)$ is a modular function and it is indeed regular everywhere in $\mathfrak{H}$.

Now, a fundamental domain of the elliptic modular group in $\mathfrak{H}$ is given by the set of $z = x + iy \in \mathfrak{H}$ for which $|z| \geq 1$ and $-1/2 \leq x \leq 1/2$. Let z tend to infinity in this fundamental domain. Then it is easy to see that the functions $\vartheta_1(w, z)e^{-\pi iz/4}$ and $f(w, z)e^{-\pi iz/4}$ tend to the same limit $-i(e^{\pi iw} - e^{-\pi iw})$. Hence $h(z) - 1$ tends to zero as z tends to infinity in the fundamental domain. But then, being regular in $\mathfrak{H}$ and invariant under modular transformations, it attains its maximum at some point in $\mathfrak{H}$. By the maximum modulus principle, $h(z) - 1$ is a constant and since $h(z) - 1$ tends to zero as z tends to infinity, $h(z) = 1$. In other words, $\vartheta_1(w, z) = f(w, z)$.

To complete the proof, all we need to do is to prove (49). We may first rewrite $f(w, z)$ as

$$f(w, z) = e^{-\pi i(w-\frac{1}{2})^2/z} \sum_{n=-\infty}^{\infty} e^{\pi iz(n+\frac{1}{2}+w/z-1/2z)^2}$$

39

Let $\xi > 0$ and let $g(x) = e^{-\pi\xi x^2}$, for $-\infty < x < \infty$. Then the series $\sum\limits_{n=-\infty}^{\infty} g(x+n)$ converges uniformly in the interval $0 \leq x \leq 1$ and by the Poisson summation formula (7),

$$\sum_{n=-\infty}^{\infty} e^{-\pi\xi(x+n)^2} = \sum_{n=-\infty}^{\infty} e^{2\pi inx} \int_{-\infty}^{\infty} e^{-\pi\xi x^2 - 2\pi inx} dx, \tag{50}$$

(provided the series on the right-hand side converges). Now

$$\int_{-\infty}^{\infty} e^{-\pi\xi x^2 - 2\pi inx} dx = \frac{e^{-\pi n^2/\xi}}{\sqrt{\xi}} \int_{-\infty}^{\infty} e^{-\pi(x+in/\sqrt{\xi})^2} dx, \quad (\sqrt{\xi} > 0).$$

By the Cauchy integral theorem, one can show that

$$\int_{-\infty}^{\infty} e^{-\pi(x+in/\sqrt{\xi})^2} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = \gamma (\text{say}).$$

It is now obvious that the series on the right-hand side of (50) converges absolutely. Hence

$$\sum_{n=-\infty}^{\infty} e^{-\pi\xi(x+n)^2} = \frac{\gamma}{\sqrt{\xi}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\xi + 2\pi inx}. \tag{51}$$

Setting $\xi = 1$ and $x = 0$ in (51), we have immediately $\gamma = 1$. Both sides of (51) represent, for complex x, analytic functions of x and since they are equal for real x, formula (51) is true even for complex x. Moreover, both sides of (51) represent analytic functions of ξ, for complex ξ with Re $\xi > 0$. Since they coincide for real $\xi > 0$, we see again that (51) is valid for all complex ξ with $\mathrm{Re}\,\xi > 0$. We have thus the **theta-transformation formula**

$$\sum_{n=-\infty}^{\infty} e^{-\pi\xi(n+v)^2} = \frac{1}{\sqrt{\xi}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\xi + 2\pi i n v} \tag{52}$$

valid for all complex v and complex ξ with $\mathrm{Re}\,\xi > 0$, $\sqrt{\xi}$ denoting that branch which is positive for real $\xi > 0$.

Setting$\xi = -iz$ and $v = (1/2) + (w/z) - (1/2z)$ in (53), we have **40**

$$f(w,z) = \frac{e^{-\pi i(w-\frac{1}{2})^2/z}}{\sqrt{\frac{z}{i}}} \sum_{n=-\infty}^{\infty} e^{-\pi i n^2/z + 2\pi i n((1/2)+(w/z)-(1/2z))}.$$

Now

$$\begin{aligned} & e^{-\pi i n^2/z + 2\pi i n((1/2)+(w/z)-(1/2z)) - \pi i(w-\frac{1}{2})^2/z} \\ &= e^{-\pi i(n+\frac{1}{2})^2/z + 2\pi i(n+\frac{1}{2})(w/z-1/2) - \pi i w^2/z + \pi i/2}. \end{aligned}$$

Thus

$$f(w,z) = \frac{ie^{-\pi i w^2/z}}{\sqrt{\frac{z}{i}}} f\left(\frac{w}{z}, -\frac{1}{z}\right),$$

which is precisely (49). Thus

Proposition 5. *For the elliptic theta-function $\vartheta_1(w,z)$ defined by the infinite product* (36)*, we have the series-expansion*

$$\vartheta_1(w,z) = \sum_{n=-\infty}^{\infty} e^{\pi i(n+\frac{1}{2})^2 z + 2\pi i(n+\frac{1}{2})(w-\frac{1}{2})}.$$

Using the method employed above, we shall also prove

Proposition 6. *The Dedekind η-function $\eta(z)$ has the infinite series expansion*

$$\eta(z) = e^{\pi i z/12} \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} e^{\pi i z(3\lambda^2-\lambda)}$$

Proof. Let us define for $z \in \mathfrak{H}$,

$$f(z) = e^{\pi i z/12} \sum_{\lambda=-\infty}^{\infty} (-1)^{*} e^{\pi i z(3\lambda^2-\lambda)}.$$

The function $f(z)$ is regular in $\mathfrak{H}$ and moreover, it is clear that $f(z+1) = f(z)e^{\pi i/12}$. Again,

$$\begin{aligned} f(z) &= e^{\pi iz/12}\sum_{\lambda=-\infty}^{\infty} e^{2\pi iz(\lambda-\frac{1}{6}(1-1/z))^2-\pi iz(1-1/z)^2/12}\\ &= e^{-\pi i/12z}e^{\pi i/6}\sum_{\lambda=-\infty}^{\infty} e^{3\pi iz(\lambda-\frac{1}{6}(1-1/z))^2}\\ &= e^{-\pi i/12z}e^{\pi i/6}\sqrt{\frac{i}{3z}}\sum_{\lambda=-\infty}^{\infty} e^{-\pi i\lambda^2/3z-\pi i\lambda(1-1/z)/3}, \end{aligned}$$

using formula (52). Here $\sqrt{i/3z}$ denotes that branch which is positive for $z = i$. **41**
Now we split up $\sum_{\lambda=-\infty}^{\infty} e^{-\pi i/3z\lambda^2-\pi i\lambda/3(1-1/z)}$ as $g_0(z)+g_1(z)+g_2(z)$, where for $k = 0, 1, 2$,

$$g_k(z) = \sum_{n=-\infty}^{\infty} e^{-\pi i(3\mu+k)^2/3z-\pi i(3\mu+k)(1-1/z)/3}.$$

Clearly,

$$\begin{aligned} g_0(z) &= \sum_{\mu=-\infty}^{\infty}(-1)^{\mu}e^{-\pi i(3\mu^2-\mu)/z},\\ g_1(z) &= e^{-\pi i/3}g_0(z),\\ g_2(z) &= \sum_{\mu=-\infty}^{\infty} e^{-\pi i(9\mu^2+12\mu+4)/3z-\pi i(3\mu+2)(1-1/z)/3}\\ &= e^{-2\pi i/3z-2\pi i/3}\sum_{\mu=-\infty}^{\infty}(-1)^{\mu}e^{-3\pi i\mu(\mu+1)/z}. \end{aligned}$$

Now

$$\sum_{\mu=-\infty}^{\infty}(-1)^{\mu}e^{-3\pi i\mu(\mu+1)/z} = -\sum_{\mu=-\infty}^{\infty}(-1)^{\mu}e^{-3\pi i\mu(\mu+1)/z},$$

on replacing μ by $-1-\mu$, and hence $g_2(z) = 0$. Thus

$$\begin{aligned} f(z) &= e^{-\pi i/12z}\sqrt{\frac{i}{3z}}(e^{\pi i/6}+e^{-\pi i/6})\sum_{\mu=-\infty}^{\infty}(-1)^{\mu}e^{-\pi i(3\mu^2-\mu)/z}\\ &= \sqrt{\frac{i}{z}}f\left(-\frac{1}{z}\right). \end{aligned}$$

Asa consequence, the function $h(z) = f(z)/n(z)$ is invariant under the transformations $z \to z+1$ and $z \to -1/z$ and hence under all modular transformations. Moreover, since $\eta(z)$ does not vanish anywhere in $\mathfrak{H}$, $h(z)$ is regular in $\mathfrak{H}$; further, as z tends to infinity in the fundamental domain, $h(z) - 1$ tends to zero. Now by the same argument as for $\vartheta_1(w,z)$ above, we can conclude that $h(z) = 1$ and the proposition is proved. ◻ 42

The integers $(1/2)(3\lambda^2 - \lambda)$ for $\lambda = 1, 2, \ldots$ are the so-called 'pentagonal numbers'.

We now give another interesting application of Kronecker's second limit formula. Let us consider, once again, formula (37). The left-hand side may be looked upon as a trigonometric series in u and v; it is true, of course, that u and v are not entirely independent. We shall now see how, using the infinite series expansion of $\vartheta_1(w,z)$, Kronecker showed that the function

$$\vartheta_1(w,z)\vartheta_1(q,-r)e^{(\pi i(w-q)^2)/(z-r)}$$

has a valid Fourier expansion in u and v and derived a beautiful formula from (37). First,

$$\vartheta_1(w,z) = \sum_{n=-\infty}^{\infty} e^{\pi i z(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})(w-\frac{1}{2})},$$

$$\vartheta_1(q,-r) = \sum_{m=-\infty}^{\infty} e^{-\pi i r(m+\frac{1}{2})^2 + 2\pi i(m+\frac{1}{2})(q-\frac{1}{2})},$$

replacing m by $-m-1$, we have

$$\vartheta_1(q,-r) = \sum_{m=-\infty}^{\infty} e^{-\pi i r(m+\frac{1}{2})^2 - 2\pi i(m+\frac{1}{2})(q-\frac{1}{2})}.$$

Hence

$$\begin{aligned}&\vartheta_1(w,z)\vartheta_1(q,-r)\\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\pi i\{z(n+\frac{1}{2})^2 + 2(n+\frac{1}{2})(w-\frac{1}{2}) - r(m+\frac{1}{2})^2 - 2(m+\frac{1}{2})(q-\frac{1}{2})\}}.\end{aligned}$$

This double series converges absolutely and replacing n by $n+m$ in the inner sum, we have

$$\begin{aligned}
&\vartheta_1(w,z)\vartheta_1(q,-r)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\pi i(z-r)(m+\frac{1}{2})^2+2\pi i(m+\frac{1}{2})(w+nz-q)+\pi in^2z+2\pi in(w-\frac{1}{2})}\\
&= \sum_{n=-\infty}^{\infty}(-1)^n e^{\pi in^2z+2\pi inw-\pi i(w+nz-q)^2/(z-r)} \sum_{m=-\infty}^{\infty} e^{\pi i(z-r)(m+\frac{1}{2}+(w+nz-q)/(z-r))^2}
\end{aligned}$$

Applying the theta-transformation formula to the inner sum, we have **43**

$$\begin{aligned}
&\vartheta_1(w,z)\vartheta_1(q,-r)e^{\pi i(w-q)^2/(z-r)}\\
&= \sqrt{\frac{i}{z-r}}\sum_{n=-\infty}^{\infty}(-1)^n e^{\pi in^2z+2\pi inw-\pi i(n^2z^2+2nz(w-q))/(x-r)}\times\\
&\quad\times\sum_{m=-\infty}^{\infty}(-1)^m e^{-\pi im^2/(z-r)-2\pi im(w-q+nz)/(z-r)}\\
&= \sqrt{\frac{i}{z-r}}\sum_{m,n}(-1)^{mn+m+n_e-\pi i(m+nz)(m+nr)/(z-r)+2\pi i((m+nz)q-(m+nr)w)/(z-r)}
\end{aligned}$$

$\sqrt{i/z-r}$ being that branch which assumes the value 1 for $z-r=i$.

Let

$$Q(\xi,\eta)=\frac{i}{z-r}(\xi+\eta z)(\xi+\eta r)=a\xi^2+b\xi\eta+c\eta^n,$$

where

$$a=\frac{i}{z-r}, b=\frac{i(z+r)}{z-r}\quad\text{and}\quad c=\frac{izr}{z-r}.$$

Then

$$\begin{aligned}
&\vartheta_1(w,z)\vartheta_1(q,-r)e^{\pi i(w-q)^2/(z-r)}\\
&= \sqrt{\frac{i}{z-r}}\sum_{m,n}(-1)^{mn+m+n}e^{-\pi Q(m,n)+2\pi i((m+nz)q-(m+nr)w)/(z-r)}
\end{aligned}$$

Formula (37) now becomes

$$-\frac{1}{2\pi}\sum_{m,n}{}' \frac{e^{2\pi i(mu+nv)}}{Q(m,n)}=\log\frac{\sqrt{\frac{i}{z-r}}\sum_{m,n}(-1)^{mn+m+n}e^{-\pi Q(m,n)+2\pi i(mu+nv)}}{\eta(z)\eta(-r)}$$

It is remarkable that the quadratic form $Q(m,n)$ emerges undisturbed on the **44** right-hand side; the right-hand side is free from z and r, except for the factors $\sqrt{i/(z-r)}$ and $\eta(z)\eta(-r)$. To eliminate z and r therefrom, we differentiate $\vartheta_1(w,z)\vartheta_1(q,-r)e^{\pi i(w-q)^2/(z-r)}$ once with respect to w and q at $w=0$, $q=0$, and then we get

$$\vartheta_1'(0,z)\vartheta_1'(0,-r) = \sqrt{\frac{i}{z-r}}\sum_{m,n}(-1)^{mn+m+n}e^{-\pi Q(m,n)}\times \frac{2\pi i(m+nz)}{z-r}\cdot\frac{-2\pi i(m+nr)}{z-r},$$

i.e.

$$\eta^3(z)\eta^3(-r) = \left(\sqrt{\frac{i}{z-r}}\right)^3\sum_{m,n}(-1)^{(m+1)(n+1)}Q(m,n)e^{-\pi Q(m,n)},$$

i.e.

$$\eta(z)\eta(-r) = \sqrt{\frac{i}{z-r}}\left\{\sum_{m,n}(-1)^{(m+1)(n+1)}Q(m,n)e^{-\pi Q(m,n)}\right\}^{\frac{1}{3}}$$

the branch of the cube root on the right-hand side being determined by the condition that it is real and positive when $r=\bar{z}$, since $\eta(z)\eta(-\bar{z})>0$.

We have then the following *formula due to Kronecker*, namely

$$-\frac{1}{2\pi}\sum_{m,n}{}' \frac{e^{2\pi i(mu+nv)}}{Q(m,n)} = \log\frac{\sum_{m,n}(-1)^{mn+m+n}e^{-\pi Q(m,n)+2\pi i(mu+nv)}}{\left\{\sum_{m,n}(-1)^{(m+1)(n+1)}Q(m,n)e^{-\pi Q(m,n)}\right\}^{\frac{1}{3}}}. \tag{53}$$

It is remarkable that, eventually, on the right-hand side of (53), z and r do not appear explicitly and only the quadratic form $Q(m,n)$ appears on the right.

The quadratic form $Q(\xi,n) = d\xi^2+b\xi\eta+c\eta^2$ introduced above is a complex quadratic form with discriminant

$$b^2-4ac = \frac{-(r+z)^2+4rz}{(z-r)^2} = -1.$$

Moreover,its real part is positive-definite for real ξ and η. For proving this, **45**
we have to show that for $(\xi,n)\neq(0,0)$, $\mathrm{Re}\left(\frac{i}{z-r}(\xi+z\eta)\cdot(\xi+r\eta)\right)>0$, i.e. $\mathrm{Im}\left(-\frac{1}{z-r}(\xi+zn)(\xi+r\eta)\right)>0$. If $\eta=0$, this is trivial. If $\eta\neq 0$, we may take $\eta=1$ without loss of generality and then we have only to show that

$$\mathrm{Im}\left(\frac{z-r}{(\xi+z)(\xi+r)}\right)>0, \text{ i.e., } \mathrm{Im}\left(\frac{1}{\xi+r}-\frac{1}{\xi-z}\right)>0.$$

But this is obvious, since z, $-r \in \mathfrak{H}$ and ξ being real, $\xi + z$, $-(\xi + r)$ and hence, $-1/(\xi + z)$, $1/(\xi + r)$ are all in $\mathfrak{H}$.

Conversely, if $Q(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$ is a complex quadratic form with real part positive-definite for real ξ and η and discriminant -1 and if Re $a > 0$, it can be written as $\dfrac{i}{z-r}(\xi+\eta z)(\xi+\eta r)$ for some z, $-r \in \mathfrak{H}$. In fact, if Re $a > 0$, $Q(\xi, \eta) = a(\xi + \eta z)(\xi + \eta r)$, where z, r are the roots of the quadratic equation $a\lambda^2 - b\lambda + c = 0$. Since the real part of $Q(\xi, \eta)$ is positive definite, z and r have both got to be complex. Now the discriminant of $Q(\xi, \eta)$ is -1 and hence $a^2(z-r)^2 = -1$, i.e. $a = \pm i/(z-r)$. We may take $a = i/(z-r)$ and then $z-r \in \mathfrak{H}$. Now $Q(\xi, \eta) = \dfrac{i}{z-r}(\xi + \eta z)(\xi + \eta r)$. We have only to show that z, $-r \in \mathfrak{H}$. Consider the expression $Q(\xi, 1) = \dfrac{i}{z-r}(\xi + z)(\xi + r)$. If both z and r are in $\mathfrak{H}$ then as ξ varies from $-\infty$ to $+\infty$, the argument of $Q(\xi, 1)$ decreases by 2π and if both $-z$ and $-r$ are in $\mathfrak{H}$ then as ξ varies from $-\infty$ to $+\infty$, the argument of $Q(\xi, 1)$ increases by 2π. But neither case is possible, since Re $Q(\xi, 1) > 0$ and so $Q(\xi, 1)$ lies always in the right half-plane for real ξ. Hence either z, $-r$ or $-z$, $r \in \mathfrak{H}$. But, again, since $z - r \in \mathfrak{H}$ we see that z, $-r \in \mathfrak{H}$.

When we subject the quadratic form $Q(\xi, \eta)$ to the unimodular transformation $(\xi, \eta) \to (\alpha\xi + \beta\eta, \gamma\xi + \delta\eta)$ where α, β, γ, δ are integers such that $\alpha\delta - \beta\gamma = \pm 1$, then $Q(\xi, \eta)$ goes over into the quadratic form $Q^*(\xi, \eta) = Q(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta)$. The quadratic forms $Q^*(\xi, \eta)$ and $Q(\xi, n)$ are said to be *equivalent*. $Q^*(\xi, \eta)$ again has discriminant -1 and its real part is positive-definite for real ξ and η.

Let $u^* = \alpha u + \gamma v$ and $v^* = \beta u + \delta v$. Then we assert that the right-hand side of (53) is invariant if we replace $Q(m, n)$, u and v respectively by$Q^*(m, n)$, **46**
u^* and v^*. This is easy to prove, for the series on the right-hand side converge absolutely and when m, n run over all integers independently, then so do $m^* = \alpha m + \beta n$, $n^* = \gamma m + \delta n$. And all we need to verify is that $(-1)^{(m^*+1)(n^*+1)} = (-1)^{(m+1)(n+1)}$. This is very simple to prove; for $(-1)^{(m+1)(n+1)} = +1$ unless m and n are both even; and m and n are both even if and only if m^* and n^* are both even.

The invariance of the left hand side of (53) under the transformation $Q(m, n)$, u, v to $Q^*(m, n)$, u^*, v^* could be also directly proved by observing that for Re $s > 1$, $\sum' \dfrac{e^{2\pi i(mu+nv)}}{(Q)(m,n)^s}$ is invariant under this transformation and hence, by analytic continuation, the invariance holds good even for $s = 1$.

5 The Epstein Zeta-function

We shall consider here a generalization of the function $\zeta_Q(s,u,v)$ introduced in § 3. Let Q be the matrix of a positive-definite quadratic form $Q(x_1,\ldots,x_n)$ in the n variables $x_1,\ldots,x_n (n \geq 2)$. Further let $\underline{x}$ denote the n-rowed column $\begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}$.

A homogeneous polynomial $\mathscr{P}(\underline{x}) = \mathscr{P}(x_1,\ldots,x_n)$ of degree g in $x_1,\ldots,x_n$ is called a **spherical function** (Kugel-funktion) **of order** g with respect to $Q(x_1,\ldots,x_n)$, if it satisfies the differential equation

$$\sum_{1\leq i,j\leq n} q^*_{ij} \frac{\partial^2 \mathscr{P}(\underline{x})}{\partial x_i \partial x_j} = 0,$$

where the matrix $(q^*_{ij}) = Q^{-1}$.

Let, for a matrix A, A' denote its transpose; we shall abbreviate $A'QA$ as $Q[A]$, for an n-rowed matrix A. By an *isotropic vector* of Q, we mean a complex column $\underline{w}$ satisfying $Q[\underline{w}] = 0$. It is easily verified that if $\underline{w}$ is an isotropic vector of Q, then the polynomial $(\underline{x}'Q\underline{w})^g$ is a spherical function of order g with respect to $Q(x_1,\ldots,x_n)$. Moreover, it is known thatif $\mathscr{P}(\underline{x})$ is a spherical 47
function of order g with respect to $Q(x_1,\ldots,x_n)$, then $\mathscr{P}(\underline{x}) = \sum_{m=1}^{M} (\underline{x}'Q\underline{w}_m)^g$, $\underline{w}_1,\ldots,\underline{w}_M$ being isotropic vectors of Q.

Let $\underline{u}$, $\underline{v}$ be two arbitrary n-rowed real columns and g, a non-negative integer. Further, let $\mathscr{P}(\underline{x})$ be a spherical function of order g with respect to $Q(x_1,\ldots,x_n)$. We define, for $\sigma > n/2$, the zeta-function

$$\zeta(s,\underline{u},\underline{v},Q,\mathscr{P}) = \sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{2\pi i \underline{m}'\underline{u}} \frac{\mathscr{P}(\underline{m}+\underline{v})}{(Q[\underline{m}+\underline{v}])^{s+g/2}}$$

where $\underline{m}$ runs over all n-rowed integral columns such that $\underline{m}+\underline{v} \neq \underline{0}$ ($\underline{0}$ being the n-rowed zero-column). By the remark on $\mathscr{P}(\underline{x})$ above, $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ is a linear combination of the series of the form

$$\sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{2\pi i \underline{m}'\underline{u}} \frac{((\underline{m}+\underline{v})'Q\underline{w})^g}{(Q[\underline{m}+\underline{v}])^{s+g/2}},$$

$\underline{w}$ being an isotropic vector of Q. These series have been investigated in detail by Epstein; in special cases as when $g = 0$ and Q is diagonal, Lerch has considered them independently and has derived for them a functional equation which he refers to as the "generalized Malmsten-Lipschitz relation."

The series introduced above converge absolutely for $\sigma > n/2$, uniformly in every half-plane $\sigma \geq n/2 + \epsilon(\epsilon > 0)$, as can be seen from the fact that they have a majorant of the form

$$c \cdot \sum_{\underline{m}+\underline{v}\neq 0} \frac{\left(\sum\limits_{i=1}^{n} |m_i + v_i|\right)^{g}}{\left(\sum\limits_{i=1}^{n} |m_i + v_i|^2\right)^{\sigma+g/2}}$$

for a constant c depending only on Q, $\underline{w}$ and g. Hence they represent analytic functions of s for $\sigma > n/2$. Thus $\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P})$ is an analytic function of s for $\sigma > n/2$.

We shall study the analytic continuation and the functional equation of $\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P})$. The proof will be based on Riemann's method of obtaining the **48**
functional equation of the ζ-function using the theta-transformation formula.

First we obtain the following generalization of (52).

Proposition 7. *If Q is the matrix of a positive-definite quadratic form in m variables with real coefficients and $\underline{v}$ an n-rowed complex column, then*

$$\sum_{\underline{m}} e^{-\pi q[\underline{m}+\underline{v}]} = |Q|^{-\frac{1}{2}} \sum_{\underline{m}} e^{-\pi Q^{-1}[\underline{m}]+2\pi \underline{m}'\underline{v}}, \tag{54}$$

where, on both sides $\underline{m}$ runs over all n-rowed integral columns and $|Q|^{\frac{1}{2}}$ is the positive square root of $|Q|$, the determinant of Q.

Proof. We shall assume formula (54) proved for Q and $\underline{v}$ of at most $n-1$ rows and uphold it for n. For $n = 1$, the formula has been proved already.

Now, it is well known that

$$A = \begin{pmatrix} P & \underline{q} \\ \underline{q}' & r \end{pmatrix} = \begin{pmatrix} P & \underline{0} \\ \underline{0}' & r - P^{-1}[\underline{q}] \end{pmatrix} \begin{bmatrix} E & P^{-1}\underline{q} \\ \underline{0}' & 1 \end{bmatrix} \tag{55}$$

where P is $(n-1)$-rowed and symmetric and E, the $(n-1)$-rowed identity matrix. Setting $\lambda = r - P^{-1}[\underline{q}]$ and writing $\underline{m} = \binom{\underline{k}}{1}$ and $\underline{v} = \binom{\underline{u}}{w}$ with $\underline{k}$ and $\underline{u}$ of $n-1$ rows, we have, in view of (55),

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]} = \sum_{1} E^{-\pi\lambda(l+w)^2} \sum_{\underline{k}} e^{-\pi P[\underline{k}+\underline{u}+P^{-1}\underline{q}(l+w)]},$$

$\underline{m}$, $\underline{k}$ running over all n-rowed and $(n-1)$-rowed integral columns respectively and l over all integers. By induction hypothesis,

$$\sum_{\underline{k}} e^{-\pi P[\underline{k}+\underline{u}+P^{-1}\underline{q}(l+w)]} = |P|^{-\frac{1}{2}} \sum_{\underline{k}} e^{-\pi P^{-1}[\underline{k}]+2\pi i \underline{k}'(\underline{u}+P^{-1}\underline{q}(l+w))}$$

Thus

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]} = |P|^{-\frac{1}{2}} \sum_{\underline{k}} e^{-\pi P^{-1}}[\underline{k}] + 2\pi i\underline{k}'\underline{u} \sum_{l} e^{-\pi\lambda(l+w)^2+2\pi i\underline{k}'P^{-1}\underline{q}^{(l+w)}}$$

Now, by (52), **49**

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty} e^{-\pi\lambda(l+w)^2+2\pi i\underline{k}'P^{-1}\underline{q}(l+w)} \\
&= e^{-\pi\lambda w^2+2\pi i\underline{k}'P^{-1}\underline{q}^{w}} \sum_{l=-\infty}^{\infty} e^{-\pi\lambda l^2-2\pi i l(\underline{k}'P^{-1}\underline{q}+i\lambda w)} \\
&= \lambda^{-\frac{1}{2}} e^{-\pi\lambda w^2+2\pi i\underline{k}'P^{-1}\underline{q}^{w}} \sum_{l=-\infty}^{\infty} e^{-\pi\lambda^{-1}(l-\underline{k}'P^{-1}\underline{q}-i\lambda\omega)^2} \\
&= \lambda^{-\frac{1}{2}} \sum_{l=-\infty}^{\infty} e^{-\pi\lambda^{-1}(l-\underline{k}'P^{-1}\underline{q})^2+2\pi i l w}
\end{aligned}$$

But $|Q| = |P|\lambda$ and $Q^{-1}[\underline{m}] = P^{-1}[\underline{k}] + \lambda^{-1}(-\underline{k}'P^{-1}\underline{q} + 1)^2$ and hence

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]} = |Q|^{-\frac{1}{2}} \sum_{\underline{k},l} e^{-\pi Q^{-1}|\underline{m}|+2\pi i(\underline{k}'\underline{u}+lw)}$$

and (54) is proved. □

Formula (54) can also be upheld by using the Poisson summation formula in n variables. Further, it can be shown to be *valid* even for *complex symmetric Q whose real part is positive* (i.e. the matrix of a positive-definite quadratic form). For, Q^{-1} again has positive real part and the absolute convergence of the series on both sides is ensured. Moreover, considered as functions of the $n(n+1)/2$ independent elements of Q, they represent analytic functions and since they are equal for all real positive Q, we see, by analytic continuation, that they are equal for all complex symmetric Q with positive real part.

Let $\underline{u}$ be another arbitrary complex n-rowed column. Replacing $\underline{v}$ by $\underline{v} - iQ^{-1}\underline{u}$ in (54), we obtain

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}-iQ^{-1}\underline{u}]} = |Q|^{-\frac{1}{2}} \sum_{\underline{m}} e^{-\pi Q^{-1}[\underline{m}]+2\pi i\underline{m}'(\underline{v}-iQ^{-1}\underline{u})},$$

i.e.

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}} = |Q|^{-\frac{1}{2}} \sum_{\underline{m}} e^{-\pi Q^{-1}[\underline{m}-\underline{u}]+2\pi i(\underline{m}-\underline{u})'\underline{v}}. \tag{56}$$

Let $\underline{w}$ be an isotropic vector of Q and λ, an arbitrary complex number. Replacing$\underline{v}$ by $\underline{v}+\lambda\underline{w}$ in (56), we have 50

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]-2\pi\lambda(\underline{m}+\underline{v})'Q\underline{w}+2\pi i\underline{m}'\underline{u}}$$
$$= |Q|^{-\frac{1}{2}} \sum_{\underline{m}} e^{-\pi Q^{-1}[\underline{m}-\underline{u}]+2\pi i(\underline{m}-\underline{u})'\underline{v}+2\pi i\lambda(\underline{m}-\underline{u})'\underline{w}}.$$

Both sides represent analytic functions of λ and differentiating them both, g times with respect to λ at $\lambda = 0$, we get

$$\sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}}((\underline{m}+\underline{v})'Q\underline{w})^g$$
$$= \frac{e^{-2\pi i\underline{u}'\underline{v}}}{i^g}|Q|^{-\frac{1}{2}} \sum_{\underline{m}} e^{-\pi Q^{-1}[\underline{m}-\underline{u}]+2\pi i\underline{m}'\underline{v}}((m-\underline{u})'\underline{w})^g. \tag{57}$$

Associated with a spherical function $\mathscr{P}(\underline{x})$ with respect to $Q[\underline{x}]$, let us define $\mathscr{P}^*(\underline{x}) = \mathscr{P}(Q^{-1}\underline{x})$; $\mathscr{P}^*(\underline{x})$ is a spherical function of order g with respect to $Q^{-1}[\underline{x}]$. Moreover $\mathscr{P}^{**}(\underline{x}) = \mathscr{P}(\underline{x})$. Further, if $\mathscr{P}(\underline{x}) = \sum\limits_{i=1}^{m}(\underline{x}'Q\underline{w}_i)^g$, then $\mathscr{P}^*(\underline{x}) = \sum\limits_{i=1}^{m}(\underline{x}'\underline{w}_i)^g$. Thus, if we set

$$f(Q,\underline{u},\underline{v},\mathscr{P}) = \sum_{\underline{m}} e^{-\pi Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v}),$$

where $\underline{m}$ runs over all n-rowed integral columns, then we see at once from (57) that $f(Q,\underline{u},\underline{v},\mathscr{P})$ satisfies the functional equation

$$i^g e^{2\pi i\underline{u}'\underline{v}} f(Q,\underline{u},\underline{v},\mathscr{P}) = |Q|^{-\frac{1}{2}} f(Q^{-1},\underline{v},-\underline{u},\mathscr{P}^*).$$

If now we replace Q by xQ for $x > 0$, then $\mathscr{P}(\underline{x})$ goes into $x^g\mathscr{P}(\underline{x})$ and $\mathscr{P}^*(\underline{x})$ remains unchanged and we have

$$i^g e^{2\pi i\underline{u}'\underline{v}} f(xQ,\underline{u},\underline{v},\mathscr{P}) = x^{-n/2-g}|Q|^{-\frac{1}{2}} f(x^{-1}Q^{-1},\underline{v},-\underline{u},\mathscr{P}^*). \tag{58}$$

Thus we have

Proposition 8. *Let Q be an m-rowed complex symmetric matrix with positive real part, $\mathscr{P}(\underline{x})$ a spherical function of order g with respect to Q and $\mathscr{P}^*(\underline{x})$ –*

$\mathscr{P}(Q^{-1}\underline{x})$. *Let u and v be two arbitrary m-rowed complex columns and* $x > 0$. *Then*

$$e^{2\pi i\underline{u}'\underline{v}_i g}\sum_{\underline{m}} e^{-\pi x Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v})$$
$$= x^{-n/2-g}|Q|^{-\frac{1}{2}}\sum_{\underline{m}} e^{-\pi x^{-1}Q^{-1}[\underline{m}-\underline{u}]+\pi i\underline{m}'\underline{v}}\mathscr{P}^*(\underline{m}-\underline{u}).$$

51

To study the analytic continuation of $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$, we obtain an integral representation of the same, by using the well-known formula due to Euler, namely, for $t > 0$ and Re $s > 0$,

$$\pi^{-s}\Gamma(s)t^{-s} = \int_0^\infty x^s e^{-\pi t x}\frac{dx}{x}.$$

For $\sigma > n/2$, we then have

$$\pi^{-(s+g/2)}\Gamma\left(s+\frac{g}{2}\right)\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$$
$$= \sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{2\pi i\underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v})\int_0^\infty x^{s+g/2}e^{-\pi x Q[\underline{m}+\underline{v}]}\frac{dx}{x}$$
$$= \int_0^\infty x^{s+g/2}\left\{\sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{-\pi x Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v})\right\}\frac{dx}{x}, \tag{59}$$

in view of the absolute convergence of the series, uniform for $\sigma \geq n/2+\epsilon(\epsilon > 0)$.

Let now, for an n-rowed real column $\underline{x}$,

$$\rho(\underline{x},g) = \begin{cases} 0, & \text{if } \underline{x} \text{ is not integral,} \\ 1, & \text{if } \underline{x} \text{ is integral and } g = 0, \\ 0, & \text{if } \underline{x} \text{ is integral and } g > 0. \end{cases}$$

Then

$$\sum_{\underline{m}+\underline{v}\neq 0} e^{-\pi x Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'u}\mathscr{P}(\underline{m}+\underline{v}) = f(xQ,\underline{u},\underline{v},\mathscr{P}) - e^{-2\pi i\underline{u}'\underline{v}}\rho(\underline{v},g).$$

From (59), we see, as a consequence, that

$$\pi^{-(s+g/2)}\Gamma\left(s+\frac{g}{2}\right)\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$$

$$= \int_1^\infty x^{s+g/2}\left\{\sum_{\underline{m}+\underline{v}\neq 0} e^{-\pi x Q[\underline{m}+\underline{v}]+2\pi i \underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v})\right\}\frac{dx}{x}+$$
$$+\int_0^1 x^{s+g/2} f(xQ,\underline{u},\underline{v},\mathscr{P})\frac{dx}{x} - \rho(\underline{v},g)e^{-2\pi i \underline{u}'\underline{v}}\int_0^1 x^{s+g/2}\frac{dx}{x}.$$

In view of (58), 52

$$\int_0^1 x^{s+g/2} f(xQ,\underline{u},\underline{v},\mathscr{P})\frac{dx}{x}$$
$$= \frac{|Q|^{-\frac{1}{2}}}{i^g e^{2\pi i \underline{u}'\underline{v}}}\int_0^1 x^{s-g/2-n/2} f(x^{-1}Q^{-1},\underline{v},-\underline{u},\mathscr{P}^*)\frac{dx}{x}.$$

Again, since

$$f(x^{-1}Q^{-1},\underline{v},-\underline{u},\mathscr{P}^*)$$
$$= \sum_{\underline{m}-\underline{u}\neq \underline{0}} e^{-\pi x^{-1}Q^{-1}[\underline{m}-\underline{u}]+2\pi i \underline{m}'\underline{v}}\mathscr{P}^*(\underline{m}-\underline{u}) + \rho(\underline{u},g)e^{2\pi i \underline{u}'\underline{v}},$$

we have

$$\int_0^1 x^{s-g/2-n/2} f(x^{-1}Q^{-1},\underline{v},-\underline{u},\mathscr{P}^*)\frac{dx}{x}$$
$$= \int_0^1 x^{s-g/2-n/2}\left\{\sum_{\underline{m}-\underline{u}\neq \underline{0}} e^{-\pi x^{-1}Q^{-1}[\underline{m}-\underline{u}]+2\pi i \underline{m}'\underline{v}}\mathscr{P}^*(\underline{m}-\underline{u})\right\}\frac{dx}{x}+$$
$$+\rho(\underline{u},g)e^{2\pi i \underline{u}'\underline{v}}\int_0^1 x^{s-(g/2)-(n/2)}\frac{dx}{x}$$
$$= \int_1^\infty x^{(n/2)-s+(g/2)}\left\{\sum_{\underline{m}-\underline{u}\neq \underline{0}} e^{-\pi x Q^{-1}[\underline{m}-\underline{u}]+2\pi i \underline{m}'\underline{v}}\mathscr{P}^*(\underline{m}-\underline{u})\right\}\frac{dx}{x}+$$
$$+\frac{\rho(\underline{u},g)e^{2\pi i \underline{u}'\underline{v}}}{s-\frac{g}{2}-\frac{n}{2}}.$$

Thus, for $\sigma > n/2$,

$$\pi^{-(s+g/2)}\Gamma\left(s+\frac{g}{2}\right)\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$$
$$= \frac{|Q|^{-\frac{1}{2}}\rho(\underline{u},g)}{i^g\left(s-\frac{g}{2}-\frac{n}{2}\right)} - \frac{\rho(\underline{v},g)e^{-2\pi i \underline{u}'\underline{v}}}{s+\frac{g}{2}}+$$

$$+\left[\int_1^\infty x^{s+g/2}\left\{\sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{-\pi x Q[\underline{m}+\underline{v}]+2\pi i\underline{m}'\underline{u}}\mathscr{P}(\underline{m}+\underline{v})\right\}\frac{dx}{x}+\right.$$

$$\left.+\frac{|Q|^{-\frac{1}{2}}}{i^g e^{2\pi i\underline{u}'\underline{v}}}\int_1^\infty x^{(n/2)-s+(g/2)}\left\{\sum_{\underline{m}-\underline{u}\neq\underline{0}} e^{-\pi x Q^{-1}[\underline{m}-\underline{u}]+2\pi i\underline{m}'\underline{v}}\times\mathscr{P}^*(\underline{m}-\underline{u})\right\}\frac{dx}{x}\right]. \tag{60}$$

53

One verifies easily that the functions within the square brackets on the right-hand side of (60) are entire functions of s. Since $\dfrac{\pi^{s+g/2}}{\Gamma(s+g/2)}$ is also an entire function of s, formula (60) gives the analytic continuation of $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ into the whole s-plane. The only possible singularities arise from those of

$$\frac{\pi^{s+g/2}}{\Gamma\left(s+\frac{g}{2}\right)}\left\{\frac{|Q|^{-\frac{1}{2}}\rho(\underline{u},g)}{i^g\left(s-\frac{g}{2}-\frac{n}{2}\right)}-\frac{\rho(\underline{v},g)e^{-2\pi i\underline{u}'\underline{v}}}{s+\frac{g}{2}}\right\}$$

Now

$$\frac{\pi^{s+g/2}}{\Gamma\left(s+\frac{g}{2}\right)\left(s+\frac{g}{2}\right)}=\frac{\pi^{s+g/2}}{\Gamma\left(s+\frac{g}{2}+1\right)}$$

is an entire function of s and so $s=-g/2$ cannot be a singularity of $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$. Moreover, if $g>0$ or if $\underline{u}$ is not integral, then $\rho(\underline{u},g)=0$ and hence in these cases, $s=(g+n)/2$ can not be a pole of $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$. If $g=0$, and $\underline{u}$ is integral, $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ has a simple pole at $s=n/2$ with residue $\pi^{n/2}|Q|^{-\frac{1}{2}}/\Gamma(n/2)$. Moreover, from (60) it is easy to verify that $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ satisfies the functional equation

$$\pi^{-s}\Gamma\left(s+\frac{g}{2}\right)\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$$

$$=\frac{e^{-2\pi i\underline{u}'\underline{v}}}{i^g}|Q|^{-\frac{1}{2}}\pi^{-(n/2-s)}\Gamma\left(\frac{n}{2}-s+\frac{g}{2}\right)\zeta\left(\frac{n}{2}-s,\underline{v},-\underline{u},Q^{-1},\mathscr{P}^*\right) \tag{61}$$

Wehave then finally. 54

Theorem 3. *The function $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ has an analytic continuation into the whole s-plane, which is an entire function of s if either $g>0$ or if $g=0$ and $\underline{u}$ is not integral. If $g=0$ and $\underline{u}$ is integral, then $\zeta(s,u,v,Q,\mathscr{P})$ is meromorphic in the entire s-plane with the only singularity at $s=n/2$ where it has a simple pole with the residue $\pi^{n/2}/(|Q|^{\frac{1}{2}}\Gamma(n/2))$. In all cases, $\zeta(s,\underline{u},\underline{v},Q,\mathscr{P})$ satisfies the functional equation* (61).

Let now $n = 2$ and $Q(x_1, x_2) = y^{-1}|x_1 + x_2 z|^2$, in our earlier notation. Let $\underline{x} = \binom{x_1}{x_2}$, $\underline{m} = \binom{m_1}{m_2}$, $\underline{u} = \binom{u_2}{-u_1}$ and $\underline{v} = \binom{v_1}{v_2}$. Both $\binom{-z}{1}$ and $\binom{-\bar{z}}{1}$ are isotropic vectors of $Q[\underline{x}]$ but we shall take the spherical function $\mathscr{P}(\underline{x}) = (-i(x_1 + x_2 z))^g$, corresponding to the former. Then by the above, the function

$$\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P}) = \frac{y^{s+g/2}}{i^g} \sum_{\underline{m}+\underline{v}\neq\underline{0}} e^{2\pi i(m_1 u_2 - m_2 u_1)} \frac{(m_1 + v_1 + z(m_2 + v_2))^g}{|m_1 + v_1 + z(m_2 + v_2)|^{2s+g}}$$

satisfies the functional equation

$$\begin{aligned}&\pi^{-s}\Gamma\left(s + \frac{g}{2}\right)\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P})\\ &\qquad = \frac{e^{-2\pi i \underline{u}'\underline{v}}}{i^g}\pi^{-(1-s)}\Gamma\left(1 - s + \frac{g}{2}\right)\zeta(1 - s, \underline{v}, -\underline{u}, Q^{-1}, \mathscr{P}^*).\end{aligned}$$

Now $Q^{-1}[\underline{x}]$ has the simple form $y^{-1}|x_1 z - x_2|^2$ and moreover, $\mathscr{P}^*(\underline{x}) = (-x_1 z + x_2)^g$. Thus for $\sigma > 1$,

$$\begin{aligned}&\zeta(s, \underline{v}, -\underline{u}, Q^{-1}, \mathscr{P}^*)\\ &= y^{s+g/2} \sum_{\underline{m}-\underline{u}\neq\underline{0}} e^{2\pi i(m_1 v_1 + m_2 v_2)} \frac{(-(m_1 - u_2)z + m_2 + u_1)^g}{|-z(m_1 - u_2) + m_2 + u_1|^{2s+g}}\\ &= y^{s+g/2} \sum_{\binom{m_1}{m_2}+\binom{u_1}{u_2}\neq 0} e^{2\pi i(m_1 v_2 - m_2 v_1)} \frac{((m_1 + u_1) + z(m_2 + u_2))^g}{|(m_1 + u_1) + z(m_2 + u_2)|^{2s+g}}.\end{aligned}$$

Let us now define for $\underline{u}^* = \binom{u_1}{u_2}$, $\underline{v}^* = \binom{v_1}{v_2}$, the function

$$\zeta(s, \underline{u}^*, \underline{v}^*, z, g) = y^s \sum_{\underline{m}+\underline{v}^*\neq\underline{0}} e^{2\pi i(m_1 u_2 - m_2 u_1)} \frac{(m_1 + v_1 + z(m_2 + v_2))^g}{|m_1 + v_1 + z(m_2 + v_2)|^{2s+g}},$$

for$\sigma > 1$. It is clear that $\zeta(s, \underline{u}^*, \underline{v}^*, z, g) = i^g y^{-g/2}\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P})$ and from 55
(**??**) we see that $\zeta(s, \underline{v}^*, \underline{u}^*, z, g) = y^{-g/2}\zeta(s, \underline{v}, -\underline{u}, Q^{-1}, \mathscr{P}^*)$. If now we defined $\varphi(s, \underline{u}^*, \underline{v}^*, z, g) = \pi^{-s}\Gamma(s + g/2)\zeta(s, \underline{u}^*, \underline{v}^*, z, g)$, we deduce from above that $\varphi(s, \underline{u}^*, \underline{v}^*, z, g)$ satisfies the nice functional equation

$$\varphi(s, \underline{u}^*, \underline{v}^*, z, g) = e^{2\pi i(u_1 v_2 - u_2 v_1)}\varphi(1 - s, \underline{v}^*, \underline{u}^*, z, g).$$

In the case $g > 0$, $\zeta(s, \underline{u}^*, \underline{v}^*, z, g)$ is an entire function of s and one can ask for analogues of Kronecker's second limit formula even here. For even g, one gets by applying the Poisson summation formula, a limit formula which is connected with elliptic functions. For odd g, the limit formulas which one gets are more complicated and involve Bessel functions. If we take the particular case $g = 1$, there occurs in the work of Hecke, a limit formula (as s tends to $1/2$) which has an interesting connection with the theory of complex multiplication.

For $n > 2$, Epstein has obtained for $\underline{u} = \underline{0}$, $\underline{v} = \underline{0}$ and $g = 0$, an analogue of the first limit formula of Kronecker. This formula involves more complicated functions that the Dedekind η-function. Perhaps, in general for $n > 2$, one cannot expect to get a limit formula which would involve analytic functions of several complex variables.

Bibliography

[1] S. Bochner: *Vorlesungen über Fouriersche Integrale*, Chelsea, New York, (1948).

[2] R. Dedekind: *Erläuterungen zu den Fragmenten XXVIII, Collected Works of Bernhard Riemann*, Dover, New York, (1953), 466-478.

[3] P. Epstein: Zur Theorie allgemeiner Zeta funktionen, I, II, *Math, Ann.* 56 (1903), 615-644, ibid, 63 (1907), 205-216.

[4] E. Hecke: Bestimmung der Perioden gewisser Integrale durch die Theorie der Klassenkorper, *Werke.* Gottingen (1959), 505-524.

[5] F. Klein: Uber die elliptischen Normalkurven der *n*-ten Ordnung, *Gesamm. Math. Abhandlungen.* Bd. III. Berlin. (1923). 198-254.

[6] Kronecker:Zur Theorie der Clliptischen Funktionen, *Werke.* Vol. 4. 347- **56**
495, vol. 5, 1-132, Leipzig, (1929).

[7] M. Lerch: Zakladové Theorie Malmstenovskych rad, *Rozpravy česke akademie Cišare Frantiska Josefa*, Bd. 1, No. 27, (1892), 525-592.

[8] C. Meyer: *Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörpern,* Akademie Verlag, Berlin, 1957.

[9] H. Rademacher: Zur Theorie der Modulfunktionen, *J. für die reine u. angew. Math.,* vol. 167 (1931), 312-336. Collected papers.

[10] B. Schoenesberg: Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen, *Math. Ann.* 116 (1939), 511-523.

[11] C. L. Siegel: A simple proof of $\eta\left(-\frac{1}{\eta}\right) = \eta(\tau)\sqrt{\frac{\tau}{i}}$, *Mathematika* 1 (1954), 4. Ges Abhand.

[12] E. T. Whittaker and G. N. Watson: *Course of Modern Analysis,* Cambridge, (1946).

Chapter 2

Applications of Kronecker's Limit Formulas to Algebraic Number Theory

1 Kronecker's solution of Pell's equation

Let K be an algebraic number field of degree n over $\mathbf{Q}$, the field of rational 57
numbers. Let, for any ideal $\mathfrak{a}$ in K, $N(\mathfrak{a})$ denote its norm. Further let $s = \sigma + it$ be a complex variable. Then for $\sigma > 1$, we define after Dedekind, the zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N(\mathfrak{a}))^{-s},$$

where the summation is over all non-zero integral ideals of K.

More generally, with a character χ of the ideal class group of K, we associate the L-series

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N(\mathfrak{a}))^{-s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1},$$

for $\sigma > 1$. The product on the right runs over all prime ideals $\mathfrak{p}$ of K.

We have h such series associated with all the h characters of the ideal class group of K. It has been proved by Hecke that these functions $L_K(s,\chi)$ can be continued analytically into the whole plane and they satisfy a functional equation for $s \to 1 - s$.

Now, χ being an ideal class character, we may write

$$L_K(s,\chi) = \sum_A \chi(A) \sum_{\mathfrak{a}\in A} (N(\mathfrak{a}))^{-s},$$

A running over all the ideal classes of K. If we define $\zeta(s,A) = \sum_{\mathfrak{a}\in A}(N(\mathfrak{a}))^{-s}$, then it can be shown that $\lim_{s\to 1}(s-1)\zeta(s,A) = \kappa$, a positive constant depending **58**
only upon the field K and *not* upon the ideal class A. (For example, in the case of an imaginary quadratic field of discriminant d, $\kappa = 2/w\sqrt{-d}$, w being the number of roots of unity in K and $\kappa = 2\log\epsilon/\sqrt{d}$ in the case of a real quadratic field of discriminant d, where ϵ is the fundamental unit and $\epsilon > 1$). Since $\zeta_K(s) = \sum_A \zeta(s,A)$, we have

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \sum_A \lim_{s\to 1}(s-1)\zeta(s,A) = \kappa\cdot h$$

where h denotes the class number of K.

From now on, we shall be concerned only with quadratic fields of discriminant d.

We have then, on direct computation,

$$\zeta_K(s) = \zeta(s)L_d(s), \tag{63}$$

where $\zeta(s)$ is the Riemann zeta-function and $L_d(s)$ is defined as follows:

$$L_d(s) = \sum_{n=1}^{\infty}\left(\frac{d}{n}\right)n^{-s}, \quad (\sigma > 1)$$

$\left(\frac{d}{n}\right)$ being the Legendre-Jacobi-Kronecker symbol.

Then

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \lim_{s\to 1}(s-1)\zeta(s)L_d(s) = \lim_{s\to 1} L_d(s) = L_d(1),$$

since $L_d(s)$ converges in the half-plane $\sigma > 0$ and $\lim_{s\to 1}(s-1)\zeta(s) = 1$.

We obtain therefore,

$$L_d(1) = \sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n} = \kappa\cdot h. \tag{64}$$

For the determination of the left-hand side, we consider more generally, for any character $\chi(\neq 1)$ of the ideal class group of K,

$$\lim_{s\to 1} L_K(s,\chi) = L_K(1,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})}.$$

Now, 59

$$\zeta(s,A) = \frac{\kappa}{s-1} + \rho(A) + \cdots,$$

and so

$$\begin{aligned} L_K(s,\chi) &= \sum_A \chi(A)\zeta(s,A) \\ &= \sum_A \chi(A)\left(\frac{\kappa}{s-1} + \rho(A) + \cdots\right) \\ &= \sum_A \chi(A)\rho(A) + \text{ terms involving higher powers of } (s-1), \end{aligned}$$

since, χ being $\neq 1$, $\sum_A \chi(A) = 0$. On taking the limit as $s \to 1$, we have

$$L_K(1,\chi) = \sum_A \chi(A)\rho(A). \tag{65}$$

Therefore, the problem of determination of $L_K(1,\chi)$ has been reduced to that of $\rho(A)$.

We now study $L_K(s,\chi)$ for a special class of characters, the so-called **genus characters** (due to Gauss) to be defined below.

We call a discriminant, a *prime discriminant*, if it is divisible by only one prime. In that case, $d = \pm p$ if d is odd, or if d is even, $d = -4$ or ± 8.

Proposition 9. *Every discriminant d can be written uniquely as a product of prime discriminants.*

Proof. It is known that if d is odd, then $d = \pm p_1 \ldots p_k$ where $p_1, \ldots, p_k$ are mutually distinct odd primes. Let $P_1 = \pm p_1$ according as $p_1 \equiv \pm 1 \pmod 4$; then P_1 is a prime discriminant and d/P_1 is again an odd discriminant. Using induction on the number k of prime factors of d (odd!) it can be shown that $d/P_1 = P_2, \ldots, P_k$ where P_i are odd prime discriminants. □

If d is even, then we know that

(a) $d = \pm 8p_2, \ldots, p_k$, if $d/4 \equiv 2 \pmod 4$,

(b) $d = \pm 4p_2, \ldots, p_k$, if $d/4 \equiv 3(\mod 4)$,

$p_2, \ldots, p_k$ being odd primes. Now we may write $d = P_1 d_1$ where P_1 is chosen to be -4, $+8$ or -8 such that d_1 is an odd discriminant. From the above it is **60**
clear that $d = P_1 P_2 \ldots P_k$ where $P_1, \ldots, P_k$ are all prime discriminants.

This decomposition is seen to be unique.

We may now introduce the genus characters.

Let d_1 be the product of any of the factors $P_1, \ldots, P_k$ of d. Then d_1 is again a discriminant and $d_1|d$. Let $d = d_1 d_2$; d_2 is also a discriminant. Indentifying the decompositions $d = d_1 d_2$ and $d = d_2 d_1$, we note that the number of such decompositions (including the trivial one, $d = 1 \cdot d$) is 2^{k-1} where k is the number of different prime factors of d.

For any such decomposition $d = d_1 d_2$ of d and for any prime ideal $\mathfrak{p}$ not dividing d, we define

$$\chi(\mathfrak{p}) = \chi_{d_1}(\mathfrak{p}) = \left(\frac{d_1}{N(\mathfrak{p})}\right),$$

where $\left(\frac{d_1}{N(\mathfrak{p})}\right)$ is the Legendre-Jacobi-Kronecker symbol. We shall show that

$$\chi_{d_2}(\mathfrak{p})\left(= \left(\frac{d_2}{N(\mathfrak{p})}\right)\right) = \left(\frac{d_1}{N(\mathfrak{p})}\right).$$

In face, since $\mathfrak{p} \times d$, $\mathfrak{p}$ belongs to one of the following types. (a) $\mathfrak{p}\mathfrak{p}' = (p)$, $N(\mathfrak{p}) = p$ or (b) $\mathfrak{p} = (p)$, $N(\mathfrak{p}) = p^2$. If $\mathfrak{p}\mathfrak{p}' = (p)$, $N(\mathfrak{p}) = p$ then

$$\left(\frac{d}{p}\right) = 1 = \left(\frac{d_1}{N(\mathfrak{p})}\right)\left(\frac{d_2}{N(\mathfrak{p})}\right)$$

which means that either both are $+1$ or both are -1. i.e. $\left(\frac{d_1}{N(\mathfrak{p})}\right) = \left(\frac{d_2}{N(\mathfrak{P})}\right)$. If $\mathfrak{p} = (p)$, $N(\mathfrak{p}) = p^2$ then $\left(\frac{d}{p}\right) = -1$; but

$$\left(\frac{d_1}{N(\mathfrak{p})}\right)\left(\frac{d_2}{N(\mathfrak{p})}\right) = \left(\frac{d}{p^2}\right) = 1,$$

so that we have again $\left(\frac{d_1}{N(\mathfrak{p})}\right) = \left(\frac{d_2}{N(\mathfrak{p})}\right)$.

When $\mathfrak{p}|d$, one of the symbols $\left(\frac{d_1}{N(\mathfrak{p})}\right)$, $\left(\frac{d_2}{N(\mathfrak{p})}\right)$ is zero, and the other non-zero, we take $\chi(\mathfrak{p})$ to be the non-zero value. In any case, $\chi(\mathfrak{p}) = \pm 1$. This definition can then be extended to all ideals $\mathfrak{a}$ of K as follows: If $\mathfrak{a} = \mathfrak{p}^k \mathfrak{q}^l, \ldots,$ **61** we define $\chi(\mathfrak{a}) = (\chi(\mathfrak{p}))^k (\chi(\mathfrak{q}))^l, \ldots$, so that

$$\chi(\mathfrak{a}\mathfrak{b}) = \chi(\mathfrak{a})\chi(\mathfrak{b})$$

for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ of K.

By a *genus character*, we mean a character χ defined as above, corresponding to any one of the 2^{k-1} different decompositions of d as product of two mutually coprime discriminants.

We shall see later that the genus characters form an abelian group of order 2^{k-1}.

Now we shall obtain an interesting consequence of our definition of the character χ, with regard to the associated L-series.

Consider the L-series, for $s = \sigma + it$, $\sigma > 1$

$$\begin{aligned} L_K(s, \chi) &= \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N(\mathfrak{a}))^{-s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1} \\ &= \prod_{p} \prod_{\mathfrak{p}|(p)} (1 - \chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1}. \end{aligned}$$

The prime ideals $\mathfrak{p}$ of K are distributed as follows:

(a) $\mathfrak{p} = (p)$, $\left(\dfrac{d}{p}\right) = -1$, $N(\mathfrak{p}) = p^2$,

(b) $\mathfrak{p}\mathfrak{p}' = (p)$, $\left(\dfrac{d}{p}\right) = +1$, $N(\mathfrak{p}) = p$,

(c) $\mathfrak{p}^2 = (p)$, $\left(\dfrac{d}{p}\right) = 0$, $N(\mathfrak{p}) = p$.

In case (a), $\left(\dfrac{d}{p}\right) = -1 = \left(\dfrac{d_1}{p}\right)\left(\dfrac{d_2}{p}\right)$ implies that one of $\left(\dfrac{d_1}{p}\right)$, $\left(\dfrac{d_2}{p}\right)$ is $+1$ and the other -1. So

$$\begin{aligned} \prod_{\mathfrak{p}|(p)} (1 - \chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1} &= (1 - p^{-2s})^{-1} \\ &= (1 - p^{-s})^{-1}(1 + p^{-s})^{-1} \\ &= \left(1 - \left(\frac{d_1}{p}\right) \cdot p^{-s}\right)^{-1} \left(1 - \left(\frac{d_2}{p}\right) \cdot p^{-s}\right)^{-1} \end{aligned}$$

In case (b), 62

$$\prod_{\mathfrak{p}|(p)}(1-\chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1} = \left(1-\left(\frac{d_1}{p}\right)\cdot p^{-s}\right)^{-1}\left(1-\left(\frac{d_1}{p}\right)\cdot p^{-s}\right)^{-1}$$
$$= \left(1-\left(\frac{d_1}{p}\right)\cdot p^{-s}\right)^{-1}\left(1-\left(\frac{d_2}{p}\right)\cdot p^{-s}\right)^{-1},$$

since $1 = \left(\frac{d}{p}\right) = \left(\frac{d_1}{p}\right)\left(\frac{d_2}{p}\right)$ implies that $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right)$. In case (c) $p|d$ implies that $p|d_1$ or $p|d_2$. We shall assume without loss of generality that $p|d_2$. Then $\left(\frac{d_2}{p}\right) = 0$ and

$$\prod_{\mathfrak{p}|(p)}(1-\chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1} = \left(1-\left(\frac{d_1}{p}\right)\cdot p^{-s}\right)^{-1}\times\left(1-\left(\frac{d_2}{p}\right)\cdot p^{-s}\right)^{-1}.$$

From all cases, we obtain

$$\begin{aligned} L_K(s,\chi) &= \prod_p \prod_{\mathfrak{p}|(p)}(1-\chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1} \\ &= \prod_p \left(1-\left(\frac{d_1}{p}\right)\cdot p^{-s}\right)^{-1}\left(1-\left(\frac{d_2}{p}\right)\cdot p^{-s}\right)^{-1} \\ &= L_{d_1}(s)L_{d_2}(s). \end{aligned}$$

We have therefore,

Theorem 4 (Kronecker). *For a genus character χ of a K corresponding to the decomposition $d = d_1 d_2$, we have,*

$$L_K(s,\chi) = L_{d_1}(s)L_{d_2}(s). \tag{66}$$

Let us consider the trivial decomposition $d = 1\cdot d$ or $d_1 = 1$ and $d_2 = d$. Then $L_{d_1}(s) = \zeta(s)$ and $L_{d_2}(s) = L_d(s)$. In other words, (66) reduces to (63). If $d_1 \neq 1$, $\zeta_{Q(\sqrt{d_1})}(s) = \zeta(s)L_{d_1}(s)$, from (63). But $L_{d_1}(s)$ can then be continued analytically into the whole plane and it is an entire function.

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent in the "narrow sense"* if $\mathfrak{a} = \mathfrak{b}(\gamma)$ with $N(\gamma) > 0$. (For $\alpha \in K, N(\alpha)$ denotes its norm over $\mathbf{Q}$.)

This definition coincides with the usual definition of equivalence in the case **63**
of an imaginary quadratic field since the norm of every element is positive. In the case of a real quadratic field, if there exists a unit ϵ in the field with $N(\epsilon) = -1$, then both the equivalence concepts are the same, as can easily be seen. Otherwise, if h_0 denotes the number of classes under the narrow equivalence and h, the number of classes under the usual equivalence, we have $h_0 = 2h$.

Proposition 10. *For a genus character χ, $\chi(\mathfrak{a}) = \chi(\mathfrak{b})$, if $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent in the narrow sense.*

Proof. We need only to prove that $\chi((\alpha)) = 1$ for α integral and $N(\alpha) > 0$.

(a) Let $(\alpha, d_1) = (1)$. We have to show that $\left(\dfrac{d_1}{N((\alpha))}\right) = 1$. Since $N(\alpha) > 0$, this is the same as proving that $\left(\dfrac{d_1}{N(\alpha)}\right) = 1$.

(1) Suppose d_1 is odd. Then $\left(\dfrac{d_1}{4}\right) = 1$. If d is even, $d = 4m$ (say). Then $\alpha = x + y\sqrt{m}$ with x, y rational integers. If not, $2\alpha = x' + y'\sqrt{m}$ with x' and y' rational integers. In any case, since $(d_1/4) = 1$, it is sufficient to prove that $\left(\dfrac{d_1}{N(2\alpha)}\right) = 1$. i.e. $\left(\dfrac{d_1}{a^2 - mb^2}\right) = 1$ for two rational integers a, b with $a^2 - mb^2 > 0$ and $(a^2 - mb^2, d_1) = 1$, in both cases. Now $d_1|d$ implies that $d_1|m$ so that from the periodicity of the Jacobi symbol, this is the same as $\left(\dfrac{d_1}{a^2}\right) = 1$ for $(a, d_1) = 1$, which is obvious.

(2) Suppose d_1 is even. Then d_1 = (even discriminant) $\times$ (odd discriminant) and we need consider only the even part, since we have already disposed of the odd part in 1). We have then three possibilities $d_1 = -4, +8$ and -8.

(i) $d_1 = -4$. Then $d = (-4)(-m)$ so that $-m$ is again a discriminant and $\equiv 1 \pmod 4$.
Consider now
$$\left(\frac{d_1}{N(\alpha)}\right) = \left(\frac{-4}{x^2 - my^2}\right);$$
$(x^2 - my^2) \equiv 1 \pmod 4$ since both x, y cannot be even or odd, for $(\alpha, d_1) = (1)$. From the periodicity of the Jacobi symbol, **64**
follows then that
$$\left(\frac{-4}{x^2 - my^2}\right) = \left(\frac{-4}{1}\right) = 1.$$

(ii) $d_1 = +8$. Here $d = 8 \times m/2$; $m/2$ is then an odd discriminant and $\equiv 1 \pmod 4$ or equivalently, $m \equiv 2 \pmod 8$.
Now,
$$\left(\frac{d_1}{N(\alpha)}\right) = \left(\frac{8}{x^2 - my^2}\right);$$

$(x^2 - my^2) \equiv \pm 1 \pmod 8$ so that

$$\left(\frac{8}{x^2 - my^2}\right) = \left(\frac{8}{1}\right) = 1 \text{ or } = \left(\frac{8}{7}\right) = \left(\frac{1}{7}\right) = 1.$$

If $d_1 = -8$, $m \equiv -2 \pmod 8$ and $x^2 - my^2 \equiv 1, 3 \pmod 8$ and $\left(\frac{d_1}{N(\alpha)}\right) = \left(\frac{-8}{3}\right) = \left(\frac{1}{3}\right) = 1$. So, if $(\alpha, d_1) = (1)$, $\chi((\alpha)) = 1$. If $(\alpha, d_1) \neq (1)$ and $(\alpha, d_2) = (1)$, we can apply the same arguments for d_2 instead of d_1 and prove $\chi((\alpha)) = 1$.

(b) If $(\alpha, d_1) \neq (1)$ and $(\alpha, d_2) \neq (1)$, we decompose (α) as follows: $(\alpha) = \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_l\mathfrak{q}$ where $\mathfrak{p}_i | d$ and $(\mathfrak{q}, d) = (1)$. We choose in the narrow class of $\mathfrak{p}_1^{-1}$, an integral ideal $\mathfrak{q}_1$ with $(d, \mathfrak{q}_1) = (1)$. Then for the element α_1 with $(\alpha_1) = \mathfrak{p}_1 \cdot \mathfrak{q}_1$, $N(\alpha_1) > 0$, $\chi((\alpha_1)) = 1$, since (α_1, d_1) or $(\alpha_1, d_2) = (1)$. Similarly for $\mathfrak{p}_2$, construct $\mathfrak{q}_2$ with $(d, \mathfrak{q}_2) = 1$ and for α_2 with $(\alpha_2) = \mathfrak{p}_2\mathfrak{q}_2$ and $N(\alpha_2) > 0$, $\chi((\alpha_2)) = 1$ and so on.

Finally, we obtain

$$(\alpha\alpha_1\alpha_2 \ldots) = \mathfrak{p}_1^2\mathfrak{p}_2^2 \ldots \mathfrak{p}_l^2\,\mathfrak{p}$$

where $(\mathfrak{b}, d) = (1)$. But $\mathfrak{p}_i | d$ imply that $\mathfrak{p}_i^2 = (p_i)$ with $p_i | d$. Therefore $(\alpha\alpha_1\alpha_2 \ldots) = (p_1p_2 \ldots)\mathfrak{b}$ so that $\mathfrak{b}$ is a principal ideal $= (\rho)$ (say). Now $N(\alpha\alpha_1\alpha_2 \ldots) = p_1^2p_2^2 \ldots p_1^2 N(\rho)$ and $\chi((\alpha\alpha_1 \ldots \alpha_l)) = \chi((\rho)) = 1$ since $N(\rho) > 0$ and $(\rho, d) = (1)$. But $\chi((\alpha_1)) = 1, \ldots \chi((\alpha_l)) = 1$ so that $\chi((\alpha)) = 1$. The proof is now **65**
complete. □

We have just proved that $\chi(\mathfrak{a})$ depends only on the narrow class of $\mathfrak{a}$, say $A \cdot \chi$ is therefore a character of the ideal class group in the "narrow sense". Then,

$$L_K(s, \chi) = \sum_A \chi(A) \sum_{\mathfrak{a} \in A} (N(\mathfrak{a}))^{-s} = \sum_A \chi(A)\zeta(s, A) \tag{67}$$

where A runs over all the ideal classes in the 'narrow sense'.

Using a method of Hecke, we shall now give an *alternative* proof of the fact that $\chi((\alpha)) = 1$ for principal ideals (α) with $N(\alpha) > 0$ and $(\alpha, d) \neq (1)$.

Denote by χ_0 that character of the ideal class group in the narrow sense, such that $\chi_0(\mathfrak{a}) = \chi(\mathfrak{a})$ for all ideals $\mathfrak{a}$ in whose prime factor decomposition, only prime ideals $\mathfrak{p}$ which do not divide d, occur. We need only to prove that $\chi(\mathfrak{a}) = \chi_0(\mathfrak{a})$ for all ideals $\mathfrak{a}$.

Let $f_{\chi_0}(s) = \sum_{\mathfrak{a}} \chi_0(\mathfrak{a})(N(\mathfrak{a}))^{-s}$ and $f_\chi(s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N(\mathfrak{a}))^{-s}$. Then

$$\frac{f_{\chi_0}(s)}{f_\chi(s)} = \prod_{\mathfrak{p}|d} \frac{(1-\chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})}{(1-\chi_0(\mathfrak{p})(N(\mathfrak{p}))^{-s})},$$

since the remaining factors cancel out. We shall show that the product on the right is = 1, or $f_{\chi_0}(s) = f_\chi(s)$.

We know that $f_\chi(s) = L_{d_1}(s) = L_{d_2}(s)$ from (66) and $f_{\chi_0}(s) = \sum_A \chi_0(A).\zeta(s, A)$ from (67). Both have functional equations of the same type, so that if we denote by $R(s)$, the quotient, the functional equation is simply $R(s) = R(1-s)$. We shall now arrive at a contradiction, by supposing that $\chi \neq \chi_0$. For, then, there exists a prime ideal $\mathfrak{p}$ with $\mathfrak{p}|d$ and $\chi(\mathfrak{p}) = \pm 1$, $\chi_0(\mathfrak{p}) = \mp 1$.

Choose $-s = \dfrac{\log(\mp 1) + 2k\pi i}{\log p}$ (k, such that $s \neq 0$). This means that $p^{-s} = \mp 1$. Consider the product

$$R(s) = \prod_{\mathfrak{p}|d} \frac{(1-\chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})}{(1-\chi_0(\mathfrak{p})(N(\mathfrak{p}))^{-s})}.$$

The above value of s is a zero of the denominator and it cannot be cancelled by any factor in the numerator, for that would mean

$$\frac{\log(\mp 1) + 2k\pi i}{\log p} = \frac{\log(\pm 1) + 2l\pi i}{\log q},$$

or in other words, $\lambda = \log p/\log q$ is rational; i.e. $q^\lambda = p$ holds for p, q **66**
primes and λ rational. This is not possible since $p \neq q$. By the same argument, the zeros and poles of $R(s)$ cannot cancel with those on the other side of the equation $R(s) = R(1-s)$. But this is a contradiction. In other words, $\chi(\mathfrak{a}) = \chi_0(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of K.

Tow ideals $\mathfrak{a}$ and $\mathfrak{b}$ are *in the same genus*, if $\chi(\mathfrak{a}) = \chi(\mathfrak{b})$ for all genus characters χ defined as above, associated with the decompositions of d.

The ideals $\mathfrak{a}$ for which $\chi(\mathfrak{a}) = 1$ for all genus characters χ, constitute the *principal genus*. Then the narrow classes A for which $\chi(A) = 1$ are the classes in the principal genus. If H denotes the group of narrow classes and G, the subgroup of classes lying in the principal genus, the quotient group $\mathfrak{G} = H/G$ gives the group of different genera. We claim that the order of $\mathfrak{G}$ is 2^{k-1}, k being the number of different prime factors of d. For, suppose we have proved that the genus characters form a group of order 2^{k-1}. From the definition of a genus and from the theory of character groups of abelian groups, we know that this group is the character group of $\mathfrak{G}$. By means of the isomorphism between $\mathfrak{G}$ and its character group, it would then follow that the order of $\mathfrak{G}$ is also 2^{k-1}.

If now remains for us to prove

Proposition 11. *The genus characters form an abelian group of order* 2^{k-1}, *where k is the number of distinct prime factors of d.*

Proof. Now, $\chi(\mathfrak{a}) = \pm 1$ implies that $\chi^2 = 1$ for all genus characters χ. So, the character χ^{-1} exists and $= \chi$. We shall prove that the genus characters and closed under multiplication and that they are all different for different decompositions of d.

Let $d = d_1 d_2 = d_1^* d_2^*$. Set $d_1 = qu_1$ and $d_1^* = qu_1^*$ so that $q = (d_1, d_1^*)$ and $(u_1, u_1^*) = 1$. □

If $d_3 = u_1 u_1^*$ then $d_1 d_1^* = q^2 d_3$. Also for the characters χ_{d_1} and $\chi_{d_1^*}$ associated with the two decompositions $d_1 d_2$ and $d_1^* d_2^*$ of d, we have $\chi_{d_1}\chi_{d_1^*} = \chi_{d_3}$ and $d_3 | d$, since both $d_1 | d$ and $u_1^* | d$ and they are coprime. Hence χ_{d_3} is again a genus character. Now, if $d_1 \neq d_1^*$, we have to show that $\chi_{d_1} \neq \chi_{d_1^*}$ or $\chi_{d_1}\chi_{d_1^*} \neq \chi_1$ (the identity character); or in other words, we need only to prove that for a proper **67** decomposition $d = d_3 d_3^*$, the associated character χ is not equal to χ_1.

We have, from (66), for $\sigma > 1$,

$$L_K(s, \chi) = L_{d_3}(s) L_{d_3^*}(s) = \sum_A \chi(A) \zeta(s, A).$$

Comparing the residue at $s = 1$ in the two forms for $L_K(s, \chi)$, we have $\sum\limits_A \chi(A) = 0$, which implies that $\chi \neq \chi_1$.

Thus the genus characters are different for different decompositions of d and they form a group. Since there are 2^{k-1} different decompositions of d, this group of genus characters is of order 2^{k-1}.

A genus character is a narrow class character of order 2. Conversely, we can show that every narrow class character of order 2 is a genus character. For this, we need the notion of an ambiguous (narrow) ideal class.

A narrow ideal class A satisfying $A = A'$ (the conjugate class of A in K) or equivalently $A^2 = E$ (the principal narrow class) is called an *ambiguous* ideal class. The ambiguous classes clearly form a group.

Proposition 12. *The group of ambiguous ideal classes in* $\mathbf{Q}(\sqrt{d})$ *is of order* 2^{k-1}, *k being the number of distinct prime factors of d.*

Proof. We shall pick out one ambiguous ideal $\mathfrak{b}$ (i.e. such that $\mathfrak{b} = \mathfrak{b}'$, the conjugate ideal) from each ambiguous ideal class A and show that there are 2^{k-1} such inequivalent ideals.

Let $\mathfrak{a}$ be any ideal in the ambiguous ideal class A. Since $\mathfrak{a} \sim \mathfrak{a}'$, we may take $\mathfrak{a}\mathfrak{a}'^{-1} = (\lambda)$ with $N(\lambda) = 1$ and λ totally positive (in symbols; $\lambda > 0$). (For $d < 0$, the condition "totally positive" implies no restriction). Let $\rho = 1 + \lambda$. Then $\rho \neq 0$ since $\lambda \neq -1$. Further $\lambda = (1 + \lambda)/(1 + \lambda') = \rho/\rho'$. If we define $\mathfrak{b} = \mathfrak{a}/(\rho)$, the $\mathfrak{b} = \mathfrak{b}'$, for $\mathfrak{b}/\mathfrak{b}' = \mathfrak{a}/\mathfrak{a}'(\rho')/(\rho) = (1)$. Further $\mathfrak{b} \in A$, since $\rho > 0$ (i.e. $\rho > 0$, $\rho' > 0$). We may take $\mathfrak{b}$ to be integral without loss of generality. We call $\mathfrak{b}$ primitive, if the greatest rational integer r dividing $\mathfrak{b}$ is 1. For any ambiguous integral ideal $\mathfrak{b}$ we have $\mathfrak{b} = r\mathfrak{b}_1$ with r, the greatest integer dividing $\mathfrak{b}$ and $\mathfrak{b}_1$ primitive. If $\mathfrak{b} \in A$, $\mathfrak{b}_1 \in A$. □

We shall now prove that any primitive integral ambiguous ideal $\mathfrak{b}$ is of the **68**
form $\mathfrak{p}_1^{\lambda_1}, \ldots, \mathfrak{p}_k^{\lambda_k}$ with $\lambda_i = 0$ or 1 and $\mathfrak{p}_i|\vartheta$, $(\mathfrak{p}_i \neq \mathfrak{p}_j)$, where ϑ is the different of $K = \mathbf{Q}(\sqrt{d})$ over $\mathbf{Q}$. Let $\mathfrak{b} = \mathfrak{q}_1^{\mu_1}, \ldots, \mathfrak{q}_s^{\mu_s}$. Now $\mathfrak{b} = \mathfrak{b}'$ implies that $\mathfrak{q}_1^{\mu_1}, \ldots, \mathfrak{q}_s^{\mu_a} = \mathfrak{q}'^{\mu_1}_1, \ldots, \mathfrak{q}'^{\mu_s}_s$. We have then $\mathfrak{q}_1^{\mu_1} = \mathfrak{q}'^{\mu_t}_t$ or $\mu_1 = \mu_t$ and $\mathfrak{q}_1 = \mathfrak{q}'_t$. We assert then $t = 1$, for if not, $\mathfrak{q}_1 = \mathfrak{q}'_t$ implies that $\mathfrak{q}'_1 = \mathfrak{q}_t$ and the factor $\mathfrak{q}_1^{\mu_1}\mathfrak{q}_t^{\mu_t} = (\mathfrak{q}_1\mathfrak{q}'_1)^{\mu_1}$ is a rational ideal $\neq (1)$. This is a contradiction to the hypothesis that $\mathfrak{b}$ is primitive. Hence $t = 1$ and $\mathfrak{q}_1 = \mathfrak{q}'_1$. The same argument applies to all prime ideals $\mathfrak{q}_i$. Since $\mathfrak{q}_i^2 = (p_i)$ with prime numbers p_i, the exponents μ_i are either 0 or 1. for otherwise they bring in rational ideals which are excluded by the primitive nature of $\mathfrak{b}$. Therefore, we have $\mathfrak{b} = \mathfrak{p}_1^{\lambda_1}, \ldots, \mathfrak{p}_k^{\lambda_k}$ with $\mathfrak{p}_i|\vartheta$ and $\lambda_i = 0$ or 1.

Now, we shall show that there are exactly 2^{k-1} inequivalent (in the narrow sense) such ideals. For the same, it is enough to prove that there is only one non-trivial relation of the form $\mathfrak{p}_1^{l_1}, \ldots, \mathfrak{p}_k^{l_k} \sim (1)$ in the narrow sense, for, then, it would imply that among the 2^k such ideals, there are 2^{k-1} inequivelent ones, which is what we require.

We shall first prove uniqueness, namely that if there is one non-trivial relation, then it is uniquely determined. Later, we show the existence of a non-trivial relation.

(a) Let $\mathfrak{p}_1^{l_1}, \ldots, \mathfrak{p}_k^{l_k} = (\rho)$ with $\rho > 0$ and $\sum_i l_i > 0$ or $(\rho) \neq (1)$. Then the set $(l_1, \ldots, l_k)$ is uniquely determined.

For, $(\rho) = (\rho')$ implies that $\rho = \eta \cdot \rho'$ with a totally positive unit η. The group of totally positive units in K being cyclic, denote by ϵ, the generator of this group. (Note that, for $d < 0$, the condition "totally positive" imposes no restrictions). Since $(\rho) = (\rho\epsilon^n)$, when ρ is replaces by $\rho\epsilon^n$, η goes over to $\eta\epsilon^{2n}$. Choosing n suitably, we may assume without loss of generality $\eta = 1$ or ϵ. We shall see that $(\rho) \neq (1)$ implies that $\eta \neq 1$. For, if $\eta = 1$, then $\rho = \rho' = a$ natural number. Now $(N(\rho)) = (\rho^2) = \mathfrak{p}_1^{2l_1}, \ldots, \mathfrak{p}_k^{2l_k} = (p_1^{l_1}, \ldots, p_k^{l_k})$, if $N(\mathfrak{p}_i) = p_i$, $p_i \neq p_j$ and ρ being a natural number, this is possible only if all $l_i = 0$, in which case $(\rho) = (1)$. But that is a contradiction to the hypothesis $(\rho) \neq (1)$.

We are therefore left with the case $\eta = \epsilon$. Then set $\mu = 1 - \eta = 1 - \epsilon \neq 0$. We have $\mu = -\eta\mu'$. Denoting by $\delta = \sqrt{d}$, $(\rho/\mu\delta)' = \rho/\mu\delta = r = a$ rational number $\neq 0$. Hence $(\rho) = (\mu\delta)(r)$ and the decomposition

$$(\mu\delta) = \left(\frac{1}{r}\right)(\rho) = \left(\frac{1}{r}\right)\mathfrak{p}_1^{l_1}, \ldots, \mathfrak{p}_k^{l_k}$$

is uniquely determined or in other words, the set $(l_1, \ldots, l_k)$ is uniquely fixed. **69**

(b) We shall now prove the existence of a non-trivial relation $\mathfrak{p}_1^{l_1}, \ldots, \mathfrak{p}_k^{l_k} \sim (1)(l_i = 0 \text{ or } 1)$ in the narrow sense.

Consider $\mu = 1 - \epsilon$. Then $\mu\delta$ is in general not primitive. Now choose a rational number r suitably so that $r\mu\delta$ is primitive and denote it by ρ. Then $r\mu\delta = \rho = \epsilon\rho'$. Now if $d < 0$, $N(\mu\delta) > 0$ clearly and if $d > 0$, $N(\mu\delta) = d\epsilon\mu'^2 > 0$, again. Hence $N(\rho) > 0$. We may assume that $\rho > 0$ and since ϵ is not a square, ρ is not a unit, so that $(\rho) \neq (1)$. Since (ρ) is primitive, $(\rho) \neq$ rational ideal. Now $(\rho) = (\rho')$, $\rho > 0$ and (ρ) is primitive so that $\mathfrak{p}_1^{l_1}, \ldots, \mathfrak{p}_k^{l_k} = (\rho) \sim (1)$ in the narrow sense. Further this relation is non-trivial as we have just shown. Proposition 12 is thus completely proved.

Now, the group of narrow class characters χ with $\chi^2 = 1$ is again of order 2^{k-1}, since it is isomorphic to the group of ambiguous (narrow) ideal classes in K. But the genus characters from a subgroup (of order 2^{k-1}) of the group of narrow class characters χ with $\chi^2 = 1$. Hence every narrow class character of order 2 is a genus character. Or, equivalently every real narrow class character is a genus character.

For any genus character χ, $\chi(i^2) = 1$ for every ideal i. In other words, i^2 is in the principal genus. Or, for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a} \sim \mathfrak{b}i^2$ (in the narrow sense), we have $\chi(\mathfrak{a}) = \chi(\mathfrak{b})$ for every genus character χ. That the converse is also true is shown by

Theorem 5 (Gauss). *If two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in the same genus, there exists an ideal i such that $\mathfrak{a} \sim \mathfrak{b}i^2$, in the narrow sense. In particular, if $\mathfrak{a}$ is in the principal genus, $\mathfrak{a} \sim i^2$ for an ideal i.*

Proof. Two narrow classes A and B are by definition, in the same genus if and only if $\chi(A) = \chi(B)$ for all genus characters χ or by the foregoing, for all narrow class characters χ with $\chi^2 = 1$. But from the duality theory of subgroups of abelian groups and their character groups, we deduce immediately that $AB^{-1} = C^2$ for a narrow ideal class C. This is precisely the assertion above. □

Remarks. (1) The notion of genus in the theory of binary quadratic forms is the same as above, if one carries over the definition by means of the correspondence between ideals and quadratic forms. 70

(2) Hilbert generalized the notion of genus to arbitrary algebraic number fields and used it for higher reciprocity laws and class field theory.

We now come back to formulas (66) and (67) ($\underline{d < 0}$). If $d = d_1 d_2$

$$L_{d_1}(s) L_{d_2}(s) = \sum_A \chi(A)\zeta(s, A),$$

the summation running over all (narrow) ideal classes A in $K = \mathbf{Q}(\sqrt{d})$.

Consider any ideal $\mathfrak{b} \in A^{-1}$. For $\mathfrak{a} \in A$, $\mathfrak{a}\mathfrak{b} = (\gamma)$ with $N(\gamma) > 0$ we have then

$$\zeta(s, A) = \frac{(N(\mathfrak{b}))^s}{w} \sum_{\mathfrak{b}|\gamma \neq 0} (N(\gamma))^{-s},$$

where w denotes the number of units in $K = \mathbf{Q}(\sqrt{d})$.

Let $[\alpha, \beta]$ be an integral basis of $\mathfrak{b}$. Then we may write $\mathfrak{b} = [\alpha, \beta] = (\alpha)[1, \beta/\alpha]$ so that $\beta/\alpha = z = x + iy$ with $y > 0$, i.e. we may suppose that $\mathfrak{b} = [1, z]$. Then, for $\gamma \in \mathfrak{b}$, if $\gamma = m + nz$ with m, n integers $\neq (0, 0)$, $N(\gamma) = |m + nz|^2$. We obtain

$$\zeta(s, A) = \frac{N([1, z])^s}{w} \sum_{m,n}{}' |m + nz|^{-2s}$$

Now, abs. $\begin{vmatrix} 1 & z \\ 1 & z' \end{vmatrix} = |z' - z| = 2y = N(b)\sqrt{|d|}$, so that

$$\begin{aligned} \zeta(s, A) &= \frac{1}{w}\left(\frac{2y}{\sqrt{|d|}}\right)^s \sum_{m,n}{}' |m + nz|^{-2s} \\ &= \frac{1}{w}\frac{2}{\sqrt{|d|}}\left(\frac{2}{\sqrt{|d|}}\right)^{s-1} y^s {\sum_{m,n}}' |m + nz|^{-2s}. \end{aligned}$$

From Kronecker's first limit formula, we obtain the following expansion:

$$y^s \sum_{m,n}{}' |m + nz|^{-2s} = \pi\left(\frac{1}{s-1} + 2C - 2\log 2 - 2\log(\sqrt{y}|\eta(z)|^2) + \cdots\right).$$

Further 71

$$\left(\frac{2}{\sqrt{|d|}}\right)^{s-1} = e^{(s-1)\log 2/\sqrt{|d|}} = \left(1 + (s-1)\log\frac{2}{\sqrt{|d|}} + \cdots\right).$$

On multiplication, we obtain

$$\zeta(s, A) = \frac{2\pi}{w\sqrt{|d|}}\left(\frac{1}{s-1} + 2C - \log|d| - 2\log(\sqrt{N([1,z])}|\eta(z)|^2) + \right.$$
$$\left. + \text{ terms with higher powers of } (s-1)\right).$$

We now set

$$F(A) = \sqrt{N([1,z])}|\eta(z)|^2. \qquad (*)$$

Clearly, $F(A)$ *depends only upon the class A.* In case $d < -4$, $w = 2$, so that

$$L_{d_1}(s)L_{d_2}(s) = \frac{-2\pi}{\sqrt{|d|}}\sum_A \chi(A)\log F(A) +$$
$$+ \text{ terms with higher powers of } (s-1).$$

Taking $s = 1$, we have

$$L_{d_1}(1)L_{d_2}(1) = \frac{-2\pi}{\sqrt{|d|}}\sum_A \chi(A)\log F(A).$$

Let us assume without loss of generality, $d_1 > 0$, $d_2 < 0$. Then, by Dirichlet's class number formula for a quadratic field, we have

$$L_{d_1}(1) = \frac{2h_1\log\epsilon}{\sqrt{d_1}}$$

whre ϵ is the fundamental unit and h_1 the class-number of $\mathbf{Q}(\sqrt{d_1})$ and

$$L_{d_2}(1) = \frac{2\pi h_2}{w\sqrt{|d_2|}},$$

h_2 denoting the class-number and w, the number of roots of unity in $\mathbf{Q}(\sqrt{d_2})$. 72
On taking the product of these two, we obtain

$$\frac{2h_1h_2}{w}\log\epsilon = -\sum_A \chi(A)\log F(A).$$

Summing up, we have the following. Let $d = d_1d_2 < -4$ be the discriminant of an imaginary quadratic field over $\mathbf{Q}$ and let $d_1 > 0$, $d_2 < 0$ be again discriminants of quadratic fields over $\mathbf{Q}$ with class-numbers h_1, h_2, respectively. Let w be the number of roots of unity in $\mathbf{Q}(\sqrt{d_2})$. Then one has

Theorem 6. *For the fundamental unit ϵ of $\mathbf{Q}(\sqrt{d_1})$, we have the formula*

$$\epsilon^{2h_1h_2/w} = \prod_A (F(A))^{-\chi(A)}, \tag{68}$$

where A runs over all the ideal classes in $\mathbf{Q}(\sqrt{d})$ and $F(A)$ has the meaning given in $(*)$ above.

If $w = 2$, the exponent of ϵ in (68) is a positive integer.

Now ϵ satisfies Pell's diophantine equation $x^2 - d_1y^2 = \pm 1$. And $F(A)$ involves the values of $|\eta(z)|$. So we have a solution of Pell's equation by means of "elliptic functions". This was found by Kronecker in 1863.

An Example. We shall take $d = -20$, $d_1 = 5$, $d_2 = -4$ so that $d = d_1d_2$. We know that $h_1 = h_2 = 1$, $w = 4$. The class number of $\mathbf{Q}(\sqrt{-20})$ is 2 and for the two ideal classes, say A_1, A_2, we can choose as representatives the ideals

$$\mathfrak{b}_1 = [1, \sqrt{-5}] = (1) \quad \text{and} \quad \mathfrak{b}_2 = \left[1, \frac{1+\sqrt{-5}}{2}\right].$$

Now $N(\mathfrak{b}_1) = 1$ and $N(\mathfrak{b}_2) = 1/2$. From the definition of $F(A)$, we have

$$F(A_1) = |\eta(\sqrt{-5})|^2 \quad \text{and} \quad F(A_2) = \frac{1}{\sqrt{2}}\left|\eta\left(\frac{1+\sqrt{-5}}{2}\right)\right|^2.$$

If $\chi(\neq 1)$ be the genus character associated with the above decomposition of d, then necessarily $\chi(A_1) = 1$ and $\chi(A_2) = \left(\frac{5}{2}\right) = -1$. Formula (68) now takes the form

$$\epsilon = \frac{1}{2}\frac{\left|\eta\left(\frac{1+\sqrt{-5}}{2}\right)\right|^4}{|\eta(\sqrt{-5})|^4}. \tag{69}$$

From the product expansion of the η-function, we obtain 73

$$\eta(\sqrt{-5}) = e^{-\pi\sqrt{5}/12}|(1-q^2)(1-q^4)(1-q^6)\ldots|,$$

where $q = e^{-\pi\sqrt{5}}$ and for $\left|\eta\left(\frac{1+\sqrt{-5}}{2}\right)\right|$, the following expansion:

$$\left|\eta\left(\frac{1+\sqrt{-5}}{2}\right)\right| = e^{-\pi\sqrt{5}/24}|(1+q)(1-q^2)(1+q^3)(1-q^4)\ldots|.$$

On substituting these expansions in (69), we have

$$\epsilon = \frac{1}{2}e^{\pi\sqrt{5}/6}(1+q)^4(1+q^3)^4(1+q^5)^4\ldots.$$

Now, $e^{\pi\sqrt{5}/3} > 10$ so that $q < 10^{-3}$ or $q^3 < 10^{-9}$. Hence the product on the right converges rapidly and if we replace the infinite product by 1, we have

$$\epsilon = \frac{1}{2}e^{\pi\sqrt{5}/6} + O(10^{-2})^{-}$$

i.e.

$$\epsilon = \frac{1+\sqrt{5}}{2} \sim \frac{1}{2}e^{\pi\sqrt{5}/6} \text{ (upto an error of the order of } 10^{-2}).$$

It is to be noted that in the expression for ϵ as an infinite product, if one cuts off at any finite stage, the resulting number on the right is always transcendental, but the limiting value ϵ on the left is algebraic.

We consider now the trivial decomposition

$$\zeta_d(s) = \zeta(s)L_d(s) = \sum_A \zeta(s, A).$$

On substituting the expansion of $\zeta(s, A)$ on the right side in powers of $(s-1)$, we have

$$\zeta_d(s) = \frac{2\pi h}{w\sqrt{|d|}}\left(\frac{1}{s-1} + 2C - \log|d| - \frac{2}{h}\sum_A \log F(A) + \cdots\right). \qquad (70)$$

On the other hand, $\zeta(s) = 1/(s-1) + C + \cdots$ and $L_d(s) = \sum_{n=1}^{\infty}\left(\frac{d}{n}\right)n^{-s} =$ 74
$L(1) + (s-1)L'(1) + \cdots$. From (70), we have now the equation,

$$\begin{aligned}\frac{2\pi h}{w\sqrt{|d|}}&\left(\frac{1}{s-1} + 2C - \log|d| - \frac{2}{h}\sum_A \log F(A) + \cdots\right)\\ &= \left(\frac{1}{s-1} + C + \cdots\right)(L(1) + (s-1)L'(1) + \cdots).\end{aligned}$$

On comparing coefficients of $1/(s-1)$ and constant terms on both sides, we obtain

$$\left.\begin{aligned}L(1) &= \frac{2\pi h}{w\sqrt{|d|}},\\ L'(1) &= \frac{2\pi h}{w\sqrt{|d|}}\left(C - \log|d| - \frac{2}{h}\sum_A \log F(A)\right),\end{aligned}\right\} \qquad (71)$$

or we have an explicit expression for

$$\frac{L'(1)}{L(1)} = C - \log|d| - \frac{2}{h}\sum_{A}\log F(A). \tag{71}'$$

We shall obtain now for the left side, an infinite series expansion.

Let us consider the infinite product

$$L(s) = \prod_p \left(1 - \left(\frac{d}{p}\right)p^{-s}\right)^{-1}.$$

this product converges absolutely for $\sigma > 1$, and on taking logarithmic derivatives, we obtain

$$\begin{aligned}\frac{L'(s)}{L(s)} &= -\sum_p \left(1 - \left(\frac{d}{p}\right)p^{-s}\right)^{-1}\left(\frac{d}{p}\right)p^{-s}\log p\\ &= -\sum_p \frac{\left(\frac{d}{p}\right)\log p}{p^s - \left(\frac{d}{p}\right)}.\end{aligned} \tag{72}$$

On passing to the limit as $s \to 1$, if the series on the right converges at $s = 1$, then by an analogue of Abel's theorem, it is equal to $L'(1)/L(1)$. But it is rather difficult to prove. One may proceed as follows: 75

Let $\pi_+(X) = \{p : p \text{ prime} \leq X \text{ and } \left(\frac{d}{p}\right) = 1\}$ and $\pi_-(X) = \{p : p \text{ prime} \leq X \text{ and } \left(\frac{d}{p}\right) = -1\}$. Then $\pi_+(X) - \pi_-(X)$ tends to ∞ less rapidly than $\pi(X)$ as $X \to \infty$. If one could show that this function has the estimate $0(X/(\log X)^r)$ where $r > 2$, then the convergence of the series on the right side of (72) at $s = 1$, can be proved. For this estimate, one requires a generalization of the proof of prime number theorem for arithmetical series.

We shall now compute $L'(1)/L(1)$ for some values of d.

Examples. $d = -4$. The quadratic field $\mathbf{Q}(\sqrt{d}) = \mathbf{Q}(\sqrt{-1})$ with discriminant -4 has class number 1. Also,

$$\left(\frac{d}{n}\right) = 1 \text{ if } n \equiv 1 \pmod 4 \text{ and } = -1 \text{ if } n \equiv 3 \pmod 4.$$

Therefore,

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \cdots$$

and

$$L(1) = 1^{-1} - 3^{-1} + 5^{-1} - \cdots$$

the so-called Leibnitz series. From Dirichlet's class number formula, we have

$$L(1) = \frac{2\pi}{4.2} = \frac{\pi}{4} = 1^{-1} - 3^{-1} + 5^{-1} - \cdots .$$

More interesting is the series for $L'(1)/L(1)$:

$$\begin{aligned}\frac{L'(1)}{L(1)} &= \frac{\log 3}{4} - \frac{\log 5}{4} + \frac{\log 7}{8} + \frac{\log 11}{12} - \frac{\log 13}{12} - \cdots \\ &= (C - \log 4 - 4\log \eta(i)).\end{aligned}$$

Now

$$\eta(i) = e^{-\pi/12} \prod_{n=1}^{\infty} (1 - e^{-2\pi n}).$$

We have $6^{-2\pi} < 1/400$ and hence $\sum\limits_{n=1}^{\infty} e^{-n\pi}$ is rapidly convergent. In other words 76
$4 \prod\limits_{n=1}^{\infty} (1 - e^{-2n\pi})$ is absolutely convergent and upto an error of 10^{-2}, it can be replaced by 1. We obtain consequently

$$\frac{L'(1)}{L(1)} \sim \left(C - \log 4 + \frac{\pi}{3}\right).$$

2 Class number of the absolute class field of $\mathbf{Q}(\sqrt{d})(d < 0)$.

We shall now apply the method outlined in § 57 to determine the class number of the absolute class field of $K_0 = \mathbf{Q}(\sqrt{d})$, with $d < 0$. But, for the time being, however, let K_0 be an arbitrary algebraic number field. The **absolute class field** K/K_0 is, by definition, the largest field K containing K_0, which is both abelian and unramified over K_0 (i.e. with relative discriminant over K_0 equal to (1)).

The absolute class field, first defined by Hilbert, is also known as the *Hilbert class field*; its existence and uniqueness were proved by Furtwängler. It can be shown that there are only finitely many abelian unramified extensions of K_0 and all these are contained in one abelian extension and this is the maximal one. It has further the property that the Galois group $G(K/K_0)$ is isomorphic to the narrow ideal class group of K_0.

Consider now any abelian unramified extension K_1 of K_0. Then $G(K_1/K_0)$ is isomorphic to a factor group of $G(K/K_0)$, i.e. to a factor group of the narrow ideal class group of K_0. The character group of $G(K_1/K_0)$ is a subgroup of the group of h_0 ideal class characters in the narrow sense.

$$\begin{array}{c} K \\ | \\ K_1 \\ | \\ K_0 \\ | \\ \mathbf{Q} \end{array}$$

We have then for the zeta function, 77

$$\zeta_{K_1}(s) = \prod_{\chi} L(s, \chi),$$

the product on the right running over all characters in this subgroup.

We shall now derive an expression for the quotient H/h of the class numbers H of K and h of K_0 in the case $K_0 = \mathbf{Q}(\sqrt{d})$ with $d < 0$, by using the above product formula for $K_1 = K$ and comparing the residue at $s = 1$ on both sides.

We have indeed

$$\zeta_{K_1}(s) = {\prod_{\chi}}' L(s, \chi) \cdot \zeta_d(s),$$

the product $\prod'$ running over all characters $\chi \neq 1$.

On comparing the residues at $s = 1$ on both sides, we obtain

$$\frac{2^{r_1}(2\pi)^{r_2}R}{W\sqrt{|D|}} H = \frac{2\pi h}{w\sqrt{|d|}} {\prod}' L(1, \chi). \tag{73}$$

Here r_1 and r_2 denote the number of real and distinct complex conjugates of the field K_1. The regulator of K_1 is denoted by R and is defined as follows: By a theorem of Dirichlet, every unit ϵ of K_1 is of the form $\epsilon = \epsilon_0^{g_0}\epsilon_1^{g_1}\dots\epsilon_r^{g_r}$ with rational integers g_i, ϵ_0 being a root of unity. Then $\epsilon_1, \dots, \epsilon_r$ (with $r = r_1 + r_2 - 1$) are called fundamental units. Define integers e_k as follows. If $K_1^{(1)}, \dots, K^{(r_1)})_1$ are the real conjugates of K_1 and $K_1^{(r_1+1)}, \dots, K_1^{(r_1+r_2)}$ the complex ones, then we define

$$e_k = \begin{cases} 1 & \text{if } k \leq r_1 \\ 2 & \text{if } r_1 < K \leq r. \end{cases}$$

Then $R = \det(e_k \log|e_l^{(k)}|)(l, k = 1 \text{ to } r)$; here $\epsilon_l^{(k)}$ denoting the conjugates of ϵ_l in the usual order.

The regulator R is then a real number $\neq 0$ and without loss of generality, one may assume R to be positive. It can then be shown that R is independent of the choice of the fundamental units. In the case of the rational number field and an imaginary quadratic field, R is by definition, equal to 1. Further in (73), D denotes the absolute discriminant of K_1 and W, the number of roots of unity in K_1.

We shall now take $K_1 = K$, i.e. the absolute class field. Then K is totally imaginary of degree $2h$, i.e. $r_1 = 0$, $r_2 = h$; since K is unramified over K_0, $D = d^h$. We shall further assume that $d < -4$. Then (73) may be rewritten as **78**

$$\frac{(2\pi)^h RH}{W|d|^{h/2}} = \frac{\pi h}{\sqrt{|d|}} \prod_{\chi}{}' \frac{2\pi}{\sqrt{|d|}} \left(-\sum_A \chi(A) \log F(A)\right)$$

or

$$\frac{2RH}{W} = h \prod_{\chi}{}' \left(-\sum_A \chi(A) \log F(A)\right). \tag{74}$$

With every element A_k of a finite group $\{A_1, \ldots, A_h\}$, we associate an indeterminate $u_{A_k}(k = 1, \ldots, h)$. Then the determinant $|u_{A_k^{-1}A_l}|$ is called the **group determinant** and was first introduced by Dedekind and extensively used by Frobenius. In the case of an abelian group we have then a decomposition of this group determinant, as follows:

$$\left|u_{A_k^{-1}A_l}\right| = \prod_{\chi}\left(\sum_A \chi(A) u_A\right) = \left(\sum_A u_A\right)\left(\prod_{\chi \neq 1} \sum_A \chi(A) u_A\right).$$

Let us suppose $A_h = E$ (the identity); we can show by an elementary transformation that

$$\left|u_{A_k^{-1}A_l}\right| = \left(\sum_A u_A\right)\left|u_{A_k^{-1}A_\iota} - u_{A_k^{-1}}\right| (k, l = 1 \text{ to } h-1).$$

Then, from the above, we deduce that

$$\left|u_{A_k^{-1}A_\iota} - u_{A_k^{-1}}\right| = \prod_{\chi \neq 1}\left(\sum_A \chi(A) u_A\right).$$

Taking $u_A = -\log F(A)$, we obtain

$$\prod_{\chi\neq 1}\left(-\sum_{A}\chi(A)\log F(A)\right)=\det\left(\log\frac{F(A_k^{-1})}{F(A_k^{-1}A_l)}\right)$$

$$=\det\left(\log\frac{\sqrt{N(b_{A_k})}|\eta(z_{A_k})|^2}{\sqrt{N(\mathfrak{b}_{A_kA_l^{-1}})}|\eta(Z_{A_kA_l^{-1}})|^2}\right)=\Delta(\text{say}).$$

From 74, we gather that 79

$$\frac{2H}{W\cdot h}=\frac{\Delta}{R}. \tag{75}$$

In other words, Δ/R is a rational number. We shall now discuss the $(h-1)$-rowed determinant Δ.

Now, $\mathfrak{b}^h_{A_l^{-1}}$ is a principal ideal $=(\beta_l)$ (say) with $\beta_l\in K_0$. Then $N(\mathfrak{b}^h_{A_l^{-1}}=|\beta_l|^2$ so that $N(\mathfrak{b}_{A_l^{-1}})^{h/2}=|\beta_l|$. Define

$$\rho_l=\frac{\eta^{24h}(z_E)}{\beta_l^{12}\eta^{24h}\left(z_{A_l^{-1}}\right)}(l=1\ \text{ to }\ h-1).$$

We shall now prove that ρ_l are all units in the absolute class field K. For the same, we first see that ρ_l depends only upon the class A_l^{-1}. If $\mathfrak{b}$ is replaced by $\mathfrak{b}(\lambda)$ with $\lambda\in K_0$, then

$$\rho_l\to\frac{\eta^{24h}(1,z_E)}{(\beta_1\lambda^{12h})\lambda^{-12h}\eta^{24h}\left(1,z_{A_l^{-1}}\right)}=\rho_l$$

since, in the "homogeneous" notation, $\eta^{24}(\lambda\omega_1,\lambda\omega_2)=\lambda^{-12}\eta(\omega_1,\omega_2)$.

Now, we pick out a prime ideal $\mathfrak{p}$ in the class A_l^{-1} with the property that $\mathfrak{p}\mathfrak{p}'=(p)$, $\mathfrak{p}\neq\mathfrak{p}'$; such prime ideals always exist in each class $(\neq E)$ as a consequence of Dirichlet's theorem. We shall show that the ρ_l defined with respect to $\mathfrak{p}$ are units in the absolute class field K.

Let $(\mathfrak{p})=\mathfrak{B}_1,\ldots,\mathfrak{B}_r$ be the decomposition of $\mathfrak{p}$ in K. If we denote $\deg_{K_0}\mathfrak{B}_i=v$, then $r\cdot v=h$. The ideal $\mathfrak{p}^v$ is principal and equal to (β) (say). Let $[\omega_1,\omega_2]$ be an integral base of K_0 and $[\omega_1^*,\omega_2^*]$ an integral base of $\mathfrak{p}$. Then we have $(\omega_1^*\omega_2^*)=(\omega_1\omega_2)P_v$ where P_v is a 2-rowed square matrix of rational integers and $|P_{\mathfrak{p}}|=p$.

Denoting by

$$\varphi_{P_{\mathfrak{p}}}(\omega)=p^{12}\frac{\eta^{24}((\omega_1,\omega_2)P_{\mathfrak{p}})}{\eta^{24}((\omega_1,\omega_2))}$$

(with $\omega=\omega_2/\omega_1$), we know from the theory of complex multiplication that 80

$\varphi_{P_\mathfrak{p}}(\omega) \in K$ and the principal ideal

$$\left(\varphi_{P_\mathfrak{p}}(\omega)\right) = (\mathfrak{p}'^{12})$$

(meaning the extended ideal in K). On taking the v^{th} power on both sides,

$$\left(\varphi_{P_\mathfrak{p}}(\omega)^v\right) = (\mathfrak{p}'^{12v}) = (\mathfrak{p}'^v)^{12} = (\overline{\beta}^{12})$$

or in other words,

$$\left(\varphi_{P_\mathfrak{p}}(\omega)\right)^v = \overline{\beta}^{12}\epsilon \text{ with } \epsilon, \text{ a unit in } K.$$

Writing out explicitly, we have, finally, since $|\beta|^2 = N\mathfrak{p}^v = p^v$,

$$\frac{\beta^{12}\eta^{24v}((\omega_1\omega_2)P_\mathfrak{p})}{\eta^{24v}((\omega_1\omega_2))} = \epsilon, \text{ a unit in } K.$$

The same also holds for a suitable unit ϵ, also with h instead of v since $v|h$. (See references (2) Deuring, specially pp. 32-33, (4) Fricke, (5) Fueter).

If $\sigma : \alpha^{(h)} \to \alpha^{(k)} (k = 1 \text{ to } h)$ denotes the automorphism of K/K_0 corresponding to the class A_k under the isomorphism between $G(K/K_0)$ and the ideal class group of K_0, it can be shown that

$$|\rho_l^{(k)}| = \left(\frac{\sqrt{N(\mathfrak{b}_{A_k})}|\eta(z_{A_k})|^2}{\sqrt{N(\mathfrak{b}_{A_kA_l^{-1}})}\cdot|\eta(z_{A_kA_l^{-1}})|^2}\right)^{12h} (k = 1 \text{ to } h-1).$$

We may then rewrite (75) as

$$12^{h-1}h^{h-2}2^hH = W\frac{\det(\log|\rho_l^{(k)}|)}{\det(\log|\epsilon_l^{(k)}|)}(k, l = 1 \text{ to } h-1), \tag{76}$$

where $\epsilon_1, \ldots, \epsilon_{h-1}$ are a system of fundamental units and $\epsilon_l^{(k)}$, $\rho_l^{(k)}$ denote the conjugates of ϵ_l and $\rho_l (l = 1 \text{ to } h-1)$ respectively, the conjugates being chosen in the manner indicated above. Since the number on the left is strictly positive, the $(h-1)$ units $\rho_1, \ldots, \rho_{h-1}$ are independent, i.e. they have no relation between them. Hence the group generated by these units is of finite index in the whole **81**
unit group and the index is precisely given by the integer on the left side of (76).

We have thus proved

Theorem 7. *Let K be the absolute class field of an imaginary quadratic field $K_0 = \mathbf{Q}(\sqrt{d})$, $d < -4$, H, h the class numbers of K and K_0 respectively, R the regulator of K and W the number of roots of unity in K. Then for H we have the formula*

$$\frac{H}{h} = \frac{W}{2} \cdot \frac{\Delta}{R}$$

where $\Delta = |(\log|\rho_l^{(k)}|^{1/12h})|$, $l, k = 1, 2, \ldots, h-1$ and $\rho_l^{(k)}$ are conjugates of $h-1$ independent units $\rho_1, \ldots, \rho_{h-1}$ in K, as found above.

Example. We shall take $d = -5$, i.e. $K_0 = \mathbf{Q}(\sqrt{-5})$.

The class number h of K_0 is 2. The absolute class field K is then a biquadratic field and is given by $K_0(\sqrt{-1}) = \mathbf{Q}(\sqrt{-5}, \sqrt{-1})$. Further,

$$\begin{array}{c} \mathbf{Q}(\sqrt{5}, \sqrt{-1}) = K \\ \Big| \,)2 \\ \mathbf{Q}(\sqrt{-5}) = K_0 \\ \Big| \,)2 \\ \mathbf{Q} \end{array}$$

$W = 4$ and the fundamental unit is $\epsilon = (1 + \sqrt{5})/2$. We take A_1, $A_2 (= E)$ to be the two ideal classes of K_0 and the unit $\epsilon_1 = \epsilon^{-1}$. The bases for the two ideal representatives are given by $\mathfrak{b}_{A_1} = [1, (1 + \sqrt{-5})/2]$ and $\mathfrak{b}_{A_2} = [1, \sqrt{-5}]$. (Refer to the example on page 71. Then from the above, we have

$$|\rho_1| = \left(\frac{2|\eta(\sqrt{-5})|^4}{\left|\eta\left(\frac{1+\sqrt{-5}}{2}\right)\right|^4} \right)^{12}$$

and from (69), we have then $|\rho_1| = |\epsilon_1|^{12}$. From (76), we gather then that

$$12.2^2 \cdot H = 4 \cdot \frac{\log(|\rho_1|)}{\log(|\epsilon_1|)} = 4.12 \quad \text{or} \quad H = 1.$$

82

We shall consider, later, class numbers of more general relative abelian extensions (which are not necessarily unramified) of imaginary quadratic fields by using Kronecker's second limit formula.

3 The Kronecker Limit Formula for real quadratic fields and its applications

In the last section, we applied Kronecker's first limit formula to determine the constant term in the expansion at $s = 1$ of the zeta-function $\zeta(s, A)$ associated with an ideal class A of an *imaginary* quadratic field over $\mathbf{Q}$. In what follows, we shall consider first, a similar problem for a *real* quadratic field $K_0 = \mathbf{Q}(\sqrt{d})$, $d > 0$. This problem which is more difficult, was solved by Hecke in his paper, "Über die Kroneckersche Grenzformel für reelle quadratische Körper und die Klassenzahl relativ-abelscher Körper". We shall, however, follow a method slightly different from Hecke's.

We start from the series

$$f(z, s) = y^s \sum_{m,n=-\infty}^{\infty}{}' -|m + nz|^{-2s},$$

with $z = x + iy$, $y > 0$ and $s = \sigma + it$, $\sigma > 1$. It is easy to verify that $f(z, s)$ is a 'non-analytic modular function', i.e. for a modular transformation $z \to z^* = (\alpha z + \beta)/(\gamma z + \delta)$, we have $f(z^*, s) = f(z, s)$.

We now make a remark which will not be used later, but which is interesting in itself, namely, $f(z, s)$ satisfies the partial differential equation

$$y^2 \Delta f = s(s - 1) f \tag{77}$$

where $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the Laplace operator.

For proving this, let us first observe that if $F(z, w)$ is a complex-valued function, twice differentiable in z and w and if $(z, w) \to (z^*, w^*)$ where $z^*(\alpha z + \beta)(\gamma z + \delta)^{-1}$, $w^* = (\alpha w + \beta)(\gamma w + \delta)^{-1}$ with $\alpha, \beta, \gamma, \delta$ real and $\alpha\delta - \beta\gamma = 0$, then 83
it can be shown directly by computation that

$$(z - w)^2 \frac{\partial^2 (F(z, w))}{\partial z \partial w} = (z^* - w^*)^2 \frac{\partial^2 F(z^*, w^*)}{\partial z^* \partial w^*}.$$

Setting $w = \bar{z}$ and $\alpha\delta - \beta\gamma > 0$, we see that $y^2\Delta$ is an operator invariant under the transformation $z \to z^*$. Now consider the function y^s. It satisfies the equation

$$y^2 \Delta (y^s) = s(s - 1) y^s.$$

If we use the invariance property of $y^2\Delta$, then we see that

$$y^2 \Delta (y^{*s}) = s(s - 1) y^{*s}.$$

Since $y^* = (\alpha\delta - \beta\gamma)y \cdot |\gamma z + \delta|^{-2}$, we have

$$y^2\Delta(y^s|\gamma z + \delta|^{-2s}) = s(s-1)y^s|\gamma z + \delta|^{-2s}$$

where γ, δ are real numbers not both zero.

It is now an immediate consequence that $f(z, s)$ satisfies the equation (77). In fact, it also follows that the series $y^s \sum\limits_{(m,n)}' |m + nz|^{-2s} e^{2\pi i(mu+nv)}$ satisfies the differential equation (77).

We shall now derive some properties of a function F which satisfies the equation (77).

Suppose the function F itself can be written in the form,

$$F = (F_{-1})/(s-1) + F_0 + F_1(s-1) + \cdots ;$$

then on setting $s(s-1)F = (s-1)^2F_- + (s-1)F$ in (77) = and by comparing coefficients, we obtain the recurrence relation

$$y^2\Delta F_n = F_{n-1} + F_{n-2}, n = 0, 1, 2, \ldots,$$

where $F_{-2} = 0$.

Now, consider the function $f(z, s)$. From Kronecker's first limit formula, $F_{-1} = \pi$ and from the above recurrence relation, we obtain $y^2\Delta F_0 = \pi$. If we write $F_0 = -\pi \log y + G$, then $y^2\Delta G = 0$. In other words, $G = F_0 + \pi \log y$ is a potential function. This is also explained by the fact that

$$G = 2\pi(C - \log 2 - \log|\eta(z)|^2).$$

84

For the function $F(z, s) = y^s \sum'_{m,n} |m + nz|^{-2s} e^{2\pi i(mu+nv)}$, from Kronecker's second limit formula, $F_{-1} = 0$ so that $y^2\Delta F_0 = 0$, i.e. F_0 is a potential function, which can also be seen directly from the expression for F_0.

These remarks stand in connection with the work of Maass and Selberg on Harmonic analysis.

Now, let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be the matrix of a hyperbolic substitution having two real fixed points ω, $\omega'(\omega \neq \omega')$, both being finite. We shall assume without loss of generality, that $\omega' < \omega$. Set $u = (z-\omega)(z-\omega')^{-1}$. Then $u^* = (z^*-\omega)(z^*-\omega')^{-1}$. The transformation $z \to z^*$ corresponds to the transformation $u \to u^* = \lambda u$ with λ, a positive real constant $\neq 1$. We can also assume without loss of generality that $\lambda > 1$ (otherwise, we may take the inverse substitution). If we introduce a new variable v,by defining $u = \lambda^v$, then the transformation $u \to u^* = \lambda u$ goes over to $v \to v^* = v + 1$.

Consider a non-analytic function f of z, invariant under the substitution $z \to z^* = (\alpha z + \beta)(\gamma z + \delta)^{-1}$. Then f considered as a function of v, has period 1. If $v = v_1 + iv_2$, then $f(z) = g(v_1, v_2)$ has period 1 in v_1.

Now $|u| = \lambda^{v_1}$ or $v_1 = \dfrac{\log |u|}{\log \lambda}$ so that $e^{2\pi i v_1} = |u|^{2\pi i/\log\lambda}$. If $g(v_1, v_2)$ possesses a valid Fourier expansion with respect to v_1, it is of the form

$$\begin{aligned} f(z) = g(v_1, v_2) &= \sum_{n=-\infty}^{\infty} c_n(v_2) e^{2\pi i n v_1} \\ &= \sum_{n=-\infty}^{\infty} c_n^* \left(\frac{u}{\overline{u}}\right) |u|^{2\pi i n/\log\lambda} \end{aligned} \tag{78}$$

since $v_2 = \dfrac{\log u/\overline{u}}{2i \log \lambda}$. In general, the Fourier coefficients c_n^* in (78), are not constants but if u has a constant argument, i.e. if $u/\overline{u}$ is a constant, then c_n^* are constants.

We shall compute the Fourier coefficients c_n^* of $f(z, s)$ in the case when ω, ω' come from a real quadratic field $K_0 = \mathbf{Q}(\sqrt{d})$ with discriminant $d > 0$. It is interesting to see that upto certain factors, *the Fourier coefficients are just* **85**
Hecke's "zeta-functions with Grössencharacters" ssociated with K_0.

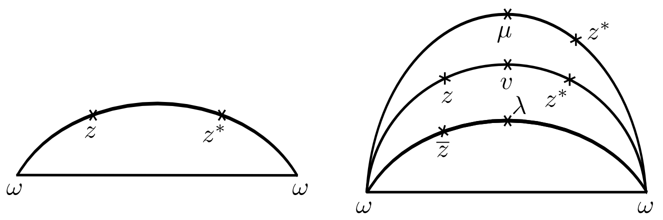
Suppose ϵ is a non-trivual ($\neq \pm 1$) unit in K_0. Let $[\omega, 1]$ be an integral basis of an ideal $\mathfrak{b}$ in K_0. Without loss of generality, we may suppose that $\omega > \omega'$ (otherwise $-\omega$ has this property!).

Now $(\epsilon)\mathfrak{b} = \mathfrak{b}$ so that $\epsilon\omega = \alpha\omega + \beta$, $\epsilon = \gamma\omega + \delta$ with rational integers α, β, γ, δ. We may also write

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \omega & \omega' \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{pmatrix}$$

It follows then that $\alpha\delta - \beta\gamma = \epsilon\epsilon' = \pm 1$. In the case $\alpha\delta - \beta\gamma = 1$, define u such that $z = \dfrac{\omega u + \omega'}{u + 1}$ or if $z^* = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, then $z^* = \dfrac{\omega u^* + \omega'}{u^* + 1}$; in other words, $u = \dfrac{z - \omega'}{\omega - z}$ and $u^* = \epsilon^2 u$. If $\alpha\delta - \beta\gamma = N(\epsilon) = -1$, define $u = \dfrac{z - \omega'}{\omega - z}$ and if $z^* = \dfrac{\alpha\overline{z} + \beta}{\gamma\overline{z} + \delta}$, $u^* = -\epsilon^2 \overline{u}$. Since $z \to z^*$ is a hyperbolic transformation, the points ω', z, z^*, ω lie on the same segment of a circle in the case $N(\epsilon) = 1$. We may assume without loss of generality, that z^* lies to the right of z, or otherwise, we can take the reciprocal substitution. We have then $\epsilon^2 > 1$, since $\epsilon > \epsilon'$, as a consequence of our assumption that z^* lies to the right of $z \cdot \epsilon$ itself might be positive or negative. If ϵ is negative, we take instead of $\alpha, \beta, \gamma, \delta$, the integers

$-\alpha$, $-\beta$, $-\gamma$, $-\delta$ so that we secure finally that $\epsilon > 1$. In the second case, when $N(\epsilon) = -1$, if z lies on the small segment $\omega'\lambda\omega$ (see figure), z^* would lie on the big segment $\omega'\mu\omega$ which is the complement of the small segment obtained by reflecting the original one on $\omega'\omega$. If we take z on the semicricle $\omega' v\omega$ on $\omega'\omega$ as diameter, then z and z^* would lie on the same segment. So, in this case, under the same argument, $\epsilon > 1$.



In either case, $\epsilon > 1$ and consequently $\epsilon' < 1$ so that $\epsilon - \epsilon' > 0$. But **86**
$\epsilon - \epsilon' = \gamma(\omega - \omega')$ implies that $\gamma > 0$.

Now, if z lies on $\omega' v\omega$, u is purely imaginary with $\arg u = \pi/2$. On setting $u = u'i$, we obtain the transformation $u' \to u'^* = \epsilon^2 u'$; u' is real and positive. Hereafter we need not distinguish between the two cases.

If we define v such that $u = \epsilon^{2v}$ (writing u instead of u') then $u \to u^*$ becomes $v \to v + 1$. The function $f(z, s) = y^s {\sum'}_{m,n} |m + nz|^{-2s}$, is invariant under the modular substitution $z \to (\alpha z + \beta)(\gamma z + \delta)^{-1}$ and hence as a function of v, it has period 1. It has a valid Fourier expansion in v of the form $f(z, s) = \sum\limits_{k=-\infty}^{\infty} a_k e^{2\pi ikv}$. Now

$$
\begin{aligned}
a_k &= \int_0^1 f(z, s) e^{-2\pi ikv} dv \\
&= \frac{1}{2\log\epsilon} \int_1^{\epsilon^2} f(z, s) u^{-\pi ik/\log\epsilon} \frac{du}{u}.
\end{aligned}
$$

On changing the variable from u to z, the series $f(z, s)$ being uniformly convergent on the corresponding segment of $\omega' v\omega$, we may interchange integration and summation and obtain

$$
a_k = \frac{1}{2\log\epsilon} \sum_{m,n} \int_1^{\epsilon^2} \frac{y^s}{|m + nz|^{2s}} u^{-\pi ik/\log\epsilon} \frac{du}{u}.
$$

From the relation $z = \dfrac{\omega ui + \omega'}{ui + 1}$ it follows that

$$y = \frac{u(\omega - \omega')}{u^2 + 1} \text{ and } m + nz = \frac{(m + n\omega)ui + (m + n\omega')}{ui + 1}$$

Setting $\beta = m + n\omega$ then $\beta \in \mathfrak{b}$ since m and n are both rational integers and $\omega - \omega' = N(\mathfrak{b})\sqrt{d}$. Then our integral becomes

$$a_k = \frac{1}{2\log\epsilon}(N(\mathfrak{b}))^s d^{s/2} \sum_{\mathfrak{b}|\beta \neq 0} \int_1^{\epsilon^2} \left(\frac{u}{\beta^2 u^2 + \beta'^2}\right)^s u^{-\pi ik/\log\epsilon} \frac{du}{u}.$$

To bring the right hand-side to proper shape again, we use an idea of Hecke. We shall now get rid of β in the integrand by introducing $U = u|\beta/\beta'|$. The **87**
integral then reduces to

$$\left|\frac{\beta}{\beta'}\right|^{\pi ik/\log\epsilon} |N(\beta)|^{-s} \int_{|\beta/\beta'|}^{|(\beta\epsilon/\beta'\epsilon')|} \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \frac{du}{u},$$

on writing u instead of U. For two elements $\beta, \gamma, \neq 0$, $(\beta) = (\gamma)$ implies that $\gamma = \pm\beta\epsilon^n$, $n = 0, \pm 1, \pm 2, \ldots$ with $\epsilon > 1$, being the fundamental unit in $\mathbf{Q}(\sqrt{d})$. Then

$$|\gamma| = |\beta\epsilon^n|, |\gamma'| = |\beta'\epsilon'^n| \text{ and } \left|\frac{\gamma}{\gamma'}\right| = \left|\frac{\beta\epsilon^n}{\beta'\epsilon'^n}\right|.$$

We have therefore, on taking into account $\gamma = \beta\epsilon^n$ of $\gamma = -\beta\epsilon^n$,

$$\sum_{\mathfrak{b}|\beta\neq 0} \int_{|\beta/\beta'|}^{|(\beta\epsilon/\beta'\epsilon')|} \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \frac{du}{u}$$
$$= 2 \sum_{\mathfrak{b}|(\beta)\neq 0} \sum_{n=-\infty}^{\infty} \int_{|(\beta\epsilon^n/\beta'\epsilon'^n)|}^{|(\beta\epsilon^{n+1}/\beta'\epsilon'^{n+1})|} \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \frac{du}{u}.$$

Now since $\epsilon > 1$, the intervals $(|\beta/\beta') \cdot \epsilon^{2n}, |\beta/\beta'| \cdot \epsilon^{2(n+1)})$ fill out the half-line $(0, \infty)$ exactly once, without gaps and overlaps, as n tends to $-\infty$ on one side and $+\infty$ on the other side. Hence we have

$$2 \sum_{n=-\infty}^{\infty} \int_{|(\beta\epsilon^n/\beta'\epsilon'^n)|}^{|(\beta\epsilon^{n+1}/\beta'\epsilon'^{n+1})|} \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \cdot \frac{du}{u} = 2\int_0^\infty \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \cdot \frac{du}{u},$$

and

$$2\int_0^\infty \frac{u^{s-(\pi ik/\log\epsilon)}}{(u^2+1)^s} \frac{du}{u} = \frac{\Gamma\left(\frac{s}{2} - \frac{\pi ik}{2\log\epsilon}\right)\Gamma\left(\frac{s}{2} + \frac{\pi ik}{2\log\epsilon}\right)}{\Gamma(s)}$$

Therefore, we have

$$\begin{aligned} a_k &= \int_0^1 f(z,s)e^{-2\pi ikv}dv \\ &= \frac{1}{2\log\epsilon}(N(\mathfrak{b}))^s d^{s/2}\frac{\Gamma\left(\frac{s}{2}-\frac{\pi ik}{2\log\epsilon}\right)\Gamma\left(\frac{s}{2}+\frac{\pi ik}{2\log\epsilon}\right)}{\Gamma(s)}\times \\ &\quad \times \sum_{\mathfrak{b}|(\beta)\neq 0}\left|\frac{\beta}{\beta'}\right|^{\pi ik/\log\epsilon}|N(\beta)|^{-s}, \end{aligned}$$

since the expression under the summation Σ does not change for associated elements β and γ. 88

Upto a product of Γ-factors, the Fourier coefficients a_k are Dirichlet series.

For $k = 0$,

$$\begin{aligned} a_0 = \int_0^1 f(z,s)dv &= \frac{1}{2\log\epsilon}\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}d^{s/2}\sum_{\mathfrak{a}\in A}(N(\mathfrak{a}))^{-s} \\ &= \frac{1}{2\log\epsilon}\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}d^{s/2}\zeta(s,A), \end{aligned}$$

where A denotes the ideal class of $\mathfrak{b}^{-1}$ in the wide sense. If $k \neq 0$, we define for $\sigma > 1$,

$$\begin{aligned} \zeta(s,\widehat{\chi},A) &= \sum_{\mathfrak{a}\in A}\widehat{\chi}(\mathfrak{a})(N(\mathfrak{a}))^{-s} \\ &= (\widehat{\chi}(\mathfrak{b}))^{-1}\sum_{\substack{\mathfrak{a}\in A\\ \mathfrak{a}\mathfrak{b}=(\beta)}}\widehat{\chi}((\beta))(N(\mathfrak{a}))^{-s}; \end{aligned}$$

The function $\zeta(s,\widehat{\chi},A)$ is called the zeta-function of the class A and associated with the Grössencharacter $\widehat{\chi}$, which is defined as follows:

The character $\widehat{\chi}$ is defined on principal ideals (β) of K_0 as

$$\widehat{\chi}((\beta)) = \left|\frac{\beta}{\beta'}\right|^{\pi ik/\log\epsilon}$$

($\widehat{\chi}((\beta))$ is independent of the generator β by definition). Then $\widehat{\chi}$ is extended to all ideals $\mathfrak{i}$ as follows: If $\mathfrak{i}^k = (\nu)$, define $\widehat{\chi}(\mathfrak{i})$ as a h^{th} root of $\widehat{\chi}((\nu))$ so that

$$\widehat{\chi}(\mathfrak{i}^h) = \widehat{\chi}((\nu)) = \left|\frac{\nu}{\nu'}\right|^{\pi ik/\log\epsilon}$$

It is clear that $\widehat{\chi}$ is multiplicative. 89

We have now

$$\widehat{\chi}(\mathfrak{b})\zeta(s,\widehat{\chi},A) = \sum_{\substack{\mathfrak{a}\in A\\ \mathfrak{a}\mathfrak{b}=(\beta)\neq(0)}} \widehat{\chi}((\beta))(N(\mathfrak{a}))^{-s}$$
$$= (N(\mathfrak{b}))^{s} \sum_{\mathfrak{b}|(\beta)\neq(0)} \widehat{\chi}((\beta))|N(\beta)|^{-s},$$

so that we may rewrite the expression for a_k as follows:

$$a_k = \frac{2\log\epsilon}{d}^{s/2} \frac{\Gamma\left(\frac{s}{2} - \frac{\pi ik}{2\log\epsilon}\right)\Gamma\left(\frac{s}{2} + \frac{\pi ik}{2\log\epsilon}\right)}{\Gamma(s)} \widehat{\chi}(b)\zeta(s,\widehat{\chi},A).$$

One can make some applications from the nature of the Fourier coefficients a_k.

We know that the function $f(z,s)$ satisfies the following functional equation, viz.

$$\pi^{-s}\Gamma(s)f(z,s) = \pi^{-(1-s)}\Gamma(1-s)f(z,1-s).$$

One can then show that from the analytic continuation of $f(z,s)$ it follows that the *Fourier coefficients a_k as functions of s have also analytic continuations into the whole s-plane and satisfy a functional equation similar to the above. In the particular* case, when $k = 0$, it follows that

$$\pi^{-s}d^{s/2}\Gamma^2\left(\frac{s}{2}\right)\zeta(s,A) = \pi^{-(1-s)}d^{(1-s)/2}\Gamma^2\left(\frac{1-s}{2}\right)\zeta(1-s,A).$$

Hence, for the zeta-function $\zeta_{k_0}(s) = \sum_A \zeta(s,A)$, we have

$$\pi^{-s}d^{s/2}\Gamma^2\left(\frac{s}{2}\right)\zeta_{K_0}(s) = \pi^{-(1-s)}d^{(1-s)/2}\Gamma^2\left(\frac{1-s}{2}\right)\zeta_{K_0}(1-s).$$

This was discovered first, by Hecke, who also introduced the zeta-functions with Grössencharacters and derived a functional equation for the same.

We shall now use Kronecker's first limit formula to study the behaviour of 90
$\zeta(s,A)$ at $s = 1$.

From Kronecker's first limit formula, we have

$$f(z,s) = \frac{\pi}{s-1} + 2\pi(C - \log 2 - \log\ \sqrt{y}|\eta(z)|^2 + \cdots).$$

It can be shown that the series on the right can be integrated term by term with respect to v (after transforming z into v) in the interval $(0, 1)$, i.e.

$$\begin{aligned}\int_0^1 f(z,s)dv &= \frac{1}{2\log\epsilon}\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}d^{s/2}\zeta(s,A)\\ &= \frac{\pi}{s-1} + 2\pi(C-\log 2) - 2\pi\int_0^1 \log(\sqrt{y}|\eta(z)|^2)dv + \cdots .\end{aligned}$$

It follows therefore that the function on the left has a pole at $s = 1$ with residue π and the constant term in the expansion is provided by the integral which cannot in general be computed. It is to be noted here that in the case of the imaginary quadratic field, we had only the integrand on the right side and the argument z was an element of the field, but here z is a (complex) variable. One cannot get rid of the integral even if one uses the series expansion for the integrand.

The above formula was found by Hecke and is the **Kronecker limit formula for a real quadratic field**. One can also consider a_k for $k \neq 0$ and obtain a similar formula.

Changing this integral to a contour integral, we shall later obtain an analogue of Kronecker's solution of Pell's equation in terms of elliptic functions.

We have now, for real v,

$$\begin{aligned}a_0 = \int_0^1 f(z,s)dv &= \int_v^{v+1} f(z,s)dv\\ &= \frac{1}{2\log\epsilon}\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}d^{s/2}\zeta(s,A).\end{aligned}$$

Using the Legendre formula 91

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \sqrt{\pi}2^{1-s}\Gamma(s),$$

we obtain

$$\frac{\Gamma(s)}{\Gamma^2\left(\frac{s}{2}\right)} = \frac{\Gamma(s)}{\Gamma^2\left(\frac{s}{2}\right)}\frac{\Gamma^2\left(\frac{s+1}{2}\right)}{\Gamma^2\left(\frac{s+1}{2}\right)} = \frac{\Gamma^2\left(\frac{s+1}{2}\right)}{\pi 2^{2(1-s)}\Gamma(s)}.$$

But on the other hand,

$$\frac{\Gamma^2\left(\frac{s+1}{2}\right)}{\Gamma(s)} = 1 + \text{ terms in } (s-1)^2,$$

so that we have

$$\begin{aligned}\zeta(s, A) &= 2\log\epsilon d^{-s/2}\frac{\Gamma(s)}{\Gamma^2\left(\frac{s}{2}\right)}\int_v^{v+1} f(z, s)dv\\ &= \frac{2\log\epsilon}{\pi\sqrt{d}}(2^2 d^{-\frac{1}{2}})^{(s-1)}(1 + \text{ terms in } (s-1)^2)\int_v^{v+1} f(z, s)dv.\end{aligned}$$

By setting $(2^2 d^{-\frac{1}{2}})^{(s-1)} = e^{(s-1)(2\log 2 - 2\log\sqrt[4]{d})}$ and expanding it in powers of $(s-1)$, and applying Kronecker's first limit formula for $f(z, s)$, we obtain finally,

$$\begin{aligned}\zeta(s, A) = \frac{2\log\epsilon}{\pi\sqrt{d}}&(1 + (s-1)(2\log 2 - 2\log\sqrt[4]{d}) + \cdots)\times\\ &\times\Bigg(\frac{1}{s-1} + 2C - (2\log 2 - 2\log\sqrt[4]{d}) -\\ &-2\int_v^{v+1}\log(\sqrt{y}\sqrt[4]{d}|\eta(z)|^2)dv + \cdots\Bigg),\end{aligned}$$

i.e.

$$\zeta(s, A) = \frac{2\log\epsilon}{\sqrt{d}}\left(\frac{1}{s-1} + 2C - 2\int_v^{v+1}\log(\sqrt{y}\sqrt[4]{d}|\eta(z)|^2)dv + \cdots\right). \tag{79}$$

From this, we deduce that $\zeta(s, A)$ has a pole at $s = 1$ with residue $\dfrac{2\log\epsilon}{\sqrt{d}}$ which **92** is independent of the ideal class A. This result was first discovered by Dirichlet.

It would be nice if one could compute the integral and express it in terms of an analytic function, but it looks impossible. We shall only simplify it to a certain extent by converting it into a contour integral.

From the substitution $ui = \dfrac{z-\omega'}{\omega - z}$ follows that

$$u = \operatorname{Im}\left(\frac{z-\omega'}{\omega - z}\right) = \frac{y(\omega-\omega')}{(z-\omega)(\bar{z}-\omega)}$$

and similarly

$$u^{-1} = \frac{y(\omega-\omega')}{(z-\omega')(\bar{z}-\omega')}.$$

On multiplying the two, we have

$$1 = \frac{y^2}{\left\{\dfrac{(z-\omega)(z-\omega')}{\omega-\omega'}\right\}\left\{\dfrac{(\bar{z}-\omega)(\bar{z}-\omega')}{\omega-\omega'}\right\}}. \tag{80}$$

Consider now the expression

$$F(z) = \frac{(z-\omega)(z-\omega')}{N(\mathfrak{b})} = az^2 + bz + c. \tag{81}$$

We then claim that a, b, c are rational with $(a, b, c) = 1$ and $b^2 - 4ac = d$. From Kronecker's generalization of Gauss's theorem on the content of a product of polynomials with algebraic numbers as coefficients, we have for the product $(\xi - \omega\eta)(\xi - \omega'\eta)$, $(1, -\omega)(1, -\omega') = (1, -(\omega + \omega'), \omega\omega')$ or in other words, $\mathfrak{b}\mathfrak{b}' = (1, \lambda, \mu)$ where λ and μ are rational, i.e.

$$\left(\frac{1}{N(\mathfrak{b})}, \frac{\lambda}{N(\mathfrak{b})}, \frac{\mu}{N(\mathfrak{b})}\right) = 1.$$

But, from our definition of $F(z)$, $a = \dfrac{1}{N(\mathfrak{b})}$, $b = \dfrac{\lambda}{N(\mathfrak{b})}$, $c = \dfrac{\mu}{N(\mathfrak{b})}$ so that $(a, b, c) = 1$. The discriminant of $F(z)$ is

$$b^2 - 4ac = \frac{(\omega + \omega')^2 - 4\omega\omega'}{N(\mathfrak{b})^2} = \frac{(\omega - \omega')^2}{N(\mathfrak{b})^2} = d.$$

Further $a = \dfrac{1}{N(\mathfrak{b})} > 0$. We can then show that the class of the quadratic form **93**
$F(z)$ is uniquely determined by the ideal class A.

From (80) it follows that $y^2 d = F(z) \cdot F(\bar{z})$ or equivalently, $\sqrt{y}\sqrt[4]{d} = \sqrt[4]{F(z) \cdot F(\bar{z})}$.

Now, for the integral in (79), we have

$$\frac{-4\log\epsilon}{\sqrt{d}} \int_v^{v+1} \log(\sqrt{u}\sqrt[4]{d}|\eta(z)|^2)dv = 2\int_z^{z^*} \log(|\sqrt[4]{F(z)}\eta(z)|^2)\frac{dz}{F(z)},$$

since, by definition of v,

$$dv = \frac{-\sqrt{d}}{2\log\epsilon} \cdot \frac{dz}{F(z)}$$

and the transformation $v \to v + 1$ corresponds to $z \to z^*$ on the segment of the orthogonal circle. On the right side, z may be taken arbitrarily on the segment.

Thus, let A be an ideal class, in the wide sense, of $K_0 = \mathbf{Q}(\sqrt{d})$, $d > 0$ and $\mathfrak{b}$ an ideal in A with an integral basis $[1, \omega]$, $\omega > \omega'$. Corresponding to $\mathfrak{b}$, let $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ be defined as on *p*. 85 and let for $z \in \mathfrak{h}$, $z^* = (\alpha z + \beta)(\gamma z + \delta)^{-1}$. Let, further, $\epsilon > 1$ be the fundamental unit in K_0. Then we have

Theorem 8 (Hecke). *For the zeta-function $\zeta(s, A)$ associated with the class A, we have the limit formula*

$$\lim_{s\to 1}\left(\zeta(s, A) - \frac{2\log\epsilon}{\sqrt{d}}\frac{1}{s-1}\right) = \frac{4C\log\epsilon}{\sqrt{d}} + 2\int_z^{z^*}\log(|\sqrt[4]{F(z)}\eta(z)|^2)\frac{dz}{F(z)},$$

where the integration is over a segment zz^* of the semicircle on $\omega'\omega$ as diameter and $F(z)$ is defined by (81).

Now, the question arises whether one could transform the integral in such a way that it is independent of the choice of z and also the path of integration. It is possible to do that, by the method of Herglotz who reduced this to an integral involving simpler functins than $\eta(z)$. **94**

If $N(\epsilon) = 1$, then the transformation $v \to v + 1$ is equivalent to $z \to z^* = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ with $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, a modular matrix.

If $N(\epsilon) = -1$, on replacing ϵ by ϵ^2, $N(\epsilon^2) = 1$ and the transformation $v \to v + 1$ goes over to the transformation $z \to \dfrac{\alpha' z + \beta'}{\gamma' z + \delta'}$ with $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)$ a modular matrix. Further, because of the periodicity of $f(z, s)$,

$$\int_v^{v+2} f(z, s)dv = 2\int_v^{v+1} f(z, s)dv$$

so that we may assume without loss of generality that $N(\epsilon) = 1$.

Now, the behaviour of $\sqrt[4]{F(z)}\cdot\eta(z)$ under a modular substitution can be studied. We know that $\eta(z^*) = \rho\sqrt{\gamma z + \delta}\,\eta(z)$ where ρ is a 24th root of unity. Further,

$$(z^* - \omega)(z^* - \omega') = \frac{(z - \omega)(z - \omega')}{(\gamma z + \delta)^2}$$

if one uses the fact that $\omega = \omega^*$ and $\gamma\omega + \delta = \epsilon$.

From these two, it follows that

$$\sqrt[4]{F(z^*)} = \kappa\frac{\sqrt[4]{F(z)}}{\sqrt{\gamma z + \delta}}$$

with κ, a 4th root of unity. Therefore

$$\sqrt[4]{F(z^*)}\eta(z^*) = \rho\kappa\sqrt[4]{F(z)}\eta(z). \tag{82}$$

(We emphasize here that (82) holds only for a hyperbolic substitution and a power of the same, since we have made essential use of the fact that ω and ω'

are fixed points of the substitution in obtaining the transformation formula for $F(z)$.)

Consider the function $\sqrt[4]{F(z)}\eta(z)$. It is an analytic function having no zeros in the upper half plane so that on choosing a single valued branch of **95**
$\log(\sqrt[4]{F(z)}\eta(z))$, we have a regular function in the upper z-half plane.

Let us denote $\log \sqrt[4]{F(z)}\eta(z)$ by $g(v)$. Then (82) implies that $g(v+1)-g(v) = 2\pi i\lambda$ (say) with λ rational. One can then express λ by means of the so-called Dedekind sums. Setting $g(v)-2\pi i\lambda v = h(v)$, we have $h(v+1) = h(v)$ possesses a Fourier development in $e^{2\pi iv}$, or in other words, $h(v) = \sum\limits_{n=-\infty}^{\infty} c_n e^{2\pi inv}$, where c_n are constants, since $h(v)$ is an analytic function of v and $c_0 = \int_v^{v+1} h(v)dv$. Here v lies in a certain strip enclosing the real axis.

Now, consider the complex integral $c_0 = \int_v^{v+1} h(v)dv$. Here v is, in general, complex and the path of integration may be any curve between v and $v+1$ lying in the strip in which the Fourier expansion is vaid. We have

$$\begin{aligned}
-\frac{\sqrt{d}}{2\log\epsilon}\int_z^{z^*} \log(|\sqrt[4]{F(z)}\eta(z)|^2)\frac{dz}{F(z)} &= 2\int_v^{v+1} \operatorname{Re}(g(v))dv \quad (v \text{ real})\\
&= 2\operatorname{Re}\left(\int_v^{v+1} h(v)dv\right)\\
&= 2\operatorname{Re}(c_0).
\end{aligned}$$

The computation of the integral on the left side therefore reduces to the determination of c_0, which in turn is independent of v and the path of integration. The trick for computing the integral is to convert it into an infinite integral by letting $z \to \infty$ and $z^* \to \alpha/\gamma$. Expanding $g(v)$ in an infinite series, one can express this infinite integral as a definite integral of an elementary function. (See G. Herglotz.)

We shall now obtain an analogue of Kronecker's solution of Pell's equation in this case. Let us define

$$g(A) = 2\int_z^{z^*} \log(|\sqrt[4]{F(z)}\eta(z)|^2)\frac{dz}{F(z)}. \tag{83}$$

Then, associated with a character χ of the ideal class group (in the wide sense), we have

$$L(s,\chi) = \sum_A \chi(A)\zeta(s,A),$$

the summation running over all wide classes A. **96**

From (79), we obtain, for $\chi \neq 1$, the expansion,

$$L(s,\chi) = \sum_A \chi(A)g(A) + \text{ terms involving } (s-1). \tag{84}$$

Before proceeding to derive the analogue of Kronecker's solution of Pell's equation from the above, we shall examine under what conditions, a genus character, which is a character of the narrow class group of $\mathbf{Q}(\sqrt{d})$, is also a character of the wide class group. Let χ be a genus character, defined as follows: If $d = d_1 d_2$, then for all ideals $\mathfrak{a}$ with $(\mathfrak{a}, d_1) = (1)$, χ is defined by $\chi(\mathfrak{a}) = \left(\dfrac{d_1}{N(\mathfrak{a})}\right)$. (We may suppose without loss of generality that d_1 is odd). Now, in general, χ is not a character of the wide class group. It will be so, if $\chi((\alpha)) = 1$ for all integers α with $(\alpha, d_1) = (1)$. For elements α with $N(\alpha) > 0$, this is true by the definition of χ.

If $N(\alpha) < 0$, then

$$\chi((\alpha)) = \left(\frac{d_1}{N((\alpha))}\right) = \left(\frac{d_1}{-N(\alpha)}\right) = 1$$

for all α, if it is true for one such α.

Take $\alpha = 1 + \sqrt{d}$ so that $N(\alpha) = 1 - d < 0$. We need examine only $\left(\dfrac{d_1}{d-1}\right)$, d_1 being odd. Then

$$\left(\frac{d_1}{d-1}\right) = \left(\frac{d-1}{d_1}\right) = \left(\frac{-1}{d_1}\right) = \begin{cases} +1 \text{ if } d_1 > 0 \\ -1 \text{ if } d_1 < 0. \end{cases}$$

Since d is positive, either both d_1, d_2 are positive or both are negative. In the former case, χ continues to be a character of the wide class group and in the latter case, it is no longer a character of the wide class group.

Case (i). Let us assume, that *both d_1 and d_2 are positive.* Then the genus character χ as defined above, is also a wide class character.

We have a decomposition of $L(s,\chi)$ due to Kronecker, from (66) as follows: $L(s,\chi) = L_{d_1}(s)L_{d_2}(s)$ and on taking the values on both sides at $s = 1$, we obtain **97**

$$L(1,\chi) = L_{d_1}(1)L_{d_2}(1).$$

From (84), we have now $L(1,\chi) = \sum_A \chi(A)g(A)$, summation running over all wide classes A.

Further

$$L_{d_1}(1) = \frac{2\log \epsilon_1 h_1}{\sqrt{d_1}}$$

if ϵ_1 denotes the fundamental unit of $\mathbf{Q}(\sqrt{d_1})$ and h_1, the wide class number of $\mathbf{Q}(\sqrt{d_1})$ and similarly for $L_{d_2}(1)$. Thus as an analogue of Kronecker's solution of Pell's equation as derived in (68), we have the following

Proposition 13. *Let χ be a genus character of $\mathbf{Q}(\sqrt{d})$, $d > 0$, corresponding to the decomposition $d = d_1 d_2 (d_1 > 0, d_2 > 0)$ of d. Let h_1, h_2 be the class numbers and ϵ_1, ϵ_2 the fundamental units of $\mathbf{Q}(\sqrt{d_1})$, $\mathbf{Q}(\sqrt{d_2})$ respectively. Then*

$$4h_1h_2 \log \epsilon_1 \log \epsilon_2 = \sqrt{d} \sum_A \chi(A) g(A), \tag{85}$$

where A runs over all the ideal classes of $\mathbf{Q}(\sqrt{d})$ in the wide sense and $g(A)$ is defined by (83).

The expression on the right side of (85) is not very simple. It is not known whether the number on the left side is rational or irrational. It is probable that the number on the left side is a complicated transcendental number, so that one cannot expect a simple value on the right side.

Case (ii). Suppose d_1 and d_2 are both negative. Then again from the decomposition formula (66), we have

$$L(1,\chi) = L_{d_1}(1) L_{d_2}(1) = \frac{2\pi h_1}{w_1\sqrt{|d_1|}} \frac{2\pi h_2}{w_2\sqrt{|d_2|}} = \frac{4\pi^2 h_1 h_2}{w_1 w_2 \sqrt{d}}$$

where h_1, h_2 denote the wide class numbers of $\mathbf{Q}(\sqrt{d_1})$ and $\mathbf{Q}(\sqrt{d_2})$, and w_1, w_2 the number of roots of unity in $\mathbf{Q}(\sqrt{d_1})$ and $\mathbf{Q}(\sqrt{d_2})$ respectively.

We shall see in § 5 that **98**

$$L(1,\chi) = \frac{\pi^2}{\sqrt{d}} \sum_B \chi(B) G(B),$$

B running over all narrow classes of $\mathbf{Q}(\sqrt{d})$ and $G(B)$ being numbers depending only on B.

We shall further prove that $G(B)$ are rational numbers which can be realized in terms of periods of certain abelian integrals of the third kind.

We then have, as an analogue of Kronecker's solution of Pell's equation, the following:

$$\frac{4h_1h_2}{w_1w_2} = \sum_B \chi(B) G(B).$$

Example. Consider the field $\mathbf{Q}(\sqrt{10})$ with discriminant $d = 40 = 5.8$; $d_1 = 5$ and $d_2 = 8$.

The class number of $\mathbf{Q}(\sqrt{10})$ is 2. Since the fundamental unit $\epsilon = 3 + \sqrt{10}$ has norm -1, the narrow classes are the same as wide classes. We can now choose the two ideal class representatives as follows

$$(1) = \mathfrak{b}_1 = [1, \sqrt{10}] \text{ and } \mathfrak{b}_2 = \left[1, \frac{1}{2}\sqrt{10}\right] \text{ with } N(\mathfrak{b}_2) = \frac{1}{2}.$$

The class numbers h_1 and h_2 of $\mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{8})$ are both 1 and the fundamental units are respectively $\dfrac{1+\sqrt{5}}{2}$ and $1+\sqrt{2}$. The only character $\chi \neq 1$ has the property that $\chi(\mathfrak{b}_1) = 1$ and $\chi(\mathfrak{b}_2) = -1$, so that we have from (85), the following:

$$2\log\left(\frac{1+\sqrt{5}}{2}\right)\cdot\log(1+\sqrt{2}) = \sqrt{10}(g(E) - g(A)).$$

We shall make a remark for more general applications. Consider the absolute class field of $\mathbf{Q}(\sqrt{d})$ with $d > 0$. For computing the class number of this field, one can proceed in the same way, as in the case of an imaginary quadratic field. For the same, one requires the computation of $L(1, \chi)$ for arbitrary narrow class characters. This will be done in § 5, for a special type of characters.

4 Ray class fields over $\mathbf{Q}(\sqrt{d})$, $d < 0$

Let K be an algebraic number field of degree n over $\mathbf{Q}$, the field of rational numbers. Let r_1 and $2r_2$ be the number of real and complex conjugates of K respectively, so that $r_1 + 2r_2 = n$. Let $K^{(1)}, \ldots, K^{(r_1)}$ be the real conjugates of K and $K^{(r_1+1)}, \ldots, K^{(n)}$ be the complex conjugates of K. Let, for $\alpha \in K$, $\alpha^{(i)} \in K^{(i)}$, $i = 1, 2, \ldots, n$, denote the conjugates of α. Further let $\mathfrak{f}$ be a given integral ideal in K. Two numbers $\gamma_1 = \dfrac{\alpha_1}{\beta_1}$, $\gamma_2 = \dfrac{\alpha_2}{\beta_2}$ with $\alpha_1, \beta_1, \alpha_2, \beta_2$ integral in K and with $\beta_1\beta_2$ coprime to $\mathfrak{f}$ are *(multiplicatively) congruent modulo* $\mathfrak{f}$ (in symbols, $\gamma_1 \equiv \gamma_2 (\bmod\ {}^*\mathfrak{f})$ if $\alpha_1\beta_2 \equiv \alpha_2\beta_1 (\bmod\ \mathfrak{f})$). If γ_1, γ_2 are integers in K, this is the usual congruence modulo $\mathfrak{f}$. 99

Let us consider non-zero fractional ideals $\mathfrak{a}$ of the form $\mathfrak{a} = \dfrac{\mathfrak{b}}{\mathfrak{c}}$ where $\mathfrak{b}$ and $\mathfrak{c}$ are integral ideals in K coprime to $\mathfrak{f}$. These fractional ideals $\mathfrak{a}$ form, under the usual multiplication of ideals, an abelian group which we shall denote by $\mathfrak{G}_\mathfrak{f}$. Let $\mathfrak{G}_\mathfrak{f}$ be the subgroup of $\mathfrak{G}_\mathfrak{f}$ consisting of all principal ideals (α) for which $\alpha > 0$ (i.e. $\alpha^{(i)} > 0$, for $i = 1, 2, \ldots, r_1$) and $\alpha \equiv 1 (\bmod\ {}^*\mathfrak{f})$. It is clear that if $(\gamma) \in \mathfrak{G}_\mathfrak{f}$ and $\gamma \equiv 1 (\bmod\ {}^*\mathfrak{f})$, then γ might be written as α/β where α and β

are integers in K satisfying the conditions, $\alpha > 0, \beta > 0, \alpha \equiv 1(\mod \mathfrak{f})$ and $\beta \equiv 1(\mod \mathfrak{f})$.

The quotient group $\mathfrak{G}_\mathfrak{f}/\mathfrak{G}_\mathfrak{f}$ is a finite abelian group called the *ray class group modulo* $\mathfrak{f}$; its elements are called *ray classes modulo* $\mathfrak{f}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{G}_\mathfrak{f}$ are *equivalent modulo* $\mathfrak{f}(\mathfrak{a} \sim \mathfrak{b} \mod \mathfrak{f})$ if they lie in the same ray class modulo $\mathfrak{f}$, i.e. $\mathfrak{a} = (\gamma)\mathfrak{b}$ with $(\gamma) \in \mathfrak{G}_\mathfrak{f}$.

If $r_1 = 0$ and $\mathfrak{f} = (1)$, equivalence modulo $\mathfrak{f}$ is the usual equivalence in the wide sense and the ray class group modulo $\mathfrak{f}$ is the class group of K in the wide sense. If $r_1 = 2$, $r_2 = 0$ (i.e. K is a real quadratic field) and $\mathfrak{f} = (1)$, then the equivalence modulo $\mathfrak{f}$ is the equivalence in the narrow sense and the ray class group is the class group of K in the narrow sense. Let χ be a character of the group of ray classes modulo $\mathfrak{f}$. Then associated with χ, we define for $\sigma > 1$, the L-series

$$L(s,\chi) = \sum_{\mathfrak{a}\neq 0} \chi(\mathfrak{a})(N(\mathfrak{a}))^{-s}$$

where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$ in K and the summation is extended over all integral ideals $\mathfrak{a}$ coprime to $\mathfrak{f}$. It is clear that $L(s,\chi)$ is a regular function of s for **100**
$\sigma > 1$. Moreover, due to the multiplicative character of χ, we have for $\sigma > 1$, an Euler-product decomposition for $L(s,\chi)$, namely

$$L(s,\chi) = \prod_{\mathfrak{p}\nmid\mathfrak{f}} (1 - \chi(\mathfrak{p})(N(\mathfrak{p}))^{-s})^{-1}$$

where the infinite product is extended over all prime ideals $\mathfrak{p}$ coprime to $\mathfrak{f}$.

Hecke has shown that $L(s,\chi)$ can be continued analytically as a meromorphic function of s in the whole s-plane and that when χ is a "proper" character, there is a functional equation relating $L(s,\chi)$ with $L(1-s,\overline{\chi})$, where $\overline{\chi}$ is the conjugate character. If $\chi \neq 1$ (the principal character), then $L(s,\chi)$ is an entire function of s. If $\chi = 1$ and $\mathfrak{f} = (1)$, then $L(s,\chi)$ is the Dedekind zeta function of K.

We are interested in determining the value at $s = 1$ of $L(s,\chi)$, in the case when K is a real or imaginary quadratic number field and χ is not the principal character. For this purpose, we need to apply Kronecker's second limit formula. Later, we shall use this for the determination of the class number of the "ray class field" of K, corresponding to the ideal $\mathfrak{f}$.

First we shall investigate the structure of a ray class character χ. Let α and $\beta(\neq 0)$ be integers coprime to $\mathfrak{f}$ such that $\alpha = \beta(\mod \mathfrak{f})$. Then $\gamma = \dfrac{\alpha^2}{\beta^2}$ satisfies $\gamma = 1(\mod {}^*\mathfrak{f})$ and $\gamma > 0$ so that $\chi((\gamma)) = 1$ i.e. $\chi((\alpha^2)) = \chi((\beta^2))$. In other words,

$$\chi((\alpha)) = \pm\chi((\beta)) \quad \text{or} \quad \chi((\alpha/\beta)) = \pm 1.$$

Let $\mathfrak{L}$ be the multiplicative group of $\lambda \neq 0$ in K. Corresponding to a ray class character χ modulo $\mathfrak{f}$, we define a character ν of $\mathfrak{L}$ as follows. For $\lambda \in \mathfrak{L}$, we find an integer $\alpha \in K$ such that $\alpha \equiv 1(\mod \mathfrak{f})$ and $\alpha\lambda > 0$ and define $\nu(\lambda) = \chi((\alpha))$. It is first clear that $\nu(\lambda)$ is well-defined; for, if $\nu(\lambda) = \chi((\beta))$ for another integer $\beta \in K$ satisfying the same conditions, then we see at once that $\dfrac{\alpha}{\beta} \equiv 1(\mod {}^*\mathfrak{f})$ and $\dfrac{\alpha}{\beta} > 0$ and so $\chi((\alpha)) = \chi((\beta))$. It is easily verified that $\nu(\lambda\mu) = \nu(\lambda)\nu(\mu)$. Moreover, $\nu(\lambda) = \pm 1$; for, $\lambda^2 > 0$ and by definition, $\nu(\lambda^2) = \chi((\alpha))$ for α satisfying $\alpha \equiv 1(\mod \mathfrak{f})$ and $\alpha > 0$ so that $(\nu(\lambda))^2 =$ **101**
$\nu(\lambda^2) = \chi((\alpha)) = +1$. We shall call $\nu(\lambda)$, a **character of signature.**

Consider the subgroup $\mathfrak{B}$ of $\mathfrak{L}$ consisting of $\lambda > 0$. Clearly $\nu(\lambda) = 1$ for all $\lambda \in \mathfrak{B}$. Thus $\nu(\lambda)$ may be regarded as a character of the quotient group $\mathfrak{L}/\mathfrak{B}$. Now $\mathfrak{L}/\mathfrak{B}$ is an abelian group of order 2^{r_1} exactly, since one can find $\lambda \in \mathfrak{L}$ for which the real conjugates $\lambda^{(i)}$, $i = 1, 2, \ldots, r_1$, have arbitrarily prescribed signs. Also, there are 2^{r_1} distinct characters u of $\mathfrak{L}/\mathfrak{B}$ defined by

$$u(\lambda) = \prod_{i=1}^{r_1} \left(\frac{\lambda^{(i)}}{|\lambda^{(i)}|}\right)^{g_i}, \quad g_i = 0 \text{ or } 1, \lambda \in \mathfrak{L}.$$

There can indeed be no more that 2^{r_1} characters of $\mathfrak{L}/\mathfrak{B}$ and hence $\nu(\lambda)$ coincides with one of these characters u, of $\mathfrak{L}/\mathfrak{B}$. We do not however assert that every character u of $\mathfrak{L}/\mathfrak{B}$ is realizable from a ray class character modulo $\mathfrak{f}$, in the manner described above.

Let now $\gamma \in K$ such that $\gamma \equiv 1(\mod {}^*\mathfrak{f})$ and $(\gamma) \in \mathfrak{S}_{\mathfrak{f}}$. By definition, $\nu(\gamma) = \chi((\delta))$ for an integer δ such that $\delta \equiv 1(\mod \mathfrak{f})$ and $\delta\gamma > 0$. But $\chi((\delta)) = \chi((\gamma))$ and hence $\nu(\gamma) = \chi((\gamma))$. Let α and $\beta (\neq 0)$ be two integers coprime to $\mathfrak{f}$ such that $\alpha = \beta(\mod \mathfrak{f})$. Then $\gamma = \alpha/\beta \equiv 1(\mod {}^*\mathfrak{f})$ and

$$\frac{\chi((\alpha))}{\chi((\beta))} = \chi((\gamma)) = \nu(\gamma) = \frac{\nu(\alpha)}{\nu(\beta)}.$$

Thus for integral $\alpha(\neq 0)$ coprime to $\mathfrak{f}$, the ratio $\chi((\alpha))/\nu(\alpha)$ depends only on the residue class of α modulo $\mathfrak{f}$ and is in fact, a character of the group $G(\mathfrak{f})$ of prime residue classes modulo $\mathfrak{f}$. We may denote it by $\chi(\alpha)$. Thus

Proposition 14. *Any character $\chi((\alpha))$ of the ray class group modulo $\mathfrak{f}$ may be written in the form*

$$\chi((\alpha)) = \nu(\alpha)\chi(\alpha)$$

where $\nu(\alpha)$ is a character of signature and $\chi(\alpha)$ is a character of the group $G(\mathfrak{f})$.

Before proceeding further, we give a simple illustration of the above. Let
us take K to be $\mathbf{Q}$, the field of rational numbers and $\mathfrak{f}$ to be the principal ideal 102
$(|d|)$ in the ring of rational integers, d being the discriminant of a quadratic field over $\mathbf{Q}$. For an integer m coprime to d, we define

$$\chi((m)) = \begin{cases} \left(\frac{d}{|m|}\right)\frac{m}{|m|} & \text{for } d > 0 \\ \left(\frac{d}{|m|}\right) & \text{for } d < 0 \end{cases}$$

where $\left(\frac{d}{|m|}\right)$ is the Legendre-Jacobi-Kronecker symbol. The group $\mathfrak{G}_\mathfrak{f}$ now consists of fractional ideals (m/n) where m and n are rational integers coprime to d. We extend χ to the ideals (m/n) in $\mathfrak{G}_\mathfrak{f}$ by setting $\chi((m/n)) = \chi((m))/\chi((n))$. Now it is known that if $m \equiv n(\mod d)$ and mn is coprime to d, then

$$\left(\frac{d}{|m|}\right) = \left(\frac{d}{|n|}\right) \text{ if } d > 0$$

and

$$\left(\frac{d}{|m|}\right)\frac{m}{|m|} = \left(\frac{d}{|n|}\right)\frac{n}{|n|} \text{ if } d < 0$$

Thus

$$\chi(m) = \left(\frac{d}{|m|}\right) \quad \text{for} \quad d > 0$$

and

$$\chi(m) = \left(\frac{d}{|m|}\right)\frac{m}{|m|} \quad \text{for} \quad d < 0$$

are characters of $G(|d|)$. Moreover $v(m) = \dfrac{m}{|m|}$ is clearly a character of signature. It is easily verified that $\chi((m/n))$ is a ray class character modulo $(|d|)$ and that $\chi((m)) = v(m)\chi(m)$.

Now $L(s,\chi) = \sum\limits_A \sum_\mathfrak{a} (\mathfrak{a})(N(\mathfrak{a}))^{-s}$, where A runs over all the ideal classes in the wide sense and $\mathfrak{a}$ over all the non-zero integral ideals in A, which are coprime to $\mathfrak{f}$. In the class A^{-1}, we can choose an integral ideal $\mathfrak{b}_A$ coprime
to $\mathfrak{f}$ and $\mathfrak{a}\mathfrak{b}_A = (\beta)$ where β is an integer divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$. 103
Conversely, and principal ideal (β) divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$ is of the form $\mathfrak{a}\mathfrak{b}_A$, where $\mathfrak{a}$ is an integral ideal in A coprime to $\mathfrak{f}$. Moreover, $\chi(\mathfrak{b}_A)\chi(\mathfrak{a}) = \chi((\beta))$ and $N(\mathfrak{b}_A) \cdot N(\mathfrak{a}) = |N(\beta)|$, where $N(\beta)$ is the norm of β. Thus

$$L(s,\chi) = \sum_A \overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \sum_{\mathfrak{b}_A|(\beta)} \chi((\beta))|N(\beta)|^{-s}$$

where the inner summation is over all principal ideals (β) divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$. Since A^{-1} runs over all classes in the wide sense when A does so, we may assume that $\mathfrak{b}_A$ is an integral ideal in A coprime to $\mathfrak{f}$. Moreover, in view of Proposition 14, we have for $\sigma > 1$,

$$L(s, \chi) = \sum_A \overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \sum_{\mathfrak{b}_A | (\beta)} \nu(\beta)\chi(\beta)|N(\beta)|^{-s} \tag{86}$$

where now A runs over all the ideal classes of K in the wide sense, $\mathfrak{b}_A$ is a fixed integral ideal in A coprime to $\mathfrak{f}$ and the inner sum is over all principal ideals (β) divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$. We may extend $\chi(\beta)$ to all residue classes modulo $\mathfrak{f}$ by setting $\chi(\alpha) = 0$ for α not coprime to $\mathfrak{f}$. Thus we may regard the inner sum in (86) as extended over all principal ideals (β) divisible by $\mathfrak{b}_A$. In order to render the series in (86) suitable for the application of Kronecker's limit formula, we have to replace $\chi(\beta)$ by an exponential of the form $e^{2\pi i(mu+nv)}$ occurring in the limit formula. We shall, in the sequel, express $\chi(\beta)$ as an exponential sum by using an idea due to Lagrange, which is as follows.

Let $x_1, \ldots, x_n$ be n distinct roots of a polynomial $f(x)$ of degree n, with coefficients in a field K_0 containing all the n^{th} roots of unity. Let, further, the field $M = K_0(x_1, \ldots, x_n)$ be an abelian extension of K_0, with galois group H. Let us assume moreover that if $\sigma_1, \ldots, \sigma_n \in H$, then $x_i = x_1^{\sigma_i} i = 1, 2, \ldots, n$. Now, if χ is a character of H, then let us define $y_\chi = \sum\limits_{i=1}^{n} \chi(\sigma_i)x_i$. It is clear that $y_\chi \in M$ and

$$y_\chi^{\sigma_j} = \sum_{i=1}^{n} \chi(\sigma_i)x_1^{\sigma_i\sigma_j} = \overline{\chi}(\sigma_j) \sum_{i=1}^{n} \chi(\sigma_i\sigma_j)x_1^{\sigma_i\sigma_j} = \overline{\chi}(\sigma_j)y_\chi.$$

Hence $y_\chi^n \in K_0$. Now, if $\overline{\chi}$ denotes the conjugate character of χ and $\sigma \in H$, then

$$y_{\overline{\chi}} = \sum_{i=1}^{n} \overline{\chi}(\sigma_i\sigma)x_1^{\sigma_i\sigma} = \overline{\chi}(\sigma) \sum_{i=1}^{n} \overline{\chi}(\sigma_i)x_i^{\sigma}.$$

If $y_{\overline{\chi}} \neq 0$, then $\chi(\sigma) = y_{\overline{\chi}}^{-1} \sum\limits_{i=1}^{n} \overline{\chi}(\sigma_i)x_i^{\sigma}$. We shall use a similar method to express **104**
$\chi(\beta)$ as an exponential sum whose terms involve β in the exponent.

First we need the following facts concerning the *different* of an algebraic number field M of finite degree over $\mathbf{Q}$. Let for $\alpha \in M$, $S(\alpha)$ denote the *trace* of α. Let $\mathfrak{a}$ be an ideal (not necessarily integral) in M and let $\mathfrak{a}^*$ be the "complementary" ideal to $\mathfrak{a}$, namely the set of $\lambda \in M$, for which $S(\lambda\alpha)$ is a rational integer for all $\alpha \in \mathfrak{a}$. It is known that $\mathfrak{a}\mathfrak{a}^*$ is independent of $\mathfrak{a}$ and in fact

$\mathfrak{a}\mathfrak{a}^* = (1)^* = \vartheta^{-1}$, where ϑ is the *different* of M. Further clearly $(\mathfrak{a}^*)^* = \mathfrak{a}$ and $N(\vartheta) = |d|$, where d is the *discriminant* of M. These facts can be verified in a simple way if M is a quadratic field over $\mathbf{Q}$, with discriminant d. Let $[\alpha_1, \alpha_2]$ be an integral basis of $\mathfrak{a}$. The ideal $\mathfrak{a}^*$ is the set of $\lambda \in M$ for which $S(\lambda\alpha_1)$ and $S(\lambda\alpha_2)$ are both rational integers. If α_1', α_2' denote the conjugates of α_1 and α_2 respectively, then it is easliy verifed that $\left[\dfrac{\alpha_2'}{\alpha_1\alpha_2' - \alpha_2\alpha_1'}, \dfrac{-\alpha_1'}{\alpha_1\alpha_2' - \alpha_2\alpha_1'}\right]$ is an integral basis of $\mathfrak{a}^*$. Now $\alpha_1\alpha_2' - \alpha_2\alpha_1' = \pm N(\mathfrak{a})\sqrt{d}$ and this means that $\mathfrak{a}^* = (1/N(\mathfrak{a})\sqrt{d})\mathfrak{a}'$. Since $(N(\mathfrak{a})) = \mathfrak{a}\mathfrak{a}'$, we see that $\mathfrak{a}^* = \mathfrak{a}^{-1}(1/\sqrt{d})$ i.e. $\mathfrak{a}\mathfrak{a}^* = (1/\sqrt{d}) = \vartheta^{-1}$. Moreover $N(\vartheta) = |d|$.

Let us consider now the ideal $\mathfrak{a} = \mathfrak{f}^{-1}\vartheta^{-1}$ in K; clearly $\mathfrak{a}^* = \mathfrak{f}$. Let us choose in the class of $\mathfrak{f}\vartheta$, an integral ideal $\mathfrak{q}$ coprime to $\mathfrak{f}$. Then $\mathfrak{q}\mathfrak{f}^{-1}\vartheta^{-1} = (\gamma)$ for $\gamma \in K$ i.e. $(\gamma)\vartheta = \mathfrak{q}\mathfrak{f}^{-1}$ has exact denominator $\mathfrak{f}$. If K were a quadratic field and $\mathfrak{f}$, a principal ideal, then $\mathfrak{f}\vartheta$ is principal and we may take $\mathfrak{q} = (1)$. We *shall consider γ fixed this way once for all, in the sequel.* Let us observe that if $\lambda \in \mathfrak{f}$, $S(\lambda\gamma)$ is a rational integer.

With a view to express $\chi(\beta)$ as an exponential sum, we now define the sum

$$T = \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\lambda\gamma)} \tag{87}$$

where λ runs over a full system of representatives of residue classes modulo $\mathfrak{f}$; for $\mathfrak{f} = (1)$, clearly $T = 1$. We see that T is defined independently of the **105**
choice of representatives λ, for, if μ runs over another system of representatives modulo $\mathfrak{f}$, then $\lambda \equiv \mu(\mod \mathfrak{f})$ in some order and in this case, $\overline{\chi}(\lambda) = \overline{\chi}(\mu)$ and $e^{2\pi i S(\lambda\gamma)} = e^{2\pi i S(\mu\gamma)}$ since $S((\lambda - \mu)\gamma)$ is a rational integer. Moreover, in this sum, λ may be supposed to run only over representatives of elements of $G(\mathfrak{f})$, since $\overline{\chi}(\alpha) = 0$, for α not coprime to $\mathfrak{f}$. The sums of this type were first investigated by Hecke; similar sums for the case of the rational number field have been studied by Gauss, Dirichlet and for complex characters χ, by Hasse.

Let α be an integer in K coprime to $\mathfrak{f}$. Then $\alpha\lambda$ runs over a complete set of prime residue classes modulo $\mathfrak{f}$ when λ λ does so. Thus

$$\begin{aligned} T &= \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\alpha\lambda) e^{2\pi i S(\alpha\lambda\gamma)} \\ &= \overline{\chi}(\alpha) \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma)}, \end{aligned}$$

i.e.

$$T\chi(\alpha) = \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma)}. \tag{88}$$

We shall presently see that (88) is true even for α not coprime to $\mathfrak{f}$ and that $T \neq 0$, when χ is a so-called *proper* character of $G(\mathfrak{f})$.

Let $\mathfrak{g}$ be a proper divisor of $\mathfrak{f}$ and ψ, a character of $G(\mathfrak{g})$. We may extend ψ into a character χ of $G(\mathfrak{f})$ by setting $\chi(\lambda) = \psi(\lambda)$ for λ coprime to $\mathfrak{f}$ and $\chi(\lambda) = 0$, otherwise. We say that a character χ of $G(\mathfrak{f})$ is *proper*, if it is not derivable in this way from a character of $G(\mathfrak{g})$ for a proper divisor $\mathfrak{g}$ of $\mathfrak{f}$.

A ray class character χ modulo $\mathfrak{f}$ is said to be *proper* if the associated character χ of $G(\mathfrak{f})$ is proper; χ is then said to have $\mathfrak{f}$ as *conductor*.

Let χ be a ray class character modulo $\mathfrak{f}$ and let $\mathfrak{g}$ be a proper divisor of $\mathfrak{f}$. Moreover let for integers α, β coprime to $\mathfrak{f}$ such that $\alpha \equiv \beta(\mod \mathfrak{g})$ and $\alpha\beta > 0, \chi(\alpha) = \chi(\beta)$. We can associate with χ, a ray class character χ_0 modulo $\mathfrak{g}$ as follows. If $\mathfrak{p}$ is a prime ideal coprime to $\mathfrak{f}$, define $\chi_0(\mathfrak{p}) = \chi(\mathfrak{p})$. If $\mathfrak{p}$ is a prime ideal coprime to $\mathfrak{g}$ but not to $\mathfrak{f}$, we can find a number α such that $\alpha \equiv 1(\mod {}^*\mathfrak{g})$, $\alpha > 0$ and $(\alpha)\mathfrak{p}$ is coprime to $\mathfrak{f}$. We then set $\chi_0(\mathfrak{p}) = \chi((\alpha)\mathfrak{p})$. We see that χ_0 is well-defined for all prime ideals coprime to $\mathfrak{g}$ and we extend χ_0 **106**
multiplicatively to all ideals in the ray classes modulo $\mathfrak{g}$. Clearly, χ_0 is a ray class character modulo $\mathfrak{G}$ and for integral α coprime to $\mathfrak{g}$, $\chi_0((\alpha)) = \chi_0(\alpha)\nu(\alpha)$, where $\chi_0(\alpha)$ is the associated character of $G(\mathfrak{g})$ and $\nu(\alpha)$ is the same signature character as the one associated with χ. We now say the ray class character χ modulo $\mathfrak{f}$ having the property described above with respect to $\mathfrak{g}$, has $\mathfrak{g}$ as *conductor*, if the associated χ_0 has $\mathfrak{g}$ as conductor. Let *from now on*, χ be a *proper* ray class character modulo $\mathfrak{f}$. Since $\chi(\alpha) = 0$ for α not coprime to $\mathfrak{f}$, all we need to prove (88) for proper χ and for α not coprime to $\mathfrak{f}$ is to show that the right hand side of (88) is zero. Let $\mathfrak{b}$ be the greatest common divisor of (α) and $\mathfrak{f}$ and let $\mathfrak{g} = \mathfrak{f}\mathfrak{b}^{-1}$. Since χ is proper, it is not derivable from any character of $G(\mathfrak{g})$. In other words, there exist integers λ and μ coprime to $\mathfrak{f}$ such that $\lambda \equiv \mu(\mod \mathfrak{g})$ and $\chi(\lambda) \neq \chi(\mu)$. Otherwise, if for all integers λ and μ coprime to $\mathfrak{f}$ and for which $\lambda \equiv \mu(\mod \mathfrak{g})$ it is true that $\chi(\lambda) = \chi(\mu)$, then we can define a character χ_0 of $G(\mathfrak{g})$ such that for λ coprime to $\mathfrak{f}$, $\chi(\lambda) = \chi_0(\lambda)$ which is a contradiction. Thus we can find integers μ and ν coprime to $\mathfrak{f}$ such that $\mu \equiv \nu(\mod \mathfrak{g})$ and $\chi(\mu) \neq \chi(\nu)$. Now clearly

$$\sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma)} = \begin{cases} \overline{\chi}(\mu) \sum\limits_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma\mu)} \\ \overline{\chi}(\nu) \sum\limits_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma\nu)}. \end{cases} \tag{89}$$

Further since $\alpha\mu \equiv \alpha\nu(\mod \mathfrak{f})$, $S(\alpha\lambda\gamma(\mu - \nu))$ is a rational integer and hence

$$\sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma\mu)} = \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma\nu)}.$$

But since $\overline{\chi}(\mu) \neq \overline{\chi}(\nu)$, we see from (89), that

$$\sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma)} = 0.$$

Hence (88) is true for *all integral* α.

We now proceed to prove the $T \neq 0$. In fact, using (89), we have

$$\begin{aligned} T\overline{T} &= T \sum_{\alpha \mod \mathfrak{f}} \chi(\alpha) e^{-2\pi i S(\alpha\gamma)} \\ &= \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) \sum_{\alpha \mod \mathfrak{f}} e^{2\pi i S(\alpha(\lambda-1)\gamma)} \qquad (90) \end{aligned}$$

Now if α runs over a complete system of representatives of residue classes 107
modulo $\mathfrak{f}$, so does $\alpha + \nu$ for every integer ν and hence

$$\sum_{\alpha \mod \mathfrak{f}} e^{2\pi i S(\mu\alpha\gamma)} = e^{2\pi i S(\mu\gamma\nu)} \sum_{\alpha \mod \mathfrak{f}} e^{2\pi i S(\mu\alpha\gamma)}$$

Thus, if there exists at least one integer ν such that $S(\mu\gamma\nu)$ is not a rational integer, $\sum_{\alpha \mod \mathfrak{f}} e^{2\pi i S(\mu\alpha\gamma)} = 0$. Now $S(\mu\gamma\nu)$ is a rational integer for all integers ν if and only if $\mu\gamma \in \vartheta^{-1}$ i.e. if and only if $(\mu)\mathfrak{q}\mathfrak{f}^{-1}$ is integral i.e. $\mu \in \mathfrak{f}$. Thus

$$\sum_{\alpha \mod \mathfrak{f}} e^{2\pi i S(\mu\alpha\gamma)} = \begin{cases} 0, & \text{if } \mu \notin \mathfrak{f}, \\ N(\mathfrak{f}), & \text{if } \mu \in \mathfrak{f}. \end{cases}$$

From this and from (90), we have then $|T|^2 = N(\mathfrak{f})$, i.e. $|T| = \sqrt{N(\mathfrak{f})}$. The determination of the exact value of $T/|T|$ is of the same order of difficulty as the corresponding problem for "generalized Gauss sums" considered by Hasse. We have, finally, for proper χ modulo $\mathfrak{f}$, as a consequence of (88),

$$\chi(\beta) = T^{-1} \sum_{\lambda \mod \mathfrak{f}} \chi(\lambda) e^{2\pi i S(\lambda\beta\gamma)} \qquad (91)$$

Let us notice that β appears in the exponent in the sum on the right-hand side of (**??**).

We now insert the value of $\chi(\beta)$ as given by (**??**) in the series on the right hand side of (86). In view of the absolute convergence of the series for $\sigma > 1$ and in view of the fact that the sum in (93) is a finite sum, we are allowed to rearrange the terms as we like. We then obtain for $\sigma > 1$ and for a ray class character χ modulo $\mathfrak{f}$ with $\mathfrak{f}$ as conductor,

$$L(s,\chi) = \frac{1}{T} \sum_{\lambda \mod \mathfrak{f}} \chi(\lambda) \sum_{A} \overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \times$$

$$\times \sum_{\mathfrak{b}_A|(\beta)} v(\beta) e^{2\pi i S(\lambda\beta\gamma)} |N(\beta)|^{-s}. \tag{92}$$

In (92), λ runs over a full system of representatives of prime residue classes modulo $\mathfrak{f}$, A over representatives of the ideal classes of K in the wide sense and the inner sum is over all principal non-zero ideals (β) divisible by $\mathfrak{b}_A$. **108**

We shall use (92) to determine the value of $L(s,\chi)$ at $s = 1$ when K is a real or imaginary quadratic field over $\mathbf{Q}$. In this section, we shall consider only the case when K is an *imaginary quadratic field* over $\mathbf{Q}$, with discriminant $D < 0$. Here $v(\beta) = 1$ identically, since there is no nontrivial character of signature. Moreover, we could assume $\mathfrak{f} \neq (1)$, for otherwise, $L(s,\chi)$ precisely the Dedekind zeta function of K. Let, therefore, $\mathfrak{f} \neq (1)$ and let w and $w_\mathfrak{f}$ denote respectively the number of all roots of unity and of roots of unity ϵ satisfying $\epsilon \equiv 1 (\mod \mathfrak{f})$. From (92), we have for $\sigma > 1$,

$$L(s,\chi) = \frac{1}{w \cdot T} \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) \sum_A \overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \times$$
$$\times \sum_{\mathfrak{b}_A|\beta\neq 0} e^{2\pi i S(\lambda\beta\gamma)} (N(\beta))^{-s} \tag{93}$$

where the inner summation is over all $\beta \neq 0$ in $\mathfrak{b}_A$.

We now contend that as λ runs over a full system of representatives of the prime residue classes modulo $\mathfrak{f}$ and $\mathfrak{b}_A$ over a complete set of representatives (integral and coprime to $\mathfrak{f}$) of the classes in the wide sense, then $(\lambda)\mathfrak{b}_A$ covers exactly $w/w_\mathfrak{f}$ times, a complete system of representatives of the ray classes modulo $\mathfrak{f}$. In fact, let $\epsilon_1, \ldots, \epsilon_v (v = w/w_\mathfrak{f})$ be a complete system of roots of unity in K, incongruent modulo $\mathfrak{f}$. The corresponding residue classes modulo $\mathfrak{f}$ form a subgroup $E(\mathfrak{f})$ of $G(\mathfrak{f})$. Let λ_i, $i = 1, \ldots, r$ be a set of integers whose residue classes modulo $\mathfrak{f}$ constitute a full system of representatives of the cosets of $G(\mathfrak{f})$ modulo $E(\mathfrak{f})$. Then it is easily verified that when λ runs over the elements $\lambda_1, \ldots, \lambda_r$ and $\mathfrak{b}_A$ over the representatives of the classes in the wide sense, $(\lambda)\mathfrak{b}_A$ covers exactly once a full system of representatives of the ray classes modulo $\mathfrak{f}$. Our assertion above is an immediate consequence. Thus, we have from (93), for $\sigma > 1$,

$$L(s,\chi) = \frac{1}{Tw_\mathfrak{f}} \sum_B \overline{\chi}(\mathfrak{b}_B)(N(\mathfrak{b}_B))^s \sum_{\mathfrak{b}_B|\beta\neq 0} e^{2\pi i S(\beta\gamma)} (N(\beta))^{-s}, \tag{94}$$

where B runs over the ray classes modulo $\mathfrak{f}$ and $\mathfrak{b}_B$ is a fixed integral ideal in B **109**
coprime to $\mathfrak{f}$.

Let $[\beta_1, \beta_2]$ be an integral basis of $\mathfrak{b}_B$; we can assume, without loss of generality that $\dfrac{\beta_2}{\beta_1} = z_B = x_B + iy_B$ with $y_B > 0$. Then if $\beta \in \mathfrak{b}_B$, $\beta = m\beta_1 + n\beta_2$ for rational integers m, n and $N(\beta) = N(\beta_1) \times |m + nz_B|^2$. Moreover,

$$N(\mathfrak{b}_B)\sqrt{|D|} = |\beta_1\overline{\beta}_2 - \beta_2\overline{\beta}_1| = 2y_B N(\beta_1).$$

Thus

$$\begin{aligned}(N(\mathfrak{b}_B))^s &\sum_{\mathfrak{b}_B|\beta\neq 0} e^{2\pi i S(\beta\gamma)}(N(\beta))^{-s}\\ &= \left(\frac{2y_B}{\sqrt{|D|}}\right)^s \sum_{m,n}{}' e^{2\pi i S((m\beta_1+n\beta_2)\gamma)}|m + nz_B|^{-2s},\end{aligned}$$

where, on the right hand side, the summation is over all ordered pairs of rational integers (m, n) not equal to $(0, 0)$. Let us set now $u_B = S(\beta_1\gamma)$ and $v_B = S(\beta_2\gamma)$ and let f be the smallest positive rational integer divisible by $\mathfrak{f}$. Then in view of the fact that $(\gamma)\vartheta$ has exact denominator $\mathfrak{f}$ and $\mathfrak{b}_B$ is coprime to $\mathfrak{f}$, it follows that u_B and v_B are rational numbers with the reduced common denominator f. Since $\mathfrak{f} \neq (1)$, u_B and v_B are not simultaneously integral. We then have

$$\begin{aligned}(N(\mathfrak{b}_B))^s &\sum_{\mathfrak{b}_B|\beta\neq 0} e^{2\pi i S(\beta\gamma)}(N(\beta))^{-s}\\ &= \left(\frac{2}{\sqrt{|D|}}\right)^s y_B^s \sum_{m,n}{}' e^{2\pi i(mu_B+nv_B)}|m + nz_B|^{-2s}.\end{aligned} \tag{95}$$

We know that the function of s defined for $\sigma > 1$ by the infinite series on the right hand side of (95) has an analytic continuation which is an entire function of s. Its value at $s = 1$ is given precisely by Kronecker's second limit formula. Indeed, by (39), we have

$$y_B {\sum_{m,n}}' e^{2\pi i(mu_B+nv_B)}|m + nz_B|^{-2} = -\pi = -\pi \log \left| \frac{\vartheta_1(v_B - u_B Z_B, z_B)}{\eta(z_B)} e^{\pi i u_B^2 z_B} \right|^2.$$

Inserting the factor $e^{-\pi i u_B v_B}$ of absolute value 1 on the right hand side, we have **110**

$$\begin{aligned}y_B &\sum_{m,n}{}' e^{2\pi i(mu_B+nv_B)}|m + nz_B|^{-2}\\ &= -\pi \log \left| \frac{\vartheta_1(v_B - u_B z_B, z_B)}{\eta(z_B)} e^{\pi i u_B(u_B z_B - v_B)} \right|^2.\end{aligned} \tag{96}$$

Now we define for real numbers u, v not simultaneously integral and $z \in \mathfrak{H}$, the function

$$\varphi(v, u, z) = e^{\pi iu(uz-v)} \frac{\vartheta_1(v - uz, z)}{\eta(z)}.$$

In each ray class B modulo $\mathfrak{f}$, we had chosen a fixed integral ideal $\mathfrak{b}_B$ with integral basis $[\beta_1, \beta_2]$ and defined $u_B = S(\beta_1\gamma)$, $v_B = S(\beta_2\gamma)$ and $z_B = \dfrac{\beta_2}{\beta_1}$. Now, the left hand side of (95) depends only on the ray class B. Hence from (95) and (96), it is clear that $\log|\varphi(v_B, u_B, z_B)|^2$ depends only on B and not on the special choice of $\mathfrak{b}_B$ or β_1, β_2.

From (94), (95) and (96) we can now deduce

Theorem 9. *If χ is a proper ray class character modulo an integral ideal $\mathfrak{f} \neq (1)$ of $\mathbf{Q}(\sqrt{D})$, $D < 0$, the associated $L(s, \chi)$ can be continued analytically into an entire function of s and its value at $s = 1$ is given by*

$$L(1, \chi) = -\frac{2\pi}{T w_{\mathfrak{f}} \sqrt{|D|}} \sum_B \chi(\mathfrak{b}_B) \log|\varphi(v_B, u_B, z_B)|^2, \tag{97}$$

where B runs over all the ray classes modulo $\mathfrak{f}$ and T is the sum defined by (87).

We shall need (97) later for the determination of the class number of the 'ray class field' of $\mathbf{Q}(\sqrt{D})$.

By the **ray class field** (modulo $\mathfrak{f}$) of an algebraic number field k, we mean the relative abelina extension K_0 of k, with Galois group isomorphic to the group of ray classes (modulo $\mathfrak{f}$) in k such that the prime divisors of $\mathfrak{f}$ are the only prime ideals which are ramified in K_0.

We are now interested first in determining the nature of the numbers
$\varphi(v_B, u_B, z_B)$. For this purpose, we observe that $\varphi(v, u, z)$ is a regular function of **111**
z in $\mathfrak{H}$ and we shall study its behaviour when z is subjected to modular transformations and the real variables v and u undergo certain linear transformations. In fact, using the transformation formula for $\vartheta_1(w, z)$ and $\eta(z)$ proved earlier and the definition of $\varphi(v, u, z)$, one easily verifies the following formulae, viz.

$$\begin{aligned} \varphi(v + i, u, z) &= -e^{-\pi iu}\varphi(v, u, z), \\ \varphi(v, u + 1, z) &= -e^{\pi iv}\varphi(v, u, z), \\ \varphi(v + u, u, z + 1) &= e^{\pi i/6}\varphi(v, u, z), \\ \varphi(-u, v, -z^{-1}) &= e^{-\pi i/2}\varphi(v, u, z). \end{aligned} \tag{98}$$

Now, corresponding to a modular transformation $z \to z^* = (az + b) \cdot (cz + d)^{-1}$ we define $v^* = av + bu$ and $u^* = cv + du$. The last two transformation formulae for $\varphi(v, u, z)$ above, merely mean that corresponding to the elementary modular transformations $z \to z^* = z + 1$ or $z \to z^* = -z^{-1}$, we have

$$\varphi(v^*, u^*, z^*) = \rho\varphi(v, u, z),$$

where ρ is a 12^{th} root of unity. Since these two transformations generate the modular group, we see that for any modular transformation $z \to z^*$,

$$\varphi(v^*, u^*, z^*) = \rho\varphi(v, u, z) \tag{98$'$}$$

where $\rho = \rho(a, b, c, d)$ is a 12^{th} root of unity which can be determined explicitly.

Let (v, u) be a pair of rational numbers with reduced common denominator $f > 1$. Let us define for $z \in \mathfrak{H}$,

$$\Phi(v, u, z) = \varphi^{12f}(v, u, z).$$

Then as a consequence of the above formulae for $\varphi(v, u, z)$, we see that $\Phi(v + 1, u, z) = \Phi(v, u, z)$, $\Phi(v, u + 1, z) = \Phi(v, u, z)$ and $\Phi(v^*, u^*, z^*) = \Phi(v, u, z)$. If $z \to z^* = (az + b)(cz + d)^{-1}$ is a modular transformation of level f, then $v^* - v$ and $u^* - u$ are rational integers and in view of the periodicity of $\Phi(v, u, z)$ in v and u, we see that

$$\Phi(v, u, z) = \Phi(v^*, u^*, z^*) = \Phi(v, u, z^*).$$

112

Let $(v_i, u_i) i = 1, 2, \ldots, q$ run over all the paits of rational numbers lying between 0 and 1 and having reduced common denominator f. Then corresponding to each pair (v_i, u_i), we have a function $\Phi_i(z) = \Phi(v_i, u_i, z)$ which, as seen above, is invariant under modular transformations of level f. Moreover, if $z \to z^* = (az + b)(cz + d)^{-1}$ is an arbitrary modular transformation, then for some i, $v_j^* \equiv v_i (\mod 1)$ and $u_j^* \equiv u_i (\mod 1)$ so that in view of the periodicity of $\Phi(v_i, u_i, z)$ in v_i and u_i we see that

$$\Phi_i(z^*) = \Phi(v_i, u_i, z^*) = \Phi(v_j^*, u_j^*, z^*) = \Phi(v_j, u_j, z) = \Phi_j(z).$$

In other words, the functions $\Phi_i(z)$ are permuted among themselves by an arbitrary modular transformation.

Now, $\Phi_i(z)$ is regular in $\mathfrak{H}$ and invariant under modular transformations of level f. Moreover, $\Phi_i(z)$ has in the local uniformizer $e^{2\pi iz/f}$ at infinity, a power-series expansion with at most a finite number of negative powers. Since $\Phi_i(z^*) = \Phi_j(z)$ for some j, we see that $\Phi_i(z)$ has at most a pole in the local uniformizers at the 'parabolic cusps' of the corresponding fundamental domain.

Thus $\Phi_i(z)$ is a modular function of level f. Since a modular transformation merely permutes the functions $\Phi_i(z)$, we see that any elementary symmetric function of the functions $\Phi_i(z)$ is a modular function. From the theory of complex multiplications, it can be shown by considering the expansions 'at infinity' of the functions $\Phi_i(z)$, that $\Phi_i(z)$ satisfies a polynomial equation whose coefficients are polynomials, with rational integral coefficients, in $j(z)$ (the elliptic modular invariant). If z lies in an imaginary quadratic field K over $\mathbf{Q}$, it is known that for $z \notin \mathbf{Q}$, $j(z)$ is an algebraic integer. Since $\Phi_i(z)$ depends integrally on $j(z)$, it follows that for such z, $\Phi_i(z)$ is an algebraic integer. Actually, from the theory of complex multiplication, one may show that for $z \in K(z \notin \mathbf{Q})$, $\Phi_i(z)$ is an algebraic integer in the ray class field modulo $\mathfrak{f}$ over K. In particular, the numbers $\varphi^{12f}(v_B, u_B, z_B)$ corresponding to the ray classes B modulo $\mathfrak{f}$ in K, are algebraic integers in the ray class field modulo $\mathfrak{f}$, over K.

Let now K_0 be the ray class field modulo $\mathfrak{f}$ over $K(= \mathbf{Q}(\sqrt{D}), D < 0)$. Let Δ, R, W and g be respectively the discriminant of K_0 relative to $\mathbf{Q}$, the regulator 113
of K_0, the number of roots of unity in K_0 and the order of the Galois group of K_0 over K. Let H and h denote respectively the class numbers of K_0 and K. It is clear that K_0 has no real conjugates over $\mathbf{Q}$ and has $2g$ complex conjugates over $\mathbf{Q}$. Moreover $|\Delta| = |D|^g N(\vartheta)$, where $N(\vartheta)$ is the norm in $K/\mathbf{Q}$ of the relative discriminant ϑ of K_0 over K.

From class field theory, we know that

$$\zeta_{K_0}(s) = \zeta_K(s) \prod_{\chi_0 \neq 1} L(s, \chi_0) \tag{$*$}$$

where χ runs over all the non-principal ray class characters modulo $\mathfrak{f}$ and if $\mathfrak{f}_\chi$ is the conductor of χ, then χ_0 is the proper ray class character modulo $\mathfrak{f}_\chi$ associated with χ. It is known that $\vartheta = \prod_\chi \mathfrak{f}_\chi$. Multiplying both sides of $(*)$ above by $s-1$ and letting s tend to 1, we have

$$\frac{(2\pi)^g \cdot H \cdot R}{W\sqrt{|\Delta|}} = \frac{2\pi h}{w\sqrt{|D|}} \prod_{\chi \neq 1} L(1, \chi_0).$$

But, from (97), we have

$$L(1, \chi_0) = -\frac{2\pi}{T_0 w_{\mathfrak{f}_\chi} \sqrt{|D|}} \sum_{B_0} \overline{\chi}_0(b_{B_0}) \log \left|\varphi(v_{B_0}, u_{B_0}, z_{B_0})\right|^2$$

where

$$T_0 = T_0(\chi_0) = \sum_{\lambda \mod \mathfrak{f}_\chi} \overline{\chi}_0(\lambda) e^{2\pi i S(\lambda\gamma_0)},$$

$\gamma_0 \in K$ is chosen such that $(\gamma_0 \sqrt{d})$ has exact denominator $\mathfrak{f}_\chi$, B_0 runs over all the ray classes modulo $\mathfrak{f}_\chi$, $\mathfrak{b}_{B_0}$ is a fixed integral ideal in B_0 coprime to $\mathfrak{f}_\chi$ and $w_{\mathfrak{f}_\chi}$ is the number of roots of unity congruent to 1 modulo $\mathfrak{f}_\chi$. Witn $N(\mathfrak{f}_\chi)$ denoting the norm in K over $\mathbf{Q}$, of the ideal $\mathfrak{f}_\chi$, we have $(\prod_\chi N(\mathfrak{f}_\chi))|D|^g = |\Delta|$. It is now easy to deduce

Theorem 10. *For the class number H of the ray class field modulo $\mathfrak{f}$ over $K = \mathbf{Q}(\sqrt{D})$, $D < 0$, we have the formula*

$$\frac{H}{h} = \frac{W}{w \cdot R} \prod_{\chi \neq 1} \left(-\frac{\sqrt{N(\mathfrak{F}_\chi)}}{T_0 w_{\mathfrak{f}_\chi}} \sum_{B_0} \overline{\chi}_0(\mathfrak{b}_{B_0}) \log \left|\varphi\left(v_{B_0}, u_{B_0}, z_{B_0}\right)\right|^2 \right)$$

When $\mathfrak{f}$ is a rational integral ideal in K, Fueter has derived a 'similar' formula for the class number of the ray class field modulo $\mathfrak{f}$ or the 'ring class field' modulo $\mathfrak{f}$ over K, by using a "generalization" of Kronecker's first limit formula. **114**

Unlike in the case of the absolute class field over K, it is not possible, in general, (for $\mathfrak{f} \neq (1)$) to write the product $\prod_{\chi \neq 1}$ occurring in the formula for H/h above, as a $(g-1)$-rowed determinant whose elements are of the form $\log|\eta_i^{(k)}|$ where $\eta_i^{(k)}$, $i = 1, 2, \ldots, g-1$ are conjugates of independent units η_i lying in K_0.

5 Ray class fields over $\mathbf{Q}(\sqrt{D})$, $D > 0$.

Let K be a real quadratic field over $\mathbf{Q}$, with discriminant $D > 0$. Let $\mathfrak{f}$ be a given integral ideal in K and $\chi(\neq 1)$, a proper character of the group of ray classes modulo $\mathfrak{f}$ in K. As in the case of the imaginary quadratic field, we shall first determine the value at $s = 1$ of the L-series $L(s, \chi)$ associated with χ and use this later to determine the class number of special abelian extensions of K.

We start from formula (92) proved earlier, namely, for $\sigma > 1$,

$$L(s,\chi) = T^{-1} \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) \sum_A \overline{\chi}(\mathfrak{b}_A) \times \\ \times (N(\mathfrak{b}_A))^s \sum_{\mathfrak{b}_A | (\beta) \neq (0)} v(\beta) e^{2\pi i S(\lambda\beta\gamma)} |N(\beta)|^{-s},$$

where A runs over all the ideal-classes of K in the wide sense, $\mathfrak{b}_A$ is a *fixed* integral ideal in A, chosen once for all and coprime to $\mathfrak{f}$ and the inner sum is extended over all non-zero principal ideals (β) divisible by $\mathfrak{b}_A$. Further, $\chi(\lambda)$ and $v(\lambda)$ are respectively the character of $G(\mathfrak{f})$ and the character of signature associated with the ray class character χ as in Proposition 14. Besides, γ is a number in K such that $(\gamma\sqrt{C})$ has exact denominator $\mathfrak{f}$ i.e. $(\gamma) = \dfrac{\mathfrak{q}}{\mathfrak{f}(\sqrt{D})}$ where $(\mathfrak{q}, \mathfrak{f}) = (1)$; *the number γ will be fixed throughout this section.*

Let, for $\alpha \in K$, α' denote its conjugate. We may, in the above, suppose that $[\beta_1, \beta_2]$ is a *fixed* integral basis of $\mathfrak{b}_A$ such that if $\omega = \dfrac{\beta_2}{\beta_1}$, then, without loss of generality, $\omega > \omega'$. We have, then, $(\omega - \omega)|N(\beta_1)| = N(\mathfrak{b}_A) \cdot \sqrt{D}$. **115**
Further, let $u_A = S(\lambda\beta_1\gamma)$ and $v_A = S(\lambda\beta_2\gamma)$; then, if $\beta = m\beta_1 + n\beta_2$, we have $e^{2\pi i S(\lambda\beta\gamma)} = e^{2\pi i(mu_A + nv_A)}$.

The character of signature $v(\lambda)$ associated with the ray class character χ may, in general, be one of the following, namely, for $\lambda \neq 0$ in K,

(i) $v(\lambda) = 1$

(ii) $v(\lambda) = \dfrac{N(\lambda)}{|N(\lambda)|}$

(iii) $v(\lambda) = \dfrac{\lambda}{|\lambda|}$

(iv) $v(\lambda) = \dfrac{\lambda'}{|\lambda'|}$.

In what follows, we shall be concerned only with such ray class characters χ, for which the corresponding $v(\lambda)$ is defined by (i) or (ii). We shall not deal with the characters χ, for which either (iii) or (iv) occurs, the determination of $L(1, \chi)$ being quite complicated in these cases.

Let us first suppose that the character of signature $v(\lambda)$ associated with χ is given by $v(\lambda) = 1$ for all $\lambda \neq 0$ in K. In other words, for an integer α in K, coprime to $\mathfrak{f}$, $\chi((\alpha)) = \chi(\alpha)$. We may, moreover, suppose that $\mathfrak{f} \neq (1)$, since otherwise, χ is a character of the wide class group and $L(s, \chi)$ is just the corresponding Dedekind zeta function of K.

Let $z = x + iy$ be a complex variable with $y > 0$. Corresponding to an ideal class A of K in the wide sense and an integer λ coprime to $\mathfrak{f}$, we define for $\sigma > 1$, the function

$$g(z, s, \lambda, u_A, v_A) = y^s \sum_{m,n}{}' e^{2\pi i(mu_A + nv_A)} |m + nz|^{-2s}$$

where the summation is over all pairs of rational integers m, n not simultaneously zero. As a function of z, $g(z, s, \lambda, u_A, v_A)$ is not regular but as a function of s, it is clearly regular for $\sigma > 1$. In fact, since u_A and v_A are not both integral, we know that $g(z, s, \lambda, u_A, v_A)$ has an analytic continuation which is an

entire function of s. For the sake of brevity, we shall denote it by $g_A(z, \lambda)$; *this does not mean that it is a function of z and λ depending only on the class A.* It does depend on the choice of the ideal $\mathfrak{b}_A$, the integral basis $[\beta_1, \beta_2]$, the residue 116
class of λ modulo $\mathfrak{f}$ and further on γ. But we have taken $\mathfrak{b}_A, \beta_1, \beta_2$ and γ to be fixed.

In order to find $L(1, \chi)$, we shall employ the idea of Hecke already referred to on p. 86 (§ 3, Chapter II) and express the series $\sum_{\mathfrak{b}_A|(\beta)\neq(0)} e^{2\pi i S(\lambda\beta\gamma)}|N(\beta)|^{-s}$ as an integral over a suitable path in $\mathfrak{H}$, with the integrand involving $g_A(z, \lambda)$.

We shall denote the group of all units in K by Γ and the subgroup of units congruent to 1 modulo $\mathfrak{f}$, by $\Gamma_{\mathfrak{f}}$. Moreover, let $\Gamma^*_{\mathfrak{f}}$ be the subgroup of $\Gamma_{\mathfrak{f}}$ consisting of ρ such that $\rho > 0$. The group $\Gamma^*_{\mathfrak{f}}$ is infinite cyclic and let ϵ be the generator of $\Gamma^*_{\mathfrak{f}}$, which is greater than 1. We see that $\epsilon \equiv 1(\mod \mathfrak{f})$, $N(\epsilon) = 1$, $\epsilon > \epsilon' > 0$.

Since $[\epsilon\beta_1, \epsilon\beta_2]$ is again an integral basis of $\mathfrak{b}_A$, $\epsilon\beta_2 = a\beta_2 + b\beta_1$, $\epsilon\beta_1 = c\beta_2 + d\beta_1$ where a, b, c, d are rational integers such that $ad - bc = N(\epsilon) = 1$. Further, if $\omega = \beta_2/\beta_1$, then $\epsilon\omega = a\omega + b$, $\epsilon = c\omega + d$ so that $\omega = (a\omega + b)(c\omega + d)^{-1}$. We also have the same thing true for ω', namely, $\epsilon\omega' = a\omega' + b$, $\epsilon' = c\omega' + d$, so that we have $\omega' = (a\omega' + b) \cdot (c\omega' + d)^{-1}$. Thus the modular transformation $z \to z^* = (az+b)\cdot(cz+d)^{-1}$ is hyperbolic, with ω, ω' as fixed points and it leaves fixed the semicircle on $\omega'\omega$ as diameter. If now we introduce the substitution $z = (\omega pi + \omega')(pi + 1)^{-1}$ (or equivalently, $po = (z - \omega')(\omega - z)^{-1}$ we see that when $z(\in \mathfrak{H})$ lies on this semicircle, p is real and positive. As a matter of fact, when $z(\in \mathfrak{H})$ describes the semicircle form ω' to ω, p runs over all positive real numbers from 0 to ∞. Let $p^* i = (z^* - \omega') \cdot (\omega - z^*)^{-1}$; it is easy to verify that $p^* = \epsilon^2 p$. This means that $p^* > p$, since $\epsilon > 1$. Consequently, whenever z lies on the semicircle on $\omega'\omega$ as diameter, z^* again lies on the semicircle and always to the right of z.

Consider now

$$F(A, \lambda, s) = \int_{z_0}^{z_0^*} g(z, s, \lambda, u_A, v_A) \frac{dp}{p},$$

the integral being extended over the arc of this semicircle, from a fixed point $z_0 \in \mathfrak{H}$ to $z_0^* = (az_0 + b)(cz_0 + d)^{-1}$. In view of the uniform convergence of the series $y^s \sum'_{m,n} e^{2\pi i(mu_A + nv_A)}|m + nz|^{-2s}$ on this zrc we have, for $\sigma > 1$,

$$\int_{z_0}^{z_0^*} g_A(z, \lambda) \frac{dp}{P} = \sum_{m,n}{}' e^{2\pi i S(\lambda\beta\gamma)} \int_{z_0}^{z_0^*} y^s |m + nz|^{-2s} \frac{dp}{P},$$

where $\beta = m\beta_1 + n\beta_2$. Setting $\mu = m + n\omega = \beta/\beta_1$, we verify easily that 117

$|m+nz|^2 = (\mu^2 p^2 + \mu'^2)(p^2+1)^{-1}$ and $y = p(\omega - \omega')(p^2+1)^{-1}$. Thus

$$\begin{aligned}
\int_{z_0}^{z_0^*} y^s |m+nz|^{-2s} \frac{dp}{p} &= (\omega-\omega')^s \int_{p_0}^{p_0^*} P^s (\mu^2 p^2 + \mu'^2)^{-s} \frac{dp}{p} \\
&= \frac{(\omega-\omega')^s}{|N(\mu)|^s} \int_{|\mu/\mu'|p_0}^{|\mu/\mu'|p_0^*} \frac{p^s}{(p^2+1)^s} \frac{dp}{p} \left(p \to \left| \frac{\mu'}{\mu} \right| p \right) \\
&= \left(\frac{N(\mathfrak{b}_A)\sqrt{D}}{|N(\beta)|} | \right)^s \int_{|(\beta\beta_2'/\beta'\beta_1)|p_0}^{|(\beta\beta_1'/\beta'\beta_1)|p_0 \epsilon^2} \frac{p^s}{(p^2+1)^s} \frac{dp}{p}
\end{aligned}$$

We have, therefore,

$$F(A,\lambda,s) = (N(\mathfrak{b}_A)\sqrt{D})^s \sum_{\mathfrak{b}_A|\beta\neq 0} e^{2\pi i S(\lambda\beta\gamma)} |N(\beta)|^{-s} \int_{|(\beta\beta_1'/\beta'\beta_1)|p_0}^{|(\beta\beta_1'/\beta'\beta_1)|\epsilon^2 p_0} \frac{p^s}{(p^2+1)^s} \frac{dp}{p} \tag{99}$$

We use once again the trick employed by Hecke (p. 86). For a fixed integer $\beta \in K$, all the integers in K which are associated with β with respect to $\Gamma_{\mathfrak{f}}^*$ are of the form $\beta\epsilon^{\pm k}$, $k = 0, 1, 2, \ldots$. Keeping β fixed, if we replace β by $\beta\epsilon^l$ in the integral on the right hand side of (99), then the integral goes over into

$$\int_{|(\beta\beta_1'/\beta'\beta_1)|\epsilon^{2l} p_0}^{|(\beta\beta_1'/\beta'\beta_1)|\epsilon^{2l+2} p_0} p^s (p^2+1)^{-s} \frac{dp}{p}.$$

Now since $\epsilon > 1$, the intervals $(|\beta\beta_1'|\beta'\beta_1|p_0\epsilon^{2k}, |\beta\beta_1'|\beta'\beta_1|p_0\epsilon^{2k+2})$ cover the entire interval $(0,\infty)$ without gaps and overlaps, as k runs over all rational integers from $-\infty$ to $+\infty$. Moreover, since $\epsilon \equiv 1 (\mod \mathfrak{f})$ and $N(\epsilon) = 1$, we have $e^{2\pi i S(\lambda\beta\epsilon^k\gamma)} = e^{2\pi i S(\lambda\beta\gamma)}$ and $|N(\beta\epsilon^k)| = |N(\beta)|$. Thus the total contribution to the sum on the right hand side of (99) from all integers associated with a fixed integer β with respect to $\Gamma_{\mathfrak{f}}^*$, is given by

$$\frac{e^{2\pi i S(\lambda\beta\gamma)}}{|N(\beta)|^s} \int_0^\infty \frac{p^s}{(p^2+1)^s} \frac{dp}{p} = \frac{e^{2\pi i S(\lambda\beta\gamma)}}{|N(\beta)|^s} \frac{\Gamma^2(s/2)}{2\Gamma(s)}$$

As a consequence, **118**

$$F(A,\lambda,s) = \frac{(N(\mathfrak{b}_A)\sqrt{D})^s \Gamma^2(s/2)}{2\Gamma(s)} \sum_{\mathfrak{b}_A|\beta(\Gamma_{\mathfrak{f}}^*)} e^{2\pi i S(\lambda\beta\gamma)} |N(\beta)|^{-s} \tag{100}$$

where, on the right hand side, the summation is over a complete set of integers $\beta \neq 0$ in $\mathfrak{b}_A$, which are not associated with respect to $\Gamma_{\mathfrak{f}}^*$.

Let $\{\rho_i\}$ denote a complete set of units incongruent modulo $\mathfrak{f}$ and let $\{\epsilon_j\}$ be a complete set of representatives of the cosets of $\Gamma_{\mathfrak{f}}$ modulo $\Gamma^*_{\mathfrak{f}}$. It is clear that $\{\rho_i\epsilon_j\}$ runs over a full set of representatives of the cosets of Γ modulo $\Gamma^*_{\mathfrak{f}}$. Thus

$$F(A,\lambda,s) = \frac{\Gamma^2(s/2)}{2\Gamma(s)}(N(\mathfrak{b}_A)\sqrt{D})^s \sum_{\rho_i,\epsilon_j}\sum_{\mathfrak{b}_A|(\beta)} e^{2\pi i S(\lambda\rho_i\epsilon_j\beta\gamma)}|N(\beta)|^{-s} \tag{101}$$

where, on the right hand side, the inner sum is over all non-zero principal ideals (β) divisible by $\mathfrak{b}_A$.

By the *signature* of an element $\alpha \neq 0$ in K, we mean the pair $(\alpha/|\alpha|, \alpha'/|\alpha'|)$. Now we can certainly find a set $\{\mu_l\}$ of integers μ_l such that $\mu_l \equiv 1 (\mod \mathfrak{f})$ and the set $\{\epsilon_j\mu_l\}$ consists precisely of 4 elements with the 4 possible different signatures. Let $\{\lambda_k\}$ be a set of integers λ_k constituting a complete system of representatives of the cosets of $G(\mathfrak{f})$ modulo $E(\mathfrak{f})$. Here $E(\mathfrak{f})$ is the group of prime residue classes $\mod \mathfrak{f}$, containing at least one unit in K. The set $\{\lambda_k\rho_i\}$ is seen to be a complete set of representatives of the prime residue classes modulo $\mathfrak{f}$. It is easy to prove

Proposition 15. *The set of ideals $\{(\lambda_k\mu_l)\mathfrak{b}_A\}$ serves as a full system of representatives of the ray classes modulo $\mathfrak{f}$.*

Proof. Obviously it suffices to show that $\{(\lambda_k\mu_l)\}$ runs over a complete set of representatives of the ray classes modulo $\mathfrak{f}$ lying in the principal wide class. In fact, if $(\alpha) \in \mathfrak{G}_{\mathfrak{f}}$, then $\alpha \equiv \rho_i\lambda_k (\mod {}^*\mathfrak{f})$ for some ρ_i and λ_k and moreover, **119** we can find $\epsilon_j\mu_l$ such that $\dfrac{\alpha}{\lambda_k\rho_i\epsilon_j\mu_l} > 0$. Since $\epsilon_j\mu_l \equiv 1 (\mod \mathfrak{f})$, we have $(\alpha) \sim (\lambda_k\mu_l\rho_i\epsilon_j) = (\lambda_k\mu_l$ modulo $\mathfrak{f}$. It is easy to verify that no two elements of the set $\{(\lambda_k\mu_l)\}$ are equivalent modulo $\mathfrak{f}$.

From (101), we then have for $\sigma > 1$

$$\begin{aligned}
&\sum_{\lambda_k,\mu_l,A} \overline{\chi}(\lambda_k\mu_l)\overline{\chi}(\mathfrak{b}_A)F(A,\lambda_k\mu_l,s)\\
&= \frac{\Gamma^2(s/2)}{2\Gamma(s)} \cdot \sum_{\lambda_k,\mu_l,A} \overline{\chi}(\lambda_k\mu_l)\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A)\sqrt{D})^s\times\\
&\quad\times \sum_{\substack{\mathfrak{b}_A|(\beta)\\ \rho_i,\epsilon_j}} e^{2\pi i S(\lambda_k\mu_l\rho_i\epsilon_j\beta\gamma)}|N(\beta)|^{-s}\\
&= \frac{\Gamma^2(s/2)}{2\Gamma(s)} \sum_{\lambda_k,\rho_i,A} \overline{\chi}(\lambda_k)\overline{\chi}(\mathfrak{b}_A)(N(\beta_A)\sqrt{D})^s\times\\
&\quad\times \sum_{\epsilon_j,\mu_l}\overline{\chi}(\mu_l)\sum_{\mathfrak{b}_A|(\beta)} e^{2\pi i S(\lambda_k\mu_l\rho_i\epsilon_j\beta\gamma)}|N(\beta)|^{-s}.
\end{aligned} \tag{102}$$

$\equiv 1(\mod \mathfrak{f})$, $e^{2\pi i S(\lambda_k\mu_l\rho_i\epsilon_j\beta\gamma)} = e^{2\pi i S(\lambda_k\rho_i\beta\gamma)}$ and since $\mu_l \equiv 1(\mod \mathfrak{f})$, $\chi(\mu_l) = 1$. Hence the sum in (102) is independent of ϵ_j, μ_l and we have

$$\begin{aligned}\sum_{\lambda_k,\mu_l,A} &\overline{\chi}(\lambda_k\mu_l)\overline{\chi}(\mathfrak{b}_A)F(A,\lambda_k\mu_l,s)\\ &= 4\cdot\frac{\Gamma^2(s/2)}{2\Gamma(s)}D^{s/2}\sum_{\lambda_k,\rho_i,A}\overline{\chi}(\lambda_k\rho_i)\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s\times\\ &\sum_{\mathfrak{b}_A|(\beta)\neq(0)} e^{2\pi i S(\lambda_k\rho_i\beta\gamma)}|N(\beta)|^{-s}.\end{aligned}$$

□

Let $(\lambda_k\mu_l)\mathfrak{b}_A$ lie in the ray class B modulo $\mathfrak{f}$. Denoting $(\lambda_k\mu_l)\mathfrak{b}_A$ by $\mathfrak{b}_B$, we know that $\mathfrak{b}_B$ runs over a full system of representatives of the ray classes modulo $\mathfrak{f}$. Also, if $\alpha_1 = \lambda_k\mu_l\beta_1$, $\alpha_2 = \lambda_k\mu_l\beta_2$, then $[\alpha_1,\alpha_2]$ is an integral basis **120**
of $\mathfrak{b}_B$ and we may now denote the function

$$g_A(z,\lambda_k\mu_l) = y^s\sum_{m,n}{}' e^{2\pi i S((m\alpha_1+n\alpha_2)\gamma)}|m+nz|^{-2s}$$

by $g_B(z,s)$ and $\int_{z_0}^{z_0^*} g_B(z,s)\dfrac{dp}{p}$ by $F(B,s)$.

The use of the notation $F(B,s)$ is justified for, from (101), we see that $F(B,s)$ *depends only on the ray class B modulo* $\mathfrak{f}$ (and on γ) and *not* on the particular integral ideal $\mathfrak{b}_B$ chosen in B. For, if we replace $\mathfrak{b}_B$ by $(\mu)\mathfrak{b}_B$ where $\mu > 0$, $\mu \equiv 1(\mod {}^*\mathfrak{f})$ and $(\mu)\mathfrak{b}_B$ is an integral ideal coprime to $\mathfrak{f}$, then it is easy to show that $S(\lambda_k\mu_l\rho_i\epsilon_j\mu\beta\gamma) - S(\lambda_k\mu_l\rho_i\epsilon_j\beta\gamma)$ is a rational integer and in addition, $(N(\mathfrak{b}_A))^s|N(\beta)|^{-s}$ again depends only on the ideal class of $\mathfrak{b}_A$.

Now $\overline{\chi}((\lambda_k\mu_l))\overline{\chi}(\mathfrak{b}_A) = \overline{\chi}(\mathfrak{b}_B)$. Moreover, $\{\lambda_k\rho_i\}$ runs over a complete set of representatives λ of the prime residue classes modulo $\mathfrak{f}$. As a consequence, we have

$$\begin{aligned}\sum_B\overline{\chi}(\mathfrak{b}_B)F(B,s) &= 2\cdot\frac{\Gamma^2(s/2)D^{s/2}}{\Gamma(s)}\sum_{\lambda \mod \mathfrak{f}}\overline{\chi}(\lambda)\times\\ &\times\sum_A\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s\sum_{\mathfrak{b}_A|(\beta)} ******(\lambda\beta\gamma)|N(\beta)|^{-s}\end{aligned}$$

where B runs over all ray classes moudlo $\mathfrak{f}$ ******** a complete system of prime residue classes modulo $\mathfrak{f}$ and A ***** ideal classes in the wide sense. In other words, we have for σ **********,

$$\sum_B\overline{\chi}(\mathfrak{b}_B)F(B,s) = *****(s/2)(\Gamma(s))^{-1}D^{s/2}T\cdot L(s,\chi). \tag{103}$$

The function $g_B(z, s)$ is an Epstein zeta-function of the type of $\zeta(s, \underline{u}, \underline{v}, z, 0)$ discussed earlier in § 5, Chapter I. Using the simple functional equation for the Epstein zeta function $\zeta(s, \underline{u}, \underline{v}, z, 0)$, we shall now derive a functional equation for $L(s, \chi)$.

Proposition 16. *If χ is a proper ray class character modulo $\mathfrak{f}(\neq (1))$ whose associated character of signature $v(\lambda) \equiv 1$, then $L(s, \chi)$ is an entire function of s satisfying the following functional equation, namely, if $\xi(s, \chi) = \pi^{-s}\Gamma^2(s/2)(DN(\mathfrak{F}))^{s/2}L(s, \chi)$, then*

$$\xi(s, \chi) = \frac{\chi(\mathfrak{q})}{T/\sqrt{N(\mathfrak{f})}}\xi(1 - s, \overline{\chi}),$$

121

Remark. The functional equation $(*)$ for ****** generalization to arbitrary algebraic number **** derived by Hecke by using the generalized **** formula' (for algebraic number ****** same, it is interesting to deduce the same **** by using the functional equation satisfied by $\zeta(s, \underline{u}, \underline{v}, \underline{z}, 0)$.

Proof. From (92), we have

$$\begin{aligned}
\pi^{-s}\Gamma^2(s/2)D^{s/2}TL(s, \chi) &= \pi^{-s}\Gamma^2(s/2)D^{s/2}\sum_{\lambda}\overline{\chi}(\lambda)\sum_{A}\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \\
&\quad \times \sum_{\mathfrak{b}_A|(\beta)\neq 0} e^{2\pi i S(\lambda\beta\gamma)}|N(\beta)|^{-s} \\
&= \frac{\pi^{-s}\Gamma^2(s/2)D^{s/2}}{e(\mathfrak{f})}\sum_{\lambda}\overline{\chi}(\lambda)\sum_{A}\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \times \\
&\quad \times \sum_{\mathfrak{b}_A|\beta(\Gamma_{\mathfrak{f}}^*)}
\end{aligned}$$

where $e(\mathfrak{f})$ is the index of $\Gamma_{\mathfrak{f}}^*$ in Γ. Now, if we set $\underline{u}^* = \binom{-v_A}{u_A}$, then, in the notation of § 5, Chapter I, we have $g_A(z, s, \lambda, u_A, v_A) = \zeta(s, \underline{u}^*, \underline{0}, z, 0)$ and by (100)

$$\begin{aligned}
\pi^{-s}\Gamma^2(s/2)D^{s/2}TL(s, \chi) &= \frac{2}{e(\mathfrak{f})}\sum_{\lambda}\overline{\chi}(\lambda)\sum_{A}\overline{\chi}(\mathfrak{b}_A)\times \\
&\quad \times \int_{z_0}^{z_0^*}\pi^{-s}\Gamma(s)\zeta(s, \underline{u}^*, \underline{0}, z, 0)\frac{dp}{p}.
\end{aligned}$$

By the method of analytic continuation of $\zeta(s, \underline{u}^*, \underline{0}, z, 0)$ discussed earlier in § 5, Chapter I, we have

$$\int^{z_0^*}\pi^{-s}\Gamma(s)\zeta(s, \underline{u}^*, \underline{0}, z, 0)\frac{dp}{p}$$

$$= \int_{z_0}^{z_0^*} \frac{dp}{p} \int_0^\infty \left(\sum_{m,n}{}' e^{-\pi i y^{-1}|m+nz|^2 + 2\pi i (mu_A + nv_A)} \right) t^s \frac{dt}{t}.$$

The inner integral $\int_0^\infty \ldots$ may be split up as $\int_0^1 \cdots + \int_1^\infty \ldots$. Now we may use 122
the theta-transformation formula for the series $\sum\limits_{m,n} e^{-\pi y^{-1} t|m+nz|^2 + 2\pi i(mu_A + nv_A)}$ and further make use of the inequality $y^{-1}|m + nz|^2 \geq c(m^2 + n^2)$ uniformly on the arc from z_0 to z_0^*, for a constant c independent of m and n. If, in addition, we keep in mind that χ is not the principal character, we can show that $L(s, \chi)$ is an entire function of s. □

Using now the functional equation of $\zeta(s, \underline{u}^*, \underline{0}, z, 0)$, we have

$$\pi^{-s}\Gamma^2(s/2) D^{s/2} T L(s, \chi) = \frac{2}{e(\mathfrak{f})} \sum_{\lambda} \overline{\chi}(\lambda) \sum_{A} \overline{\chi}(\mathfrak{b}_A) \times$$
$$\times \int_{z_0}^{z_0^*} \pi^{-(1-s)} \Gamma(1-s) \zeta(1-s, \underline{0}, \underline{u}^*, z, 0) \frac{dp}{p}.$$

For $\sigma < 0$, $\zeta(1 - s, \underline{0}, \underline{u}^*, z, 0) = y^{1-s} \sum\limits_{m,n} |m - v_A + (n + u_A)z|^{-2(1-s)}$ and by the same arguments as above, we can show that

* * * * * * * * * * * * * * * **

$$= \Gamma^2 \left(\frac{1-s}{2} \right) (N(\mathfrak{b}_A) \sqrt{D})^{1-s} \sum_{\substack{\mu = \beta + \nu \\ \beta \in \mathfrak{b}'_A}} |N(\mu)|^{-(1-s)}$$

where, on the right hand side, $\mu = -\beta'_1 S(\lambda\beta_2\gamma) + \beta'_2 S(\lambda\beta_1\gamma) = -\sqrt{D}\lambda\gamma N(\mathfrak{b}_A)$ and μ runs over a complete set of numbers of the form $\beta + \nu$ with $\beta \in \mathfrak{b}'_A$, such that they are not mutually associated with respect to $\Gamma^*_{\mathfrak{f}}$.

Let δ be an integer in K such that $\mathfrak{a}\mathfrak{f} = (\delta)$ with an integral ideal $\mathfrak{a}$ coprime to $\mathfrak{f}$. Then, for $\sigma < 0$, we obtain from above that

$$\pi^{-s}\Gamma^2(s/2) D^{s/2} T L(s, \chi) = \frac{\pi^{-(1-s)}}{e(\mathfrak{f})} D^{(1-s)/2} \Gamma^2((1-s)/2) \times$$
$$\times \sum_{A} \overline{\chi}(\mathfrak{b}_A) (N(\mathfrak{b}_A))^{1-s} (N(\mathfrak{a}\mathfrak{f}))^{1-s} \sum_{\lambda} \overline{\chi}(\lambda) \times$$
$$\times \sum_{\substack{\beta \equiv \nu\delta (\bmod \mathfrak{a}\mathfrak{b}'_A \mathfrak{f}) \\ \beta \neq 0, \mathfrak{a}\mathfrak{b}'_A | \beta (\Gamma^*_{\mathfrak{f}})}} |N(\beta)|^{-(1-s)},$$

where, in the inner sum on the right hand side, β runs over a complete set of integers in $\mathfrak{a}\mathfrak{b}'_A$ which are not mutually associated with respect to $\Gamma^*_{\mathfrak{f}}$ and which satisfy $\beta \equiv \nu\delta (\mod \mathfrak{a}\mathfrak{b}'_A\mathfrak{f})$. If $\beta \equiv \nu\delta \mod \mathfrak{a}\mathfrak{b}'_A\mathfrak{f}$, then **123**

$$\begin{aligned}\chi(\beta) &= \chi(\nu\delta) = \chi(-\gamma\sqrt{D}\lambda\delta N(\mathfrak{b}_A)) = \chi((N(\mathfrak{b}_A)\gamma\sqrt{D}\lambda\delta))\\ &= \chi(\mathfrak{b}_A)\chi(\mathfrak{b}'_A)\chi((\gamma\sqrt{D}\delta))\chi(\lambda) = \chi(\mathfrak{b}_A)\chi(\mathfrak{b}'_A)\chi(\mathfrak{a}\mathfrak{q})\chi(\lambda).\end{aligned}$$

Hence $\overline{\chi}(\lambda) = \overline{\chi}(\beta)\chi(\mathfrak{b}_A\mathfrak{b}'_A\mathfrak{a}\mathfrak{q})$. Further if λ runs over representatives of prime residue classes modulo $\mathfrak{f}$ and β runs independently over a complete set of integers in $\mathfrak{a}\mathfrak{b}'_A$ congruent modulo $\mathfrak{a}\mathfrak{b}'_A\mathfrak{f}$ to $-\sqrt{D}\lambda\gamma\delta N(\mathfrak{b}_A)$ and not mutually associated with respect to $\Gamma^*_{\mathfrak{f}}$, then β runs over a complete set of integers in $\mathfrak{a}\mathfrak{b}'_A$ not mutually associated with respect to $\Gamma^*_{\mathfrak{f}}$ and satisfying $((\beta), \mathfrak{a}\mathfrak{b}'_A\mathfrak{f}) = \mathfrak{a}\mathfrak{b}'_A$, If we note, in addition, that $\chi(\beta) = 0$ for those β in $\mathfrak{a}\mathfrak{b}'_A$ for which $(\beta)(\mathfrak{a}\mathfrak{b}'_A)^{-1}$ is not coprime to $\mathfrak{f}$, we can show finally that

$$\sum \overline{\chi}(\lambda) \sum_{\substack{\mathfrak{a}\mathfrak{b}'_A|\beta(\Gamma^*_1)\\ \beta\equiv\nu\delta(\mod \mathfrak{a}\mathfrak{b}'_A\mathfrak{F})}} |N(\beta)|^{-(1-s)} = \chi(\mathfrak{b}_A\mathfrak{b}'_A\mathfrak{a}\mathfrak{q})\times$$

$$\times \sum_{\substack{\mathfrak{a}\mathfrak{b}'_A|\beta(\Gamma^*_{\mathfrak{f}})\\ \beta\neq 0}} \overline{\chi}(\beta)|N(\beta)|^{-(1-s)}.$$

Thus, for $\sigma < 0$,

$$\begin{aligned}\pi^{-s}\Gamma^2(s/2)D^{s/2}TL(s,\chi) &= \frac{\pi^{-(1-s)}}{(N(\mathfrak{f}))^{s-1}}D^{(1-s)/2}\Gamma^2((1-s)/2)\chi(\mathfrak{q})\times\\ &\quad\times\sum_A \chi(\mathfrak{a}\mathfrak{b}'_A)N((\mathfrak{a}\mathfrak{b}'_A))^{1-s}\times\\ &\quad\sum_{\mathfrak{a}\mathfrak{b}'_A|(\beta)\neq(0)} \overline{\chi}(\beta)|N(\beta)|^{-(1-s)}\\ &= \pi^{-(1-s)}D^{(1-s)/2}\Gamma^2((1-s)/2)\frac{\chi(\mathfrak{q})}{(N(\mathfrak{f}))^{s-1}}L(1-s,\overline{\chi}),\end{aligned}$$

since $\mathfrak{a}\mathfrak{b}'_A$ again runs over a system of representatives of the ideal classes similar to $\mathfrak{b}_A$. Since $L(s,\chi)$ is an entire function of s, the above functional relation is **124**
clearly valid for all s. We have now only to set $\xi(s,\chi) = \pi^{-s}\Gamma^s(s/2)(DN(\mathfrak{f}))^{s/2}L(s,\chi)$, to see that the equation $(*)$ is true.

Let now $[\alpha_1, \alpha_2]$ be an integral basis for the ideal $\mathfrak{b}_B$ in the ray class B and let $\omega = \alpha_2/\alpha_1$, $u_B = S(\alpha_1\gamma)$, $v_B = S(\alpha_2\gamma)$. By Kronecker's second limit formula, we have

$$g_B(z,s) = -\pi \log|\varphi(v_B, u_B, z)|^2$$

$$+ \cdots \text{ terms involving higher powers of } (s-1)\ldots.$$

Letting s tend to 1, we have from (103),

$$L(1,\chi) = -\frac{1}{2T\sqrt{D}} \sum_B \overline{\chi}(\mathfrak{b}_B) \int_{z_0}^{z_0^*} \log|\varphi(v_B, u_B, z)|^2 \frac{dp}{p}.$$

Now it is easy to verify that

$$\frac{dp}{p} = -\frac{(\omega-\omega')}{(z-\omega)(z-\omega')}dz = -\frac{\sqrt{D}}{F_B(z)}dz,$$

where

$$F_B(z) = \frac{\sqrt{D}}{\omega-\omega'}(z-\omega)(z-\omega') = a_1z^2 + b_1z + c_1 \qquad (*)$$

has the property that a_1, b_1, c_1 are rational integers with the greatest common divisor 1, $a_1 > 0$ and $b_1^2 - 4a_1c_1 = D$. Thus we have

Theorem 11. *For a proper ray class character modulo $\mathfrak{f} \neq (1)$ in $\mathbf{Q}(\sqrt{D})$, $D > 0$, with the associated $v(\lambda) \equiv 1$, the value of $L(s,\chi)$ at $s = 1$ is given by*

$$L(1,\chi) = \frac{1}{2T} \sum_B \overline{\chi}(\mathfrak{b}_B) \int_{z_0}^{z_0^*} \log|\varphi(v_B, u_B, z)|^2 \frac{dz}{F_B(z)},$$

where B runs over all the ray classes modulo $\mathfrak{f}$ and $F_B(z)$ is given by $(*)$ above.

As remarked earlier, the terms of the sum on the right hand side depend only on γ (fixed!) and on the ray class B and not on the choice of the ideal $\mathfrak{b}_B$ in B. We observe further that it does not seem to be possible to simplify this **125** formula any further, in an elementary way. One might try to follow the method of Herglotz to deal with the integrals but then the invariance properties of the integrand are lost in the process.

We proceed to discuss the case when *the character of signature $v(\lambda)$ associated with the given ray class character χ is of type* (ii), i.e. for integral $\alpha(\neq 0)$ coprime to $\mathfrak{f}$, we have $\chi((\alpha)) = \chi(\alpha) \cdot \dfrac{N(\alpha)}{|N(\alpha)|}$. In this case, we shall now see that the value at $s = 1$ of $L(s,\chi)$ can be determined in terms of elementary functions, using the first or second limit formula of Kronecker, according as $\mathfrak{f} = (1)$ or $\mathfrak{f} \neq (1)$.

The fact that the character of signature $v(\lambda)$ associated with χ is defined by

$$v(\lambda) = \frac{N(\lambda)}{|N(\lambda)|} \quad \text{for} \quad \lambda \neq 0 \qquad (104)$$

implies automatically a condition on $\mathfrak{f}$. For instance, if ρ is a unit congruent to 1 modulo, $\mathfrak{f}$, then $\dfrac{N(\rho)}{|N(\rho)|} = \chi(\rho)v(d) = \chi((\rho)) = 1$. In other words, every unit congruent to 1 modulo $\mathfrak{f}$ should necessarily have norm 1.

To determine $L(1,\chi)$, we start as before from formula (92), namely, for $\sigma > 1$,

$$L(s,\chi) = T^{-1} \sum_{\lambda \mod \mathfrak{f}} \overline{\chi}(\lambda) \sum_{A} \overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s \times$$
$$\times \sum_{\mathfrak{b}_A|(\beta)\neq(0)} e^{2\pi i S(\lambda\beta\gamma)} \frac{N(\beta)}{|N(\beta)|} |N(\beta)|^{-s}.$$

We shall try to express the inner sum on the right hand side as an integral, as before. We shall follow the same notation as in the case treated above.

Let $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\dfrac{\partial}{\partial x} - \dfrac{i}{2}\dfrac{\partial}{\partial y}$; then in view of the uniform convergence of the series defining the function $g_A(z,s,\lambda,u_A,v_A)$ (denoted by $g_A(z,\lambda)$ for brevity) for $\sigma > 1$, we have

$$\frac{\partial g_A(z,\lambda)}{\partial z} = \sum_{m,n}{}' e^{2\pi i(mu_A+nv_A)} \frac{\partial}{\partial_z}\{y^s|m+nz|^{-2s}\}$$
$$= \frac{s}{2i} y^{s-1} \sum_{m,n}{}' e^{2\pi i(mu_A+nv_A)}|m+nz|^{-(2s-2)}(m+nz)^{-2}. \tag{105}$$

126

Let us now consider the integral $\int_{z_0}^{z_0^*} \dfrac{\partial g_A(z,\lambda)}{\partial z} dz$, extended over the same arc of the semi-circle in $\mathfrak{H}$ as considered earlier. Due to the uniform convergence of the series (105) on this arc, we have

$$\int_{z_0}^{z_0^*} \frac{\partial g_A(z,\lambda)}{\partial z} dz = \frac{s}{2i} \sum_{m,n}{}' e^{2\pi i(mu_A+nv_A)} \int_{z_0}^{z_0^*} \frac{y^{s-1}}{|m+nz|^{2s-2}(m+nz)^2} dz$$
$$= \frac{s}{2} D^{s/2} (N(\mathfrak{b}_A))^s \times$$
$$\times \sum_{\mathfrak{b}_A|\beta\neq 0} e^{2\pi i S(\lambda\beta\gamma)} |N(\beta_1)|^{-s} \int_{p_0}^{p_0^*} \frac{p^{s-1}dp}{(\mu^2p^2+\mu'^2)^{s-1}(\mu p i+\mu')^2}.$$

Effecting the substitution $p \to |\mu'/\mu|p$, we see that

$$\int_{p_0}^{p_0^*} \frac{p^{s-1}}{(\mu^2p^2+\mu'^2)^{s-1}(\mu p i+\mu')^2} dp$$

$$= |N(\mu)|^{-s}\int_{|\mu/\mu'|p_0}^{|\mu/\mu'|p_0^*}\frac{p^{s-1}}{(p^2+1)^{s-1}(pi\tau+1)^2}dp,$$

where

$$\tau=\frac{\mu/\mu'}{|\mu/\mu'|}=\frac{N(\mu)}{|N(\mu)|}=\frac{N(\beta)}{|N(\beta)|}\frac{N(\beta_1)}{|N(\beta_1)|}=\nu(\beta)\nu(\beta_1).$$

Further $(\pi\tau+1)^2=(pi+\tau)^2$. We have, therefore,

$$\int_{z_0}^{z_0^*}\frac{\partial g_A(z,\lambda)}{\partial z}dz=\frac{s}{2}D^{s/2}(N(\mathfrak{b}_Z))^s\times$$
$$\times\sum_{\mathfrak{b}_A|\beta\neq 0}e^{2\pi iS(\lambda\beta\gamma)}|N(\beta)|^{-s}\int_{|(\beta\beta_1'/\beta'\beta_1)|p_0}^{|(\beta\beta_1'/\beta'\beta_1)|p_0\epsilon^2}\frac{p^{s-1}}{(p^2+1)^{s-1}(pi+\tau)^2}dp.$$

Now, since $\nu(\epsilon)=1$, the value of τ corresponding to β and $\beta\epsilon^k (k=\pm1,\pm2,\ldots)$ **127**
is the same. Moreover $e^{2\pi iS(\lambda\beta\gamma)}=e^{2\pi iS(\lambda\beta\epsilon^k\gamma)}$. If then we apply the same analysis as in the former case, we obtain

$$\int_{z_0}^{z_0^*}\frac{\partial g_A(z,\lambda)}{\partial z}dz=\frac{s}{2}D^{s/2}(N(\mathfrak{b}_A))^s\times$$
$$\times\sum_{\mathfrak{b}_A|\beta(\Gamma_\dagger^*)}e^{2\pi iS(\lambda\beta\gamma)}|N(\beta)|^{-s}\int_0^\infty\frac{p^{s-1}dp}{(p^2+1)^{s-1}(pi+\tau)^2}\qquad(106)$$

where, on the right hand side, the summation is over a complete set of integers $\beta\neq 0$ in $\mathfrak{b}_A$, which are not associated with respect to $\Gamma_\dagger^*$.

Effecting the transformation $p\to p^{-1}$, we see that

$$\int_0^\infty\frac{p^{s-1}}{(p^2+1)^{s-1}(\tau+ip)^2}dp=-\int_0^\infty\frac{p^{s-1}}{(p^2+1)^{s-1}(\tau-ip)^2}dp.$$

Taking the arithmetic mean, we have

$$\begin{aligned}&\int_0^\infty\frac{p^{s-1}}{(p^2+1)^{s-1}(\tau+ip)^2}dp\\&=\frac{1}{2}\int_0^\infty\frac{p^{s-1}}{(p^2+1)^{s-1}}\left\{\frac{1}{(\tau+ip)^2}-\frac{1}{(\tau-ip)^2}\right\}dp\\&=-2\tau i\int_0^\infty\frac{p^{s+1}}{(p^2+1)^{s+1}}\frac{dp}{p}\\&=\frac{\nu(\beta)\nu(\beta_1)}{i}\frac{\Gamma^2((s+1)/2)}{\Gamma(s+1)}.\end{aligned}\qquad(107)$$

From (106) and (107), we obtain for $\sigma > 1$,

$$v(\beta_1)\int_{z_0}^{z_0^*}\frac{\partial g_A(z,\lambda)}{\partial z}dz = \frac{D^{s/2}}{2i}\frac{\Gamma^2((s+1)/2)}{\Gamma(s)}(N(\mathfrak{b}_A))^s\times$$
$$\times\sum_{\mathfrak{b}_A|\beta(\Gamma_{\mathfrak{f}}^*)} v(\beta)e^{2\pi i S(\lambda\beta\gamma)}|N(\beta)|^{-s}.$$

Since $\{\rho_i\epsilon_j\}$ is a system of representatives of Γ modulo $\Gamma_{\mathfrak{f}}^*$, we have **128**

$$v(\beta_1)\int_{z_0}^{z_0^*}\frac{\partial g_A(z,\lambda)}{\partial z}dz = \frac{D^{s/2}}{2i}\frac{\Gamma^2((s+1)/2)}{\Gamma(s)}(N(\mathfrak{b}_A))^s\times$$
$$\times\sum_{\substack{\mathfrak{b}_A|(\beta)\\ \rho_i,\epsilon_j}} v(\beta\rho_i\epsilon_j)e^{2\pi i S(\lambda\beta\rho_i\epsilon_j\gamma)}|N(\beta)|^{-s}, \tag{108}$$

where the summation on the right hand side is extended over all non-zero principal ideals divisible by $\mathfrak{b}_A$ and over the finite sets of representatives $\{\rho_i\}$ and $\{\epsilon_j\}$. Summing over all ideal classes A and the elements of the sets $\{\lambda_k\}$ and $\{\mu_l\}$ we obtain from (108),

$$\sum_{\lambda_k,\mu_l,A}\overline{\chi}(\lambda_k\mu_l)\overline{\chi}(\mathfrak{b}_A)v(\beta_1)\int_{z_0}^{z_0^*}\frac{\partial g_A(z,\lambda_k\mu_l)}{\partial z}dz$$
$$= \frac{D^{s/2}}{2i}\frac{\Gamma^2((s+1)/2)}{\Gamma(s)}\sum_{\lambda_k,\mu_l,A}\overline{\chi}(\lambda_k\mu_l)\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s\times$$
$$\times\sum_{\substack{\mathfrak{b}_A|(\beta)\\ \rho_i,\epsilon_j}} v(\beta\rho_i\epsilon_j)e^{2\pi i S(\lambda_k\mu_1\beta\rho_i\epsilon_j\gamma)}|N(\beta)|^{-s}. \tag{109}$$

It is quite easy to verify that

$$v(\rho_i\epsilon_j) = \chi((\rho_i\epsilon_j))\overline{\chi}(\rho_i\epsilon_j) = \overline{\chi}(\rho_i\epsilon_j) = \overline{\chi}(\rho_i),$$
$$e^{2\pi i S(\lambda_k\mu_l\beta\rho_i\epsilon_j\gamma)} = e^{2\pi i S(\lambda_k\rho_i\beta\gamma)}.$$

Using these facts and applying the same arguments as in the earlier situation, we see that the right hand side of (109) is precisely

$$-2iD^{s/2}\frac{\Gamma^2((s+1)/2)}{\Gamma(s)}\sum_{\lambda \mod \mathfrak{f}}\overline{\chi}(\lambda)\sum_{A}\overline{\chi}(\mathfrak{b}_A)(N(\mathfrak{b}_A))^s\times$$
$$\times\sum_{\mathfrak{b}_A|(\beta)} v(\beta)e^{2\pi i S(\lambda\beta\gamma)}|N(\beta)|^{-s}$$

$$= -2iD^{s/2}\frac{\Gamma^2((s+1)/2)}{\Gamma(s)} T \cdot L(s,\chi).$$

To simplify the expression on the left hand side of (109), we first observe **129**
that $\overline{\chi}(\lambda_k\mu_1) = \overline{\chi}((\lambda_k\mu_l))v(\lambda_k\mu_l)$. Moreover, if we denote, as before, the ideal $(\lambda_k\mu_l)\mathfrak{b}_A$ by $\mathfrak{b}_B$ where B is the ray class modulo $\mathfrak{f}$ containing it, then, by Proposition 15, $\mathfrak{b}_B$ runs over a full system of representatives of ray classes modulo $\mathfrak{f}$. We may assume, without risk of confusion, that $[\beta_1, \beta_2]$ is still an integral basis of $\mathfrak{b}_B$ and set $u_B = S(\beta_1\gamma)$, $v_B = S(\beta_2\gamma)$. Further we may denote $g_A(z, s, \lambda_k\mu_l, u_A, v_A) = y^s \sum_{m,n}{}' e^{2\pi i(mu_B+nv_B)}|m+nz|^{-2s}$ by $g_B(z) = g_B(z,s)$. The function $g_B(z,s)$ depends not just on the ray class B but also on the choice of the ideal $\mathfrak{b}_B$ in B and on the integral basis $[\beta_1, \beta_2]$ of $\mathfrak{b}_B$. Since $\mathfrak{b}_B$ and $[\beta_1, \beta_2]$ are fixed, the modular transformation $z \to (az+b)(cz+d)^{-1}$ is uniquely determined by $\epsilon\beta_2 = a\beta_2 + b\beta_1$, $\epsilon\beta_1 = c\beta_2 + d\beta_1$; $\omega = \beta_2/\beta_1$, $\omega' = \beta_2'/\beta_1'$ are the two fixed points of this transformation and $z_0^* = (az_0+b)(cz_0+d)^{-1}$. From (109), we have now, for $\sigma > 1$,

$$L(s,\chi) = \frac{iD^{-s/2}}{2T}\frac{\Gamma(s)}{\Gamma^2((s+1)/2)}\sum_B \overline{\chi}(\mathfrak{b}_B)v(\beta_1)\int_{z_0}^{z_0^*}\frac{\partial g_B(z)}{\partial z}dz. \qquad (110)$$

Let us denote $-\dfrac{1}{2\pi^2 i}v(\beta_1)\int_{z_0}^{z_0^*}\dfrac{\partial g_B(z)}{\partial z}dz$ by $G(B,s)$. We see from (108) that, for $\sigma > 1$,

$$G(B,s) = \frac{D^{s/2}\Gamma^2((s+1)/2)}{4\pi^2\Gamma(s)}(N(\mathfrak{b}_B))^s \times \\ \times \sum_{\rho_i,\epsilon_j}\sum_{\mathfrak{b}_B|(\beta)\neq(0)} v(\beta\rho_i)e^{2\pi\sqrt{-1}S(\beta\rho_i\gamma)}|N(\beta)|^{-s}.$$

Obviously, the right hand side remains unchanged, when $\mathfrak{b}_B$ is replaces by an- **130**
other integral ideal in B. Thus, $G(B,s)$ clearly depends *only on B* (and on γ and s) but *not* on the special choice of $\mathfrak{b}_B$ or its integral basis. Writing $\overline{\chi}(B)$ for $\overline{\chi}(\mathfrak{b}_B)$, we see that (110) goes over into

$$L(s,\chi) = \frac{\pi^2}{TD^{s/2}}\frac{\Gamma(s)}{((s+1)/2)}\sum_B \overline{\chi}(B)G(B,s). \qquad (111)$$

The function $\dfrac{\partial g_B(z,s)}{\partial z}$ is essentially an Epstein zeta-function of the type $\zeta(s, \underline{u}, \underline{v}, Q, \mathscr{P})$ discussed in § 5, Chapter I and it can be shown as before that $\int_{z_0}^{z_0^*}\dfrac{\partial g_B(z,s)}{\partial z}dz$

is an entire function of s, by using the method of analytic continuation of the Epstein zeta-function. Thus formula (110) gives the analytic continuation of $L(s, \chi)$ into the entire s-plane, as an entire function of s. Moreover, if we use the functional equation satisfied by $\dfrac{\partial g_B(z,s)}{\partial z}$, we can show as in Proposition 16 that $L(s, \chi)$ satisfies the following functional equation, viz. if we set

$$\Lambda(s, \chi) = \pi^{-s}\Gamma^2((s+1)/2)(DN(\mathfrak{f}))^{s/2}L(s, \chi),$$

then

$$\Lambda(s, \chi) = \frac{\chi(\mathfrak{q})\nu(\gamma\sqrt{D})}{T/\sqrt{N(\mathfrak{f})}}\Lambda(1-s, \overline{\chi}). \tag{112}$$

Let us observe that the constant on the right hand side is of absolute value 1.

In order to determine the value of $L(1, \chi)$, we distinguish between the two cases $\mathfrak{f} \neq (1)$ and $\mathfrak{f} = (1)$.

(i) *Let first* $\mathfrak{f} \neq (1)$. Then the numbers u_B and v_B corresponding to a ray class B modulo $\mathfrak{f}$ are rational numbers which are not both integral. For the function $g_B(z,s)$ we have the following power-series expansion at 131
$s = 1$, by Kronecker's second limit formula, namely

$$g_B(z,s) = -\pi \log |\varphi(v_B, u_B, z)|^2$$
$$\cdots \text{ (terms involving higher powers of } (s-1)).$$

Applying $\partial/\partial z$ to $g_B(z,s)$ and noting that $(\partial/\partial z)\log \overline{\varphi(v_B, u_B, z)} = 0$, we have for $\dfrac{\partial g_B(z,s)}{\partial z}$ the power-series expansion,

$$\frac{\partial g_B(z,s)}{\partial z} = -\pi \frac{\partial}{\partial z}\log \varphi(v_B, u_B, z)$$
$$+ \cdots \text{ (terms involving higher powers of } (s-1)).$$

Now since $\varphi(v_B, u_B, z)$ is regular and non-vanishing in $\mathfrak{H}$, we can choose a fixed branch of $\log \varphi(v_B, u_B, z)$ in $\mathfrak{H}$ and $(\partial/\partial z)\log \varphi(v_B, u_B, z)$ is just $(d/dz)\log \varphi(v_B, u_B, z)$. The coefficients of the higher powers of $(s-1)$ might involve $\bar{z}$. Letting s tend to 1, we have

$$G(B, 1) = \frac{\nu(\beta_1)}{2\pi i}\left[\log \varphi(v_B, u_B, z)\right]_{z_0}^{z_0^*}$$

We shall denote $G(B, 1)$ by $G(B)$ and then we have, from (111)

$$L(1, \chi) = \frac{\pi^2}{T\sqrt{D}}\sum_B \overline{\chi}(B)G(B).$$

(ii) *Let now* $\mathfrak{f} = (1)$. Then, for all B, we see that $g_B(z,s) = y^s \sum_{m,n} |m + nz|^{-2s}$ and by Kronecker's first limit formula, we have the following power-series expansion for $g_B(z,s)$ at $s = 1$, namely

$$g_B(z,s) - \frac{\pi}{s-1} = 2\pi(C - \log 2) - \pi \log(y|\eta(z)|^4) + \cdots .$$

The application of $\partial/\partial z$ does away with the terms $\pi/(s-1)$ and $2\pi(C - \log 2)$. Noting that

$$\frac{\partial}{\partial z} \log y = \frac{1}{2iy} = \frac{1}{z - \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial z}(\overline{\eta(z)})^2 = 0,$$

we have for $\dfrac{\partial g_B(z,s)}{\partial z}$, the following power-series expansion at $s = 1$, namely 132

$$\frac{\partial g_B(z,s)}{\partial z} = -\frac{\pi}{z - \bar{z}} - \pi \frac{d}{dz} \log(\eta(z))^2 + \\ + \cdots \text{ terms involving higher powers of } (s-1).$$

We wish to replace $1/(z - \bar{z})$ if possible by a regular function of z; but we shall see now that we can do this when z lies on the semi-circle on $\omega'\omega$ as diameter. In fact, the equation to the semi-circle on $\omega'\omega$ as diameter is

$$\frac{z - \omega}{z - \omega'} + \frac{\bar{z} - \omega}{\bar{z} - \omega'} = 0, \quad z \in \mathfrak{h}.$$

This means that $z\bar{z} + p(z + \bar{z}) + q = 0$ with $p = -\dfrac{\omega + \omega'}{2}$, $q = \omega\omega'$, when z lies on the semi-circle. But then

$$\begin{aligned} \frac{1}{z - \bar{z}} &= \frac{z + p}{z^2 + 2pz + q} \\ &= \frac{1}{2} \frac{d}{dz} \log(z^2 + 2pz + q) \\ &= \frac{d}{dz} \log \sqrt{(z - \omega)(z - \omega')}. \end{aligned}$$

Hence, for z on the semi-circle, we have the power-series expansion

$$\frac{\partial g_B(z,s)}{\partial z} = -\pi \frac{d}{dz} \log \sqrt{(z - \omega)(z - \omega')}\, \eta^2(z) + \\ + \cdots \text{ (higher powers of } s - 1).$$

Thus letting s tend to 1,

$$G(B,1) = \frac{\nu(\beta_1)}{2\pi i}\left[\log\sqrt{(z-\omega)(z-\omega')}\eta^2(z)\right]_{z_0}^{z_0^*}$$

for a fixed branch of $\log(\sqrt{(z-\omega)(z-\omega')}\eta^2(z))$. Denoting $G(B,1)$ by $G(B)$ again, we have from (111),

$$L(1,\chi) = \frac{\pi^2}{\sqrt{D}}\sum_B \overline{\chi}(B)G(B).$$

We are thus led to **133**

Theorem 12. *For a proper ray class character $\chi(\neq 1)$ modulo $\mathfrak{f}$ in $\mathbf{Q}(\sqrt{D})(D > 0)$, with associated $\nu(\lambda)$ defined by* (104), *we have*

$$L(1,\chi) = \frac{\pi^2}{T\sqrt{D}}\sum_B \overline{\chi}(B)G(B)$$

where the summation is over all ray classes B modulo $\mathfrak{f}$ and

$$G(B) = \begin{cases} \dfrac{\nu(\beta_1)}{2\pi i}\left[\log\varphi(v_B,u_B,z)\right]_{z_0}^{z_0^*}, & \textit{for } \mathfrak{f}\neq(1),\\ \dfrac{\nu(\beta_1)}{2\pi i}\left[\log(\sqrt{(z-\omega)(z-\omega')}\eta^2(z))\right]_{z_0}^{z_0^*}, & \textit{for } \mathfrak{f}=(1). \end{cases} \tag{113}$$

We are now interested in the *explicit determination of the values of* $G(B)$ corresponding to the various ray classes B, *in terms of elementary arithmetical functions.* The expressions inside the square brackets in (113) represent analytic functions of z and we know that $G(B)$ itself does not depend on the point z_0 chosen on the semi-circle on $\omega'\omega$ as diameter. Thus $G(B)$ is, in both cases, the value of an analytic function z, which is a constant on the semi-circle on $\omega'\omega$ as diameter; as a consequence, $G(B)$ is a constant independent of z_0 and in order to calculate $G(B)$, we might replace z_0 in (113) by any point z in $\mathfrak{H}$, not necessarily lying on the semi-circle on $\omega'\omega$ as diameter.

First we observe that $(1/2\pi i)[\log\varphi(v_B,u_B,z)]_z^{z^*}$ *is a rational number with at most* $12f$ *in the denominator,* where f is the smallest positive rational integer divisible by $\mathfrak{f}$. This can be verified as follows. Let $z^* = (az+b)(cz+d)^{-1}$ where the rational integers a, b, c, d satisfying the condition $ad-bc=1$ are uniquely determined by the unit ϵ and a fixed integral basis $[\beta_1,\beta_2]$ of a fixed integral ideal $\mathfrak{b}_B$ in B, coprime to $\mathfrak{f}$. Corresponding to z^*, let $v_B^* = av_B + bu_B$ and $u_B^* = cv_B + du_B$. We know already from the properties of $\varphi(v_B,u_B,z)$ that

$\varphi(v_V^*, u_B^*, z^*) = \rho\varphi(v_B, u_B, z)$ where ρ is a 12^{th} root of unity depending only on a, b, c, d. On the other hand $v_B^* = aS(\beta_2\gamma) + bS(\beta_1\gamma) = S(\epsilon\beta_2\gamma)$ and hence $v_B^* - v_B = S((\epsilon - 1)\beta_2\gamma)$ is a rational integer k, say; similarly, $u_B^* - u_B$ is again a rational integer l, say. But we know from (99) that

$$\begin{aligned}\varphi(v_B^*, u_B^*, z^*) &= \varphi(v_B + k, u_B + l, z^*)\\ &= \tau\varphi(v_B, u_B, z^*),\end{aligned}$$

where τ is a $2f$-th root of unity. Hence 134

$$\varphi(v_B, u_B, z^*) = \theta\varphi(v_B, u_B, z),$$

where θ is a $12f$-th root of unity. Choosing a fixed branch of $\log\varphi(v_B, u_B, z)$, we see that $\dfrac{1}{2\pi i}[\log\varphi(v_B, u_B, z)]_z^{z^*}$ is a rational number with at most $12f$ in the denominator. Now, we know that for $z \in \mathfrak{H}$, the function $\varphi^{12f}(v_B, u_B, z)$ is a modular function of level f and $[\log\varphi(v_B, u_B, z)]_z^{z^*}$ is just a 'period' of the abelian integral $\log\varphi(v_B, u_B, z)$. The explicit computation of these 'periods' of the abelian integral $\log\varphi(v_B, u_B, z)$ has been essentially considered by Hecke, in connection with the determination of the class number of a biquadratic field obtained from a real quadratic field k over $\mathbf{Q}$, by adjoining the square root of a "totally negative" number in k; employing the ideal of the well-known Riemann-Dedekind method, Hecke used for this purpose, the asymptotic behaviour of $\log\vartheta_{11}(w, z)$ as z tends to infinity and z^* to the corresponding 'rational point' a/c on the real axis. It is to be remarked, however, that from Hecke's considerations, one can at first sight conclude only that the quantities $\dfrac{1}{2\pi i}[\log\varphi(v_B, u_B, z)]_z^{z^*}$ are rational numbers with at most $24f^2$ in the denominator.

Regarding $\dfrac{1}{2\pi i}[\log\sqrt{(z-\omega)(z-\omega')}\eta^2(z)]_z^{z^*}$, we can conclude again that they are rational numbers with at most $12f$ in the denominator. For this purpose, we notice that

$$z^* - \omega = \frac{z-\omega}{\epsilon(cz+d)} \quad \text{and} \quad z^* - \omega' = \frac{z-\omega'}{\epsilon'(cz+d)}$$

and $\eta^2(z^*) = \rho(cz+d)\eta^2(z)$, ρ being a 12^{th} root of unity depending only on a, b, c, d. Hence

$$\sqrt{(z^*-\omega)(z^*-\omega')}\eta^2(z^*) = \rho\sqrt{(z-\omega)(z-\omega')}\eta^2(z).$$

Choosing a fixed branch of $\log(\sqrt{(z-\omega)(z-\omega')}\eta^2(z))$ in $\mathfrak{H}$, we see that our assertion is true.

We may now proceed to obtain the value of $G(B)$, first for $\mathfrak{f} \neq (1)$.

We make a few preliminary simplifications. We can suppose without loss of generality that $0 \leq u_B, v_B < 1$, since, in view of (98), $\varphi(v_B, u_B, z)$ merely picks up a $2f^{\text{th}}$ root of unity as a factor when v_B is replaced by $v_B \pm 1$ or u_B by $u_B \pm 1$ and this gets cancelled, when we consider $[\log\varphi(v_B, u_B, z)]_z^{z^*}$. Moreover, since we know already that $G(B)v(\beta_1)$ is real (in fact, even rational) it is enough to find the real part of $\dfrac{1}{2\pi i}[\log\varphi(v_B, u_B, z)]_z^{z^*}$. Now **135**

$$\begin{aligned}
\varphi(v, u, z) &= e^{\pi iu(uz-v)-\pi i/2+\pi iz/6}(e^{\pi i(v-uz)} - e^{-\pi i(v-uz)})\times \\
&\quad \times\prod_{m=1}^{\infty}(1 - e^{2\pi i(v-uz)+2\pi imz})\prod_{m=1}^{\infty}(1 - e^{-2\pi i(v-uz)+2\pi imz}) \\
&= e^{\pi iu(uz-v)-\pi i/2+\pi iz/6+\pi i(v-uz)}\times \\
&\quad \times\prod_{m=1}^{\infty}(1 - VQ^{m-u})\prod_{m=0}^{\infty}(1 - V^{-1}Q^{m+u}),
\end{aligned}$$

where we have set $V = e^{2\pi iv}$, $Q = e^{2\pi iz}$ and $Q^{-u} = e^{-2\pi iuz}$. Moreover, if $u = 0$, then $0 < v < 1$ and in the second infinite product above we notice that $m+u \geq 0$, in any case. We define the branch of $(1/2\pi i)\cdot\log\varphi(v, u, z)$ by

$$\begin{aligned}
\frac{1}{2\pi i}\log\varphi(v, u, z) &= \frac{z}{2}\left(u^2 - u + \frac{1}{6}\right) - \frac{1}{4} + \frac{v}{2}(1 - u)+ \\
&\quad + \frac{1}{2\pi i}\sum_{m=1}^{\infty}\log(1 - VQ^{m-u})+ \\
&\quad + \frac{1}{2\pi i}\sum_{m=0}^{\infty}\log(1 - V^{-1}Q^{m+u}),
\end{aligned}$$

where, on the right hand side, we take the principal branches of the logarithms. Noting that the series $\sum\limits_{n=1}^{\infty}\dfrac{(V^{-1}Q^u)^n}{n}$ converges even if $u = 0$, since then $0 < v < 1$ and further the series $\sum\limits_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{(VQ^{m-u})^n}{n}$ and $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\dfrac{(V^{-1}Q^{m+u})^n}{n}$ converge absolutely, we conclude that

$$\begin{aligned}
\frac{1}{2\pi i}\log\varphi(v, u, z) &= \frac{1}{2}\left(u^2 - u + \frac{1}{6}\right)z - \frac{1}{4} + \frac{v}{2}(1 - u)- \\
&\quad - \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(VQ^{-u})^nQ^n + (V^{-1}Q^u)^n}{1 - Q^n}
\end{aligned}$$

Thus, for $\mathfrak{f} \neq (1)$, 136

$$G(B)v(\beta_1) = \frac{1}{2}\left(u_B^2 - u_B + \frac{1}{6}\right)(z^* - z) - \\ - \left[\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(VQ^{\frac{1}{2}-u_B})^n + (V^{-1}Q^{-\frac{1}{2}+u_B})^n}{Q^{-n/2} - Q^{n/2}}\right]_z^{z^*} \tag{114}$$

where we have used V to stand for $e^{2\pi i v_B}$ without risk of confusion.

Now $c = (\epsilon - \epsilon')/(\omega - \omega') > 0$; so, if we set $z = -(d/c) + (i/cr)$ with $r > 0$, then $z^* = (a/c) + (ir/c)$. As r tends to zero, z^* tends to the rational point a/c vertically and z tends to infinity. In view of our earlier remarks, we may let r tend to zero in (114), in order to find the value of $G(B)v(\beta_1)$; moreover, it suffices to find the real part of the right hand side of (114), since $G(B)v(\beta_1)$ is rational. Thus, we have for $\mathfrak{f} \neq (1)$,

$$G(B)v(\beta_1) = \frac{1}{2}\left(u_B^2 - u_B + \frac{1}{6}\right)\frac{(a+d)}{c} + \sigma_1 + \sigma_2$$

where

$$\sigma_1 = \lim_{r\to 0}\left(\text{real part of } \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(VQ_1^{-u_B})^n Q_1^n + (V^{-1}Q_1^{u_B})^n}{1 - Q_1^n}\right),$$

$$\sigma_2 = \lim_{r\to 0}\left(\text{real part of } \frac{-1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(VQ_2^{\frac{1}{2}-u_B}) + (V^{-1}Q_2^{-\frac{1}{2}+u_B})^n}{Q_2^{-n/2} - Q_2^{n/2}}\right)$$

with $Q_1 = e^{2\pi i z}$ and $Q_2 = e^{2\pi i z^*}$.

As r tends to zero, Q_1 tends to zero exponentially and it is easy to see that $\sigma_1 = 0$ unless $u_B = 0$ and in this case

$$\sigma_1 = \sum_{n=1}^{\infty}\frac{V^{-n} - V^n}{4\pi i n}$$

So, if we define $\lambda(u_B)$ to be zero if $0 < u_B < 1$ and equal to 1 if $u_B = 0$, then 137

$$\sigma_1 = \lambda(u_B)\sum_{n=1}^{\infty}\frac{V^{-n} - V^n}{4\pi i n}.$$

In order to evaluate σ_2, we need the identity

$$\frac{q^c - q^{-c}}{q - q^{-1}} = \sum_{k=1}^{c} q^{c-2k+1} = \sum_{k=1}^{c} q^{-(c-2k+1)}, \quad (q \neq q^{-1})$$

i.e.

$$\frac{1}{q-q^{-1}} = \sum_{k=1}^{c} \frac{q^{c-2k+1}}{q^c - q^{-c}} = \sum_{k=1}^{c} \frac{q^{-(c-2k+1)}}{q^c - q^{-c}}. \tag{115}$$

Employing the identities (115) with $q = Q_2^{-n/2}$, we have

$$\sigma_2 = -\lim_{r\to 0}\left(\text{real part of } \sum_{k=1}^{c}\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(VQ_2^{k-c/2-u_B})^n + (VQ_2^{k-c/2-u_B})^{-n}}{Q_2^{-cn/2} - Q_2^{+cn/2}}\right).$$

Let now $t_k = 1-(2/c)(k-u_B)$ for $k = 1, 2, \ldots c$; clearly $-1 \leq t_k < 1$ for $0 \leq u_B < 1$. Further, let for $k = 1, \ldots, c$, $V_k = e^{2\pi i(v_B + (a/c)(k-u_B))}$ and let $P = e^{\pi r}$. It is clear that $Q_2^c = e^{2\pi i a - 2\pi r} = P^{-2}$ and P tends to 1 as r tends to zero. Now it can be verified that

$$\sigma_2 = -\lim_{P\to 1}\left(\text{real part of } \sum_{k=1}^{c}\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k P^{t_k})^n + (V_k P^{t_k})^{-n}}{P^n - P^{-n}}\right)$$

i.e.

$$\sigma_2 = -\lim_{P\to 1}\sum_{k=1}^{c}\sum_{n=1}^{\infty} \frac{1}{4\pi i n} \frac{(V_k^n - V_k^{-n})(P^{t_k n} - P^{-t_k n})}{P^n - P^- n}. \tag{116}$$

We know that $\lim_{P\to 1} \dfrac{P^{t_k n} - P^{-t_k n}}{P^n - P^{-n}} = t_k$ and if we can establish the uniform convergence of the inner series in (116) with respect to P for $1 \leq P < \infty$, then we can interchange the passage to the limit and the summations in (116). For this purpose, we remark that, for fixed k with $1 \leq k \leq c$, the function 138
$f_k(x) = \dfrac{x^{t_k} - x^{-t_k}}{x - x^{-1}}$ is a bounded monotone function of x for $1 \leq x < \infty$. In fact, it is monotone decreasing for $0 < t_k < 1$, identically zero for $t_k = 0$, monotone increasing for $-1 < t_k < 0$ and identically ± 1 for $t_k = \pm 1$. Moreover if $|t_k| < 1$, $f_k(x)$ tends to 0 as x tends to infinity and $f_k(x)$ tends to t_k as x tends to 1. Thus the sequence $\{f_k(P^n)\}$, $n = 1, 2, \ldots$ is a bounded monotone sequence for $1 \leq k \leq c$. Further the series $\sum\limits_{n=1}^{\infty} \dfrac{V_k^n - V_k^{-n}}{4\pi i n}$ is convergent. By Abel's criterion, the inner series in (116) converges uniformly with respect to P for $1 \leq P < \infty$. As a consequence, we have from (116),

$$\sigma_2 = \sum_{k=1}^{c}\left(\frac{k-u_B}{c} - \frac{1}{2}\right)\sum_{n=1}^{\infty} \frac{V_k^n - V_k^{-n}}{2\pi i n}$$

Let now for real x,

$$\mathscr{P}_1(x) = -\sum_{n=1}^{\infty} \frac{e^{2\pi i n x} - e^{-2\pi i n x}}{2\pi i n};$$

as is well-known, we have

$$\mathscr{P}_1 = \begin{cases} x - [x] - \dfrac{1}{2} & \text{for } x \text{ not integral,} \\ 0 & \text{for } x \text{ integral,} \end{cases}$$

where $[x]$ denotes the integral part of x. Further, let for real x,

$$\mathscr{P}_2(x) = -\sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi inx}}{(2\pi in)^2}.$$

Clearly $\mathscr{P}_2(x) = \frac{1}{2}\{(x-[x])^2 - (x-[x])\} + 1/12$. With this notation then, we have for $\mathfrak{f} \neq (1)$,

$$\begin{aligned} G(B)\nu(\beta_1) = {} & \mathscr{P}_2(u_B)\frac{(a+d)}{c} + \frac{1}{2}\lambda(u_B)\mathscr{P}_1(v_B) - \\ & - \sum_{k=1}^{c}\left(\frac{k-u_B}{c} - \frac{1}{2}\right)\mathscr{P}_1\left(a \cdot \frac{k-u_B}{c} + v_B\right). \end{aligned}$$

Now, for $1 \leq k \leq c$ and $0 \leq u_B \leq 1$, $0 < (k-u_B)/c < 1$ and $(k-u_B)/c = 1$ only when $k = c$ and $u_B = 0$. Therefore, for $1 \leq k < c$,

$$\mathscr{P}_1\left(\frac{k-u_B}{c}\right) = \frac{k-u_B}{c} - \frac{1}{2}$$

and for $k = c$, 139

$$\mathscr{P}_1\left(\frac{k-u_B}{c}\right) = \frac{k-u_B}{c} - \frac{1}{2} - \frac{1}{2}\lambda(u_B).$$

Further when $k = c$ and $u_B = 0$,

$$\mathscr{P}_1\left(a \cdot \frac{k-u_B}{c} + v_B\right) = \mathscr{P}_1(a + v_B) = \mathscr{P}_1(v_B).$$

Thus

$$\begin{aligned} G(B)\nu(\beta_1) = {} & \mathscr{P}_2(u_B)\frac{(a+d)}{c} + \frac{1}{2}\lambda(u_B)\mathscr{P}_1(v_B) - \\ & - \sum_{k=1}^{c}\mathscr{P}_1\left(\frac{k-u_B}{c}\right)\mathscr{P}_1\left(a \cdot \frac{k-u_B}{c} + v_B\right) - \frac{1}{2}\lambda(u_B)\mathscr{P}_1(v_B) \\ = {} & \mathscr{P}_2(u_B)\frac{(a+d)}{c} - \sum_{k=1}^{c}{}' \mathscr{P}_1\left(\frac{k-u_B}{c}\right)\mathscr{P}_1\left(a \cdot \frac{k-u_B}{c} + v_B\right). \end{aligned} \tag{117}$$

Since $\mathscr{P}_1(x)$ is an odd function of x,

$$\mathscr{P}_1\left(\frac{k-u_B}{c}\right) = -\mathscr{P}_1\left(\frac{-k+u_B}{c}\right)$$

and

$$\mathscr{P}_1\left(a\cdot\frac{k-u_B}{c}+v_B\right) = -\mathscr{P}_1\left(a\cdot\frac{-k+u_B}{c}-v_B\right).$$

Moreover, in the sum in (117), we can lek k run over any complete set of residues modulo c, in view of the periodicity of $\mathscr{P}_1(x)$. Thus

$$\begin{aligned} G(B)v(\beta_1) &= \mathscr{P}_2(u_B)\frac{(a+d)}{c} - \sum_{k=1}^{c}\mathscr{P}_1\left(\frac{-k+u_B}{c}\right)\mathscr{P}_1\left(a\cdot\frac{-k+u_B}{c}-v_B\right) \\ &= \mathscr{P}_2(u_B)\frac{(a+d)}{c} - \sum_{k=1}^{c}\mathscr{P}_1\left(\frac{k+u_B}{c}\right)\mathscr{P}_1\left(a\cdot\frac{k+u_B}{c}-v\right) \\ &= \mathscr{P}_2(u_B)\frac{(a+d)}{c} - \sum_{k=0}^{c-1}\mathscr{P}_1\left(\frac{k+u_B}{c}\right)\mathscr{P}_1\left(a\cdot\frac{k+u_B}{c}-v_B\right). \quad (118) \end{aligned}$$

140

It is surprising that even though, prima facie, it appears from (118) that $G(B)$ is a rational number with a high denominator, say $12c^2f^2$, the factor c^2 in the denominator drops out and eventually, $G(B)$ is a rational number with at most $12f$ in the denominator.

The determination of the asymptotic development of $\log\vartheta_{11}(v_B-u_Bz,z)$ as z tends to a/c is done in a rather more complicated manner by Hecke in his work referred to earlier.

We now take up the calculation of $G(B)$ in the case $\mathfrak{f}=(1)$. We set $z^*=a/c+ir/c$ and $z=-(d/c)+i/cr$ with $r>0$ and as before, $G(B)v(\beta_1)$ is

$$\lim_{r\to 0}\left(\text{real part of }\frac{1}{2\pi i}\left[\log\sqrt{(z-\omega)(z-\omega')}\eta^2(z)\right]_z^{z^*}\right).$$

We shall show first that

$$\lim_{r\to 0}\left(\text{real part of }\frac{1}{2\pi i}\left[\log\sqrt{(z-\omega)(z-\omega')}\right]_z^{z^*}\right) = -\frac{1}{4}.$$

For this purpose, we remark in the first place that $-(d/c)<\omega'<\omega<a/c$, since $\omega(a-c\omega)=1$ implies $a/c>\omega$ and $\epsilon(c\omega'+d)=1$ implies $-d/c<\omega'$. Now as $z=-(d/c)+i/cr$ and r tends to zero, arg $(z-\omega)(z-\omega')$ tends to π and similarly $\arg(z^* - \omega)(z^* - \omega')$ tends to zero as $z^* = a/c + ir$ and r tends to zero. Thus our assertion above is proved.

Defining the branch of $\log \eta^2(z)$ by

$$\log \eta^2(z) = \frac{\pi i z}{6} + 2 \sum_{m=1}^{\infty} \log(1 - e^{2\pi i m z}),$$

where on the right hand side we have taken the principal branches, we see that 141

$$\log \eta^2(z) = \frac{\pi i z}{6} - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i m n z}$$

and in view of absolute convergence of the series, we have once again,

$$\log \eta^2(z) = \frac{\pi i z}{6} - 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n}{1 - Q^n},$$

where $Q = e^{2\pi i z}$. Now

$$\left[\log \eta^2(Z)\right]_z^{z^*} = \frac{\pi i}{6}(z^* - z) - 2 \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n}{1 - Q^n}\right]_z^{z^*}$$

By our remarks above,

$$G(B)v(\beta_1) = -\frac{1}{4} + \frac{1}{12} \frac{(a+d)}{c} - \sigma_3,$$

where

$$\sigma_3 = 2 \cdot \lim_{r \to 0} \left(\text{real part of } \left[\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{Q^n}{1 - Q^n}\right]_z^{z^*}\right).$$

It is easily seen that

$$\lim_{r \to 0} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{Q_1^n}{1 - Q_1^n} = 0, \quad \text{for} \quad Q_1 = e^{2\pi i z}.$$

Thus

$$\frac{1}{2}\sigma_3 = \lim_{r \to 0} \left(\text{real part of } \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{Q_2^n}{1 - Q_2^n}\right) \text{ with } Q_2 = e^{2\pi i z^*}.$$

To determine σ_3, we first use the identity,

$$\frac{1}{q-q^{-1}} = \sum_{k=1}^{c} \frac{q^{c-2k+1}}{q^c - q^{-c}},$$

to make the denominators of the terms $Q_2^n/(1-Q_2^n)$ real. Setting $q = Q_2^{-n/2}$ in 142
this identity, we see that

$$\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{Q_2^n}{1-Q_2^n} = \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \sum_{k=1}^{r} \frac{Q_2^{-n/2(c-2k)}}{Q_2^{-nc/2} - Q_2^{nc/2}}.$$

Now $Q_2^{-nc/2} = (-1)^{an} P^n$ with $P = e^{\pi r}$ and $Q_2^k = V_k P^{-2k/c}$ where $V_k = e^{2\pi i (a/c)k}$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{Q_2^n}{1-Q_2^n} = \sum_{k=1}^{c} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{V_k^n P^{n(1-2k/c)}}{P^n - P^{-n}}.$$

As a consequence,

$$\sigma_3 = \lim_{r\to 0} \left[\sum_{k=1}^{c} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k^n - V_k^{-n}) P^{t_k n}}{P^n - P^{-n}} \right], \tag{119}$$

where $t_k = 1 - 2k/c$. In (119), the sum $\sum\limits_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k^n - V_k^{-n}) P^{t_k n}}{P^n - P^{-n}}$ corresponding to $k = c$ is zero since $V_c^n - V_c^{-n} = 0$. Hence effectively k runs from 1 to $c-1$; but then $c-k$ also does the same when k does so. On the other hand, if k is replaced by $c-k$, then V_k goes to V_k^{-1} and t_k to $-t_k$ so that

$$\sum_{k=1}^{c} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k^n - V_k^{-n}) P^{t_k n}}{P^n - P^{-n}} = -\sum_{k=1}^{c} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k^n - V_k^{-n}) P^{-t_k n}}{P^n - P^{-n}}$$

Taking the arithmetic mean, we see that

$$\sigma_3 = \frac{1}{2} \lim_{r\to 0} \sum_{k=1}^{c} \sum_{n=1}^{\infty} \frac{1}{2\pi i n} \frac{(V_k^n - V_k^{-n})(P^{t_k n} - P^{-t_k n})}{P^n - P^{-n}}$$

We are now in the same situation as in (116) but with u_B and v_B replaced by 0. By the same analysis as in the former case, we can show that

$$\sigma_3 = \sum_{k=1}^{c} \mathscr{P}_1\left(\frac{k}{c}\right) \mathscr{P}_1\left(a \cdot \frac{k}{c}\right) = \sum_{k=0}^{c-1} \mathscr{P}_1\left(\frac{k}{c}\right) \mathscr{P}_1\left(a \cdot \frac{k}{c}\right).$$

Thus, for $\mathfrak{f} = (1)$,

$$G(B)\nu(\beta_1) = -\frac{1}{4} + \mathscr{P}_2(0)\frac{(a+d)}{c} - \sum_{k=0}^{c-1} \mathscr{P}_1\left(\frac{k}{c}\right)\mathscr{P}_1\left(a\cdot\frac{k}{c}\right). \tag{120}$$

We now define 143

$$\nu(\mathfrak{f}) = \begin{cases} \frac{1}{4}, & \text{if } \mathfrak{f} = (1) \\ 0, & \text{otherwise.} \end{cases}$$

Consolidating (118) and (120) into a single formula, we see that the value of $G(B)$ is given explicitly in terms of elementary arithmetical functions by

Theorem 13. *Corresponding to a proper ray class character χ modulo $\mathfrak{f}$ in $\mathbf{Q}(\sqrt{D})(D>0)$, with associated $\nu(\lambda)$ defined by* (104) *and a ray class B modulo $\mathfrak{f}$, we have the formula*

$$G(B) = \nu(\beta_1)\left\{\mathscr{P}_2(u_B)\frac{a+d}{c} - \sum_{k=0}^{c-1}\mathscr{P}_1\left(\frac{k+u_B}{c}\right)\mathscr{P}_1\left(a\frac{k+u_B}{c} - v_B\right) - \nu(\mathfrak{f})\right\}, \tag{121}$$

where $u_B = v_B = 0$ for $\mathfrak{f} = (1)$.

Note. Here $[\beta_1, \beta_2]$ is an integral basis of an integral ideal $\mathfrak{b}_B$ in B and $u_B = S(\beta_1\gamma)$, $v_B = S(\beta_2\gamma)$. We recall that $G(B)$ depends only on B and *not* on the special choice of $\mathfrak{b}_B$ or of its integral basis. The rational integers a, b, c, d are determined by $\epsilon\beta_2 = a\beta_2 + b\beta_1$, $\epsilon\beta_1 = c\beta_2 + d\beta_1$. The Riemann-Dedekind method used above for the explicit determination of $G(B)$ does not make any specific use of the fact that $z \to z^*$ is a hyperbolic substitution.

Coming back to our formula for $L(1,\chi)$ for a proper ray class character χ modulo $\mathfrak{f}$ in $\mathbf{Q}(\sqrt{D})$ for which the associated $\nu(\lambda)$ is defined by (104), we have as a consequence of Theorems 12 and 13,

$$L(1,\chi) = \frac{\pi^2}{T\sqrt{D}}\Lambda, \tag{122}$$

where $\Lambda = \sum\limits_{B} \overline{\chi}(B)G(B)$ with B running over all ray classes modulo $\mathfrak{f}$. $G(B)$ is given by (121), and

$$T = \sum_{\lambda \bmod \mathfrak{f}} \overline{\chi}(\lambda)e^{2\pi i S(\lambda\gamma)}$$

with $\gamma \in K$ such that $(\gamma\sqrt{D})$ has exact denominator $\mathfrak{f}$. 144

We shall now use the elementary formula for $L(1,\chi)$ derived above, in order to determine the class number of special abelian extensions of the real quadratic field $\mathbf{Q}(\sqrt{D})$.

Let F be a relative abelian extension of degree n over $K = \mathbf{Q}(\sqrt{D})$ and let $\mathfrak{f}$ be the 'Führer' (conductor) of F relative to K. Let $\zeta(s,K)$ and $\zeta(s,F)$ be the Dedekind zeta functions of K and F respectively. From class field theory, we know that the galois group of F over K is isomorphic to a subgroup of the character group of the group of ray classes modulo $\mathfrak{f}$ in K. Moreover, the law of splitting in F of prime ideals of K is equivalent to the following decomposition of $\zeta(s,F)$, namely

$$\zeta(s,F) = \zeta(s,K)\prod_{\chi\neq 1} L(s,\chi) \tag{123}$$

where χ runs over a complete set of $n-1$ non-principal ray class characters modulo $\mathfrak{f}_\chi$ with conductor $\mathfrak{f}_\chi$ dividing $\mathfrak{f}$ and $L(s,\chi)$ is the L-series in K associated with χ. Multiplying both sides of (123) by $s-1$ and letting s tend to 1, we have in analogy with Theorem 10,

$$\frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot R\cdot H}{W\sqrt{|\Delta|}} = \frac{4rh}{w\sqrt{D}}\prod_{\chi\neq 1} L(1,\chi). \tag{124}$$

In (124), r_1 and $2r_2$ are respectively the number of real and complex conjugates of F over $\mathbf{Q}$, H is the class number of F, W is the number of roots of unity in F, R is the regulator of F, Δ is the discriminant of F over $\mathbf{Q}$, h is the class number of K, r is the regulator of K and $w(=2)$ the number of roots of unity in K.

We have been able to obtain a formula for $L(1,\chi)$ involving elementary arithmetical functions only in the case when the character of signature $v(\lambda)$ associated with χ is defined by (104). Thus, if we are to obtain for H, a formula involving purely elementary arithmetical functions, then we ought to consider only such abelian extensions F over K, for which, on the right hand side of (124), there occur in the product only characters χ whose associated $v(\lambda)$ is given by (104). Moreover, suppose that χ_1 and χ_2 are two such distinct characters occuring in that product, then $\chi_1\chi_2^{-1}$ also occurs in the product and its associated $v(\lambda)$ is defined by $v(\lambda) = 1$ for all $\lambda \neq 0$ in K. But this, again, is **145**
a situation which we should avoid, in view of our aim. Thus, in order that we could obtain a formula of elementary type for the class number H of F, we are obliged to consider only those abelian extensions F over K, for which we have the decomposition $\zeta(s,F) = \zeta(s,K)L(s,\chi)$ where χ is a ray class character modulo $\mathfrak{f}_\chi$ with $\mathfrak{f}_\chi$ as conductor and its associated character of signature is given by (104). This, in the first place, means that F is a quadratic extension of K, i.e. $F = \mathbf{Q}(\sqrt{D},\sqrt{\theta})$ where θ is a non-square number in K. Let ϑ be the

relative discriminant of F over K. We could assume that $(\theta) = \vartheta i^2$ for an ideal i in K, coprime to ϑ.

From class field theory, we know that a prime ideal $\mathfrak{p}$ in K not dividing $\mathfrak{f}$, the conductor of F relative to K, either splits into a product of distinct prime factors or stays prime in F and moreover

$$\chi(\mathfrak{p}) = \begin{cases} 1, & \text{if } \mathfrak{p} \text{ splits in } F \\ -1, & \text{if } \mathfrak{p} \text{ stays prime in } F. \end{cases} \tag{125}$$

We also know that $\mathfrak{f}_\chi$ divides $\mathfrak{f}$.

On the other hand, let $\mathfrak{p}$ be a prime ideal in K not dividing ϑ. We can then find $c \in K$ such that $\theta^* = \theta c^2$ is prime to $\mathfrak{p}$. Let us define then

$$\psi(\mathfrak{p}) = \left(\frac{\theta^*}{\mathfrak{p}}\right),$$

where, for $\mathfrak{p} \nmid (2)$, $\left(\frac{\theta^*}{\mathfrak{p}}\right)$ is the quadratic residue symbol in K and if $\mathfrak{p}^a$ is the highest power of $\mathfrak{p}$ dividing (2), then $\left(\frac{\theta^*}{\mathfrak{p}}\right) = +1$ or -1, according as the congruence $\theta^* \equiv \xi^2 (\mod \mathfrak{p}^{2a+1})$ is solvable in K or not. We see that $\psi(\mathfrak{p})$ is unambiguously defined for all prime ideals $\mathfrak{p}$ not dividing ϑ and we extend ψ multiplicatively to all ideals $\mathfrak{a}$ in the ray classes modulo ϑ in K.

From the theory of relative quadratic extensions of algebraic number fields, it follows that for prime ideals $\mathfrak{p}$ in K not dividing ϑ, $\chi(\mathfrak{p}) = \psi(\mathfrak{p})$ and hence $\chi(\mathfrak{a})$ and $\psi(\mathfrak{a})$ coincide on the ideals in the ray classes modulo ϑ.

Now, by the law of quadratic reciprocity in K, it can be shown that for all numbers α in K for which $\alpha \equiv 1(\mod {}^*\vartheta)$ and $\alpha > 0$, $\psi((\alpha)) = 1$. In other words, ψ, and hence χ, is a ray class character modulo ϑ. Incidentally 146
we note that $\mathfrak{f}_\chi$ divides ϑ, since χ is a ray class character modulo $\mathfrak{f}_\chi$, with $\mathfrak{f}_\chi$ as conductor. Again, using the law of quadratic reciprocity, we can show that for integral $\lambda \neq 0$ and coprime to ϑ, $\chi((\lambda)) = \psi((\lambda)) = \psi(\lambda)v(\lambda)$ where $\psi(\lambda)$ is a prime residue class character modulo ϑ and $v(\lambda)$ is given by

$$v(\lambda) = \begin{cases} 1, \text{ if } \theta > 0 \\ \dfrac{\lambda}{|\lambda|}, \text{ if } \theta > 0, \theta' < 0 \\ \dfrac{\lambda'}{|\lambda'|}, \text{ if } \theta < 0, \theta' > 0 \\ \dfrac{N(\lambda)}{|N(\lambda)|}, \text{ if } -\theta > 0. \end{cases}$$

But, for our purposes, we require $\chi((\lambda))$ to be of the form $\psi(\lambda)\cdot\dfrac{N(\lambda)}{|N(\lambda)|}$. Thus, finally, in order that we could calculate H in terms of elementary arithmetical functions, we conclude that F should be of the form $K(\sqrt{\theta})$ where $\theta \in K$ and $-\theta > 0$. In other words, F is a biquadratic field over $\mathbf{Q}$, which is realized as an imaginary quadratic extension of the real quadratic field $\mathbf{Q}(\sqrt{D})$. We have in this case, the following formula for the class number H of F, viz.

$$\begin{aligned}\frac{4\pi^2\cdot R}{W\sqrt{|\Delta|}}H &= \frac{4rh}{w\sqrt{D}}L(1,\chi)\\ &= \frac{4rh}{w\sqrt{D}}\frac{\pi^2}{T\sqrt{D}}\Lambda.\end{aligned} \tag{126}$$

In order to determine the value of T explicitly, we use the functional equations of $\zeta(s,F)$, $L(s,\chi)$ and $\zeta_D(s) = \zeta(s,\mathbf{Q}(\sqrt{D}))$, viz.

$$\begin{gathered}(2\pi)^{-2s}\Gamma^2(s)(|\Delta|)^{s/2}\zeta(s,F) = (2\pi)^{-2(1-s)}\Gamma^2(1-s)\times(|\Delta|)^{(1-s)/2}\zeta(1-s,F),\\ \pi^{-s}\Gamma^2(s/2)D^{s/2}\zeta_D(s) = \pi^{-(1-s)}\Gamma^2\left(\frac{1-s}{2}\right)D^{(1-s)/2}\zeta_D(1-s)\end{gathered} \tag{127}$$

$$\begin{aligned}&\pi^{-s}\Gamma^2((s+1)/2)(DN(\mathfrak{f}_\chi))^{s/2}L(s,\chi) =\\ &= \pi^{-(1-s)}\Gamma^2\left(\frac{1-s}{2}\right)(DN(\mathfrak{f}_\chi))^{(1-s)/2}L(1-s,\overline{\chi})\times\\ &\quad\frac{\chi(\mathfrak{q})\nu(\gamma\sqrt{D})\sqrt{N(\mathfrak{f}_\chi)}}{T},\end{aligned}$$

where $\gamma \in K$ such that $(\gamma\sqrt{D}) = \mathfrak{q}/\mathfrak{f}_\chi$ has exact denominator $\mathfrak{f}_\chi$. Further **147**
$\zeta(s,F) = \zeta_D(s)L(s,\chi)$ and $\Gamma(s/2)\Gamma((s+1)/2) = \pi^{\frac{1}{2}}2^{1-s}\Gamma(s)$. These, together with (127), give us

$$\left(\frac{D^2N(\mathfrak{f}_\chi)}{|\Delta|}\right)^{s-\frac{1}{2}} = \frac{\chi(\mathfrak{q})\nu(\gamma\sqrt{D})\sqrt{N(\mathfrak{f}_\chi)}}{T}. \tag{128}$$

Since the right hand side is independent of s, we see by setting $s = 1/2$, that $T = \sqrt{N(\mathfrak{f}_\chi)}\chi(\mathfrak{q})\nu(\gamma\sqrt{d})$. Since the right-hand side of (128) is 1, we see again by setting $s = 3/2$ in (128), that $D^2N(\mathfrak{f}_\chi) = |\Delta|$. But we know that $D^2N(\vartheta) = |\Delta|$. This means that $N(\vartheta) = N(\mathfrak{f}_\chi)$ and since $\mathfrak{f}_\chi$ divides ϑ, we obtain incidentally that $\mathfrak{f}_\chi = \vartheta$. Thus we have

$$T = \sqrt{N(\vartheta)}\chi(\mathfrak{q})\nu(\gamma\sqrt{D}).$$

But

$$(\gamma\sqrt{D}) = \frac{\mathfrak{q}}{\mathfrak{f}_\chi} = \frac{\mathfrak{q}}{\vartheta} = \frac{\mathfrak{q}\mathfrak{i}^2}{\vartheta\mathfrak{i}^2} = \frac{(\kappa)}{(\theta)}$$

where $\kappa = \theta\gamma\sqrt{D}$. Moreover, since χ is a real character, $\chi(\mathfrak{i}^2) = 1$. Hence $\chi(\mathfrak{q}) = \chi(\mathfrak{q}\mathfrak{i}^2) = \chi(\kappa)\nu(\kappa) = \chi(\kappa)\nu(\theta\gamma\sqrt{D}) = \nu(\gamma\sqrt{D})\chi(\kappa)$. Thus $T = \sqrt{N(\vartheta)}\chi(\kappa)$. As a result, from (126), we obtain

$$\begin{aligned}\frac{H}{h}\frac{R}{r}\frac{w}{W} &= \chi(\kappa)\Lambda \\ &= \chi(\theta\gamma\sqrt{D})\sum_B \overline{\chi}(B)G(B). \qquad (129)\end{aligned}$$

We may summarise the above in

Proposition 17. *Let $F = \mathbf{Q}(\sqrt{D}, \sqrt{\theta})$ be an imaginary quadratic extension of $K = \mathbf{Q}(\sqrt{D})$, $D > 0$, with an integral ideal $\mathfrak{f}$ in K for its "conductor", θ being a totally negative number in K. Then with our earlier notation, the class number H of F is given by*

$$H = h\frac{r}{R}\frac{W}{w}\chi(\theta\gamma\sqrt{D})\sum_B \overline{\chi}(B)G(B),$$

where B runs over all the ray classes modulo $\mathfrak{f}$ in K, $G(B)$ is given by (121) *and* 148
χ is the non-principal ray class character modulo $\mathfrak{f}$, associated with F.

Remark. We know that $G(B)$ are rational numbers with at most $12f$ (where f is the smallest positive rational integer divisible by $\mathfrak{f}_\chi = \mathfrak{f} = \vartheta$) in the denominator, it follows that $12f\frac{R}{r}\frac{w}{W}\frac{H}{h}$ is a rational integer.

Let η and μ be respectively the fundamental units in F and $\mathbf{Q}(\sqrt{D})$. Then $R = 2\log|\eta|$ and $r = \log|\mu|$. Now either $\mu = \eta$ or $\mu = \pm\eta^2$. In the former case $R = 2r$; in the latter case, F may be generated over $\mathbf{Q}(\sqrt{D})$ by adjoining the square root of $\mp\mu$ (whichever is totally negative) to $\mathbf{Q}(\sqrt{D})$ and in this case, we have $R = r$. Thus, in any case, since w divides W, we see that $24f(H/h)$ is a rational integer.

If h is coprime to $24f$, we obtain the interesting result that h divides H.

Let us now assume that F *is an unramified imaginary quadratic extension of* $\mathbf{Q}(\sqrt{D})$. Then $\vartheta = (1)$ and hence the ray class group modulo ϑ is just the narrow class group of $\mathbf{Q}(\sqrt{D})$ and χ is a genus character. It is known that all totally complex unramified quadratic extensions of $\mathbf{Q}(\sqrt{D})$ are of the form $\mathbf{Q}(\sqrt{D_1}, \sqrt{D_2})$ where D_1 and D_2 are coprime negative discriminants (of quadratic fields over **Q**) which satisfy $D_1D_2 = D$. Hence F is obtained from $\mathbf{Q}(\sqrt{D})$ by adjoining either $\sqrt{D_1}$ or $\sqrt{D_2}$, for such a decomposition of D as product of two coprime negative discriminants.

On the other hand, since $F = \mathbf{Q}(\sqrt{D_1}, \sqrt{D_2})$ is an abelian extension of **Q** with galois group of order 4, we have the decomposition

$$\zeta(s, F) = \zeta(s)L_{D_1}(s)L_{D_2}(s)L_D(s),$$

where $L_{D_1}(s)$, $L_{D_2}(s)$ and $L_D(s)$ are the Dirichlet L-series in **Q** associated with the Legendre-Jacobi-Kronecker symbols $\left(\frac{D_1}{n}\right)$, $\left(\frac{D_2}{n}\right)$, and $\left(\frac{D}{n}\right)$ respectively. Moreover

$$\zeta_D(s) = \zeta(s)L_D(s)$$

and hence 149

$$L(s, \chi) = L_{D_1}(s)L_{D_2}(s),$$

a result due to Kronecker, which we have met already (§1, Chapter II), χ being a genus character in $\mathbf{Q}(\sqrt{D})$. Therefore

$$L(1, \chi) = L_{D_1}(1)L_{D_2}(1).$$

On the other hand, if h_1 and h_2 are the class numbers of $\mathbf{Q}(\sqrt{D_1})$ and $\mathbf{Q}(\sqrt{D_2})$ respectively and w_1 and w_2 the respective number of roots of unity in $\mathbf{Q}(\sqrt{D_1})$ and $\mathbf{Q}(\sqrt{D_2})$, then

$$L_{D_1}(1) = \frac{2\pi h_1}{w_1\sqrt{|D_1|}}, \quad L_{D_2}(1) = \frac{2\pi h_2}{w_2\sqrt{|D_2|}}.$$

Hence

$$\frac{\pi^2}{\sqrt{D}}\Lambda = L(1, \chi) = \frac{4\pi^2 h_1 h_2}{w_1 w_2\sqrt{|D_1|}\sqrt{|D_2|}}$$

and as a consequence, we have

Proposition 18. *If $D_1 < 0$, $D_2 < 0$, $D = D_1D_2$ are discriminants of quadratic fields over* **Q** *and if h_1, h_2 are the class numbers of and w_1, w_2 the number of roots of unity in* $\mathbf{Q}(\sqrt{D_1})$, $\mathbf{Q}(\sqrt{D_2})$ *respectively, then we have*

$$\frac{4h_1h_2}{w_1w_2} = \Lambda = \sum_B \overline{\chi}(B)G(B). \tag{130}$$

(In (130), *χ is the genus character in* $\mathbf{Q}(\sqrt{D})$ *corresponding to the decomposition $D = D_1D_2$ and B runs over all the narrow ideal classes in* $\mathbf{Q}(\sqrt{D})$*).*

Remark. Formula (130) is interesting in that it gives another arithmetical significance for Λ. Besides, it is the analogue of Kronecker's solution of Pell's equation which we referred to in § 3, Chapter II earlier. It has not appeared in literature so far.

From (129) and (130), we have

$$H = \frac{4hh_1h_2W}{w_1w_2w}\frac{r}{R}. \tag{131}$$

If we exclude the special cases when $-4 \leq D_1, D_2 < 0$, then $w_1 = w_2 = 2 = w$. 150
Moreover, r/R is 1 or $\frac{1}{2}$, as we have seen earlier. Further if we exclude F from being the cyclotomic fields of the 8th or 10th or 12th roots of unity or the fields $\mathbf{Q}(\sqrt{D}, \sqrt{-1})$ or $\mathbf{Q}(\sqrt{D}, \sqrt{-3})$, then $W = 2$. Thus barring these special cases, we have for the class number H of F the formula

$$H = h\frac{r}{R}\Lambda$$

and from (131)

$$H = \frac{r}{R}hh_1h_2. \tag{132}$$

In general, $R/r = 2$ and hence we have, except for some special cases,

$$2\frac{H}{h} = \Lambda = h_1h_2.$$

For $D_1 = -4$, formula (131) was obtained by Dirichlet and in the general case, this was discovered by Hilbert. A generalization of (132) has been considered by Herglotz. One breaks up D as the product of $r(\geq 2)$ mutually coprime discriminants. The field $K = \mathbf{Q}(\sqrt{D_1}, \ldots, \sqrt{D_r})$ is an unramified abelian extension of $\mathbf{Q}(\sqrt{D})$ and for the Dedekind zeta function of F, we have the decomposition as a product of L-series for the corresponding genus-characters in $\mathbf{Q}(\sqrt{D})$. One then proceeds as above to obtain the required generalization of (132), involving the class numbers $h_1, \ldots h_r$ of $\mathbf{Q}(\sqrt{D_1}), \ldots \mathbf{Q}(\sqrt{D_r})$ respectively.

6 Some Examples

This section is devoted to giving a few interesting examples pertaining to the determination of the class number of totally complex biquadratic extensions of $\mathbf{Q}$ with particular reference to Proposition 17 and 18.

Examples 1-6 deal with the case when F is a totally complex number field 151 which is an unramified imaginary quadratic extension of $\mathbf{Q}(\sqrt{D})$, $D > 0(\mathfrak{f} = (1))$. In fact, then $F = \mathbf{Q}(\sqrt{D_1}, \sqrt{D_2})$ where $D = D_1 D_2$ is a decomposition of D as product of coprime negative discriminants D_1, D_2. Formula (129) gives the class number H of F, on computing R, r, h, w, W and $G(B)$. Incidentally we also check (130), by finding h_1, h_2, w_1, w_2.

On the other hand we know that for any algebraic number field, the class number can be found directly, as follows. Since, in every class of ideals, there exists an integral ideal of norm not exceeding $\sqrt{|\Delta|}$ (Δ being the absolute discriminant), it suffices to test for mutual equivalence, the integral ideals of norm not exceeding $\sqrt{|\Delta|}$ and obtain a maximal set of inequivalent ideals from among these. This will give us the class number.

Our notation here will be the same as in the last section.

The following remark is to effect a simplification of the computation, in the case of Examples 1-6. The ray classes modulo $\mathfrak{f}(= (1))$ in $\mathbf{Q}(\sqrt{D})$ are just the narrow ideal classes. Looking at the definition of $G(B)$, we see that for two narrow classes B_1 and B_2 lying in the same wide class, $G(B_1) = -G(B_2)$ and further $\chi(B_1) = -\chi(B_2)$ so that $\chi(B_1)G(B_1) = \chi(B_2)G(B_2)$. Thus

$$\Lambda = 2\sum_{A} \overline{\chi}(B)G(B) = 2\Lambda^*(\text{say})$$

where A runs over the wide classes in $\mathbf{Q}(\sqrt{D})$ and B is a narrow class contained in A. From (129), we have

$$H = 2h \cdot \frac{r}{R} \cdot \frac{W}{w} \cdot \Lambda^*.$$

Further, formula (130) becomes

$$\Lambda^* = \frac{2h_1 h_2}{w_1 w_2}.$$

As before, $\epsilon(> 1)$ is the generator of the group $\Gamma^*_{\mathfrak{f}}$ in $\mathbf{Q}(\sqrt{D})$.

Example 1. $D = 12$, $D_1 = -3$, $D_2 = -4$

$$h = 1, \text{ since } (2) = (1+\sqrt{3})(1-\sqrt{3}), (3) = (\sqrt{3})^2.$$

Similarly $h_1 = 1$, $h_2 = 1$. Further $w_1 = 6$, $w_2 = 4$, $w = 2$, $\epsilon = 2 + \sqrt{3} > 0$, 152
$W = 12$. Since $\epsilon = i\left(\frac{1+\sqrt{3}}{1+i}\right)^2$, $r/R = 1$.

Since $h = 1$, $\Lambda^* = G(B_1)$, B_1 being the principal narrow class in $\mathbf{Q}(\sqrt{12})$. We take $\mathfrak{b}_{B_1} = (1)$; $\mathfrak{b}_{B_1} = [1, \sqrt{3}], \beta_1 = 1, \beta_2 = \sqrt{3}$, $v(\beta_1) = 1$.

$$(2+\sqrt{3})\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \quad a = 2, b = 3, c = 1, d = 2.$$

$$G(B_1) = \frac{1}{12}\cdot\frac{2+2}{1} - \frac{1}{4} = \frac{1}{12}.$$

Thus

$$H = 2.1.1\cdot\frac{12}{2}\cdot\frac{1}{12} = 1.$$

Further

$$\frac{2h_1h_2}{w_1w_2} = \frac{2.1.1}{6.4} = \frac{1}{12} = \Lambda^*.$$

Example 2. $D = 24$, $D_1 = -3$, $D_2 = -8$, $h = h_1 = h_2 = 1$, $w = 2$, $w_1 = 6$, $w_2 = 2$, $W = 6$, $\epsilon = 5 + 2\sqrt{6} > 0$.

Since $-\epsilon = (\sqrt{-3} + \sqrt{-2})^2$, $r/R = 1$. Also $\Lambda^* = G(B_1)$, B_1 being the principal narrow class in $\mathbf{Q}(\sqrt{24})$; $\mathfrak{b}_{B_1} = (1) = [1, \sqrt{6}]$, $\beta_1 = 1$, $\beta_2 = \sqrt{6}$, $v(\beta_1) = 1$.

$$\epsilon\begin{pmatrix}\sqrt{6}\\1\end{pmatrix} = \begin{pmatrix}5 & 12\\2 & 5\end{pmatrix}\begin{pmatrix}\sqrt{6}\\1\end{pmatrix}, \quad a = 5, b = 12, c = 2, d = 5,$$

$$\Lambda^* = G(B_1) = \frac{1}{12}\cdot\frac{5+5}{2} - \frac{1}{4} = \frac{1}{6},$$

$$H = 2.1\cdot\frac{6}{2}\cdot\frac{1}{6} - 1,$$

$$\frac{2h_1h_2}{w_1w_2} = \frac{2.1.1}{6.2} = \Lambda^*.$$

Example 3. $D = 140$, $D_1 = -4$, $D_2 = -35$, $h_1 = 1$, $w_1 = 4$, $h_2 = 2$, $w_2 = 2$, **153**
$h = 2$, $w = 2$, $\epsilon = 6 + \sqrt{35} > 0$, $W = 4$, $r/R = 1/2$. We take $\mathfrak{p}_1 = (1)$, $\mathfrak{p}_2 = (2, 1 + \sqrt{35})$ as representatives of the two wide classes in $\mathbf{Q}(\sqrt{35})$ and denote the ray classes containing them by B_1, B_2 respectively. Since $\mathfrak{p}_2^2 = (2)$, $N(\mathfrak{p}_2) = 2$ and $\chi(B_2) = \left(\frac{-35}{2}\right) = -1$.

For B_1, $\mathfrak{b}_{B_1} = \mathfrak{p}_1, \beta_1 = 2, \beta_2 = \sqrt{35}$, $v(\beta_1) = 1$, $a = 6$, $b = 35$, $c = 1$, $d = 6$, $G(B_1) = 1/12\cdot(6+6)/1 - 1/4 = 3/4$.

For B_2, $\mathfrak{b}_{B_2} = \mathfrak{p}_2, \beta_1 = 2, \beta_2 = 1 + \sqrt{35}$, $v(\beta_1) = 1$, $\omega = (1+\sqrt{35})/2$,

$$\epsilon = 2\omega + 5, \epsilon\omega = 7\omega + 17, a = 7, b = 17, c = 2, d = 5,$$

$$G(B_2) = \frac{1}{12}\cdot\frac{7+5}{2} - \frac{1}{4} = \frac{1}{4},$$

$$\Lambda^* = \sum_A \overline{\chi}(B)G(B) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2},$$

$$H = 2.2\cdot\frac{1}{2}\cdot\frac{4}{2}\cdot\frac{1}{2} = 2,$$

$$\frac{2h_1h_2}{w_1w_2} = \frac{2.1.2}{4.2} = \Lambda^*.$$

Example 4. $D = 140$, $D_1 = -7$, $D_2 = -20$, $h_1 = 1$, $w_1 = 2$, $h_2 = 2$, $w_2 = 2$, $h = 2$, $w = 2$, $W = 2$, $r/R = \frac{1}{2}$, $\epsilon = 6 + \sqrt{35} > 0$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$, B_1, B_2 have the same significance as in Example 3. Then $\chi(\mathfrak{p}_2) = \left(\frac{-7}{2}\right) = 1$, $G(B_1) = 3/4$, $G(B_2) = 1/4$, $\Lambda^* = 3/4 + 1/4 = 1$,

$$H = 2.2\cdot\frac{1}{2}\cdot 1.1 = 2,$$

$$\frac{2h_1h_2}{w_1w_2} = \frac{2.1.2}{2.2} = \Lambda^*.$$

Example 5. $D = 21$, $D_1 = -3$, $D_2 = -7$, $h_1 = 1$, $w_1 = 6$, $h_2 = 1$, $w_2 = 2$, $h = 1$, $w = 2$, $W = 6$, $\epsilon = (5 + \sqrt{21})/2 > 0$; $r/R = 1$, since $-\epsilon = ((\sqrt{-3} + \sqrt{-7})/2)^2$.

For the principal narrow class B_1 in $\mathbf{Q}(\sqrt{21})$, we take $\mathfrak{b}_{B_1} = (1)$ with integral basis $[1, \epsilon]$; $\beta_1 = 1$, $\beta_2 = \epsilon$, $\omega = \epsilon$, $\epsilon\omega = 5\omega - 1$, $a = 5$, $b = -1$, $c = 1$, $d = 0$.

$$\Lambda^* = G(B_1) = \frac{1}{12}\cdot\frac{5}{1} - \frac{1}{4} = \frac{1}{6},$$

$$H = 2.1.1\cdot\frac{6}{2}\cdot\frac{1}{6} = 1,$$

$$\frac{2h_1h_2}{w_1w_2} = \frac{2.1.1}{6.2} = \Lambda^*.$$

154

We might show that $H = 1$, directly. The discriminant of F is $(21)^2$ and we have therefore to examine the splitting of (2), (3), (5), (7), (11), (13), (17), (19) in F. The ideals (2), (5), (11) and (17) stay prime in $\mathbf{Q}(\sqrt{3})$ and (3), (13), (19) stay prime in $\mathbf{Q}(\sqrt{-7})$. Further $(7) = (\sqrt{7})^2$ in $\mathbf{Q}(\sqrt{-7})$. The further splitting in F is seen to be as follows:

$$(2) = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right),$$

$$
\begin{aligned}
(3) &= (\sqrt{-3})^2,\\
(5) &= (2\sqrt{-3} + \sqrt{-7})(2\sqrt{-3} - \sqrt{-7}),\\
(7) &= (1+\rho)^2(1-\rho^{-1})^{-2} \text{ with } \rho = \frac{\sqrt{-3}+\sqrt{-7}}{2},\\
(11) &= (2+\sqrt{-7})(2-\sqrt{-7}),\\
(13) &= (1+2\sqrt{-3})(1-2\sqrt{-3}),\\
(17) &= \left(\frac{5\sqrt{-3}+\sqrt{-7}}{2}\right)\left(\frac{5\sqrt{-3}-\sqrt{-7}}{2}\right),\\
(19) &= (4+\sqrt{-3})(4-\sqrt{-3}).
\end{aligned}
$$

Thus all prime ideals of norm ≤ 21 are principal in F and hence $H = 1$.

Example 6. $D = 2021$, $D_1 = -43$, $D_2 = -47$, $h_1 = 1$, $w_1 = 2$.

To find h_2, we observe that $\left(\frac{-47}{5}\right) = -1$ and so (5) stays prime in $\mathbf{Q}(\sqrt{-47})$, while $(2) = \mathfrak{p}_2\mathfrak{p}'_2$, $(3) = \mathfrak{p}_3\mathfrak{p}'_3$ where

$$\mathfrak{p}_2 = \left(2, \frac{1+\sqrt{-47}}{2}\right), \quad \mathfrak{p}_3 = \left(3, \frac{1\pm\sqrt{47}}{2}\right).$$

The integral ideals of norm $\leq \sqrt{|47|}$ in $\mathbf{Q}(\sqrt{47})$ are $\mathfrak{p}_2$, $\mathfrak{p}'_2$, $\mathfrak{p}_3$, $\mathfrak{p}'_3$, $\mathfrak{p}_2^2$, $\mathfrak{p}'^2_2$, $\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}'_2\mathfrak{p}_3$, $\mathfrak{p}_2\mathfrak{p}'_3$, $\mathfrak{p}'_2\mathfrak{p}'_3$. Now

$$N\left(\frac{1+\sqrt{-47}}{2}\right) = 12 = \mathfrak{p}_2^2\mathfrak{p}'^2_2\mathfrak{p}_3\mathfrak{p}'_3.$$

Let $\mathfrak{p}_3$ be so chosen that 155

$$\mathfrak{p}_2^2\mathfrak{p}'_3 = \left(\frac{1\pm\sqrt{-47}}{2}\right).$$

Then $\mathfrak{p}_2^2 \sim \mathfrak{p}_3$ (equivalent in the wide sense). Let now

$$\mathfrak{p}_2^k = \left(\frac{x+y\sqrt{-47}}{2}\right),$$

k being the order of the wide class containing $\mathfrak{p}_2$ and x, y being rational integers. Then $x^2 + 47y^2 = 2^{k+2}$ and certainly $y \neq 0$, for then we would have $\mathfrak{p}_2^k = \mathfrak{p}'^k_2$

i.e. $\mathfrak{p}_2 = \mathfrak{p}_2'$, which is not true. But now the smallest $k > 0$ for which the diophantine equation $x^2 + 47y^2 = 2^{k+2}$ is solvable with $y \neq 0$ is $k = 5$ (in fact, $9^2 + 47 = 128$). Hence $\mathfrak{p}_2^5 = \left(\frac{9 + \sqrt{-47}}{2}\right)$ and $\mathfrak{p}_2^4 \nsim (1)$. Since $\mathfrak{p}_2^2 \sim \mathfrak{p}_3$, $\mathfrak{p}'^2_2 \sim \mathfrak{p}_3'$, $\mathfrak{p}_2\mathfrak{p}_3 \sim \mathfrak{p}_2^3 \sim \mathfrak{p}_2^2 \sim \mathfrak{p}_3'$, $\mathfrak{p}_2'\mathfrak{p}_3 \sim \mathfrak{p}_2$, $\mathfrak{p}_2\mathfrak{p}_3' \sim \mathfrak{p}_2'$, $\mathfrak{p}_2'\mathfrak{p}_3' \sim \mathfrak{p}_3$, we see that (1), $\mathfrak{p}_2$, $\mathfrak{p}_2'$, $\mathfrak{p}_3$, $\mathfrak{p}_3'$ form a complete set of mutually inequivalent integral ideals of norm $\leq \sqrt{|47|}$ and thus $h_2 = 5$.

To find h, we have to consider integral ideals of norm $\leq \sqrt{2021}$ (i.e. ≤ 44). Since

$$\left(\frac{2021}{p}\right) = -1 \quad \text{for} \quad p = 2, 3, 7, 11, 13, 23, 29, 31, 37, 41,$$

these rational primes stay prime also in $\mathbf{Q}(\sqrt{2021})$. We need to consider therefore only the decomposition of (5), (17), (19) and (43) in $\mathbf{Q}(\sqrt{2021})$. Now

$$\left(\frac{2021}{43}\right) = 0 \quad \text{and} \quad (43) = \left(\frac{43 + \sqrt{2021}}{2}\right)\left(\frac{43 - \sqrt{2021}}{2}\right).$$

Further

$$\left(\frac{2021}{5}\right) = \left(\frac{2021}{17}\right) = \left(\frac{2021}{19}\right) = 1$$

and so $(5) = \mathfrak{p}_5 \cdot \mathfrak{p}_5'$, $(17) = \mathfrak{p}_{17} \cdot \mathfrak{p}_{17}'$, $(19) = \mathfrak{p}_{19} \cdot \mathfrak{p}_{19}'$ (say). Actually

$$\mathfrak{p}_5 = \left(5, \frac{1 + \sqrt{2021}}{2}\right).$$

Now, **156**

$$\mathfrak{p}_5^2 = \left(\frac{39 - \sqrt{2021}}{2}, -25\right) \quad \text{and} \quad \mathfrak{p}_5^3 = \left(\frac{39 - \sqrt{2021}}{2}\right).$$

Hence $\mathfrak{p}_5^3 \sim (1)$ but $\mathfrak{p}_5^2 \nsim (1)$, for this would mean $\mathfrak{p}_5 \sim (1)$ which is not true, since ± 5 is not the norm of any integer in $\mathbf{Q}(\sqrt{2021})$. Now

$$(44 + \sqrt{2021})(44 - \sqrt{2021}) = (5)(17)$$

and

$$(46 + \sqrt{2021})(46 - \sqrt{2021}) = (5)(19)$$

and by properly choosing $\mathfrak{p}_{17}$, $\mathfrak{p}_{19}$, we can suppose that $\mathfrak{p}\mathfrak{p}_{17}'- \sim (1)$ and $\mathfrak{p}_5\mathfrak{p}_{19}' \sim (1)$. But 1, $\mathfrak{p}_5$, $\mathfrak{p}_5^2$ are inequivalent and $\mathfrak{p}_{17} \sim \mathfrak{p}_5 \sim \mathfrak{p}_{19}$, $\mathfrak{p}_{17}' \sim \mathfrak{p}_{19}' \sim \mathfrak{p}_5^2$.

Hence $h = 3$. Further $w = 2$, $\epsilon = \frac{1}{2}(45 + \sqrt{2021})$; $w_2 = W = 2$. Moreover, $r/R = 1$ since

$$-\epsilon = \left(\frac{\sqrt{-43} + \sqrt{-47}}{2}\right)^2.$$

Let B_1, B_2, B_3 be the narrow classes in $\mathbf{Q}(\sqrt{2021})$ containing (1), $\mathfrak{p}_5$, $\mathfrak{p}_5^2$. Then

$$\mathfrak{b}_{B_1} = (1), \beta_1 = 1, \beta_2 = \frac{1+\sqrt{2021}}{2}, \omega_1 = \frac{\beta_2}{\beta_1} = \frac{1+\sqrt{2021}}{2},$$
$$v(\beta_1) = 1,$$

$$\left.\begin{aligned} \epsilon\omega_1 &= 23\omega_1 + 505 \\ \epsilon &= \omega_1 + 22 \end{aligned}\right\} a = 23, b = 505, c = 1, d = 22$$

$$\mathfrak{b}_{B_2} = \mathfrak{p}_5, \beta_1 = 5, \beta_2 = \frac{1+\sqrt{2021}}{2}, v(\beta_1) = 1,$$
$$\omega_2 = \frac{\beta_2}{\beta_1} = \frac{1+\sqrt{2021}}{10},$$

$$\left.\begin{aligned} \epsilon\omega_2 &= 23\omega_2 + 101 \\ \epsilon &= 5\omega_2 + 22 \end{aligned}\right\} a = 23, b = 101, c = 5, d = 22$$

157

$$\mathfrak{b}_{B_3} = \mathfrak{p}_5^2, \beta_1 = \frac{1}{2}(39 - \sqrt{2021}), \beta_2 = -25, v(\beta_1) = -1,$$
$$\omega_3 = \frac{\beta_2}{\beta_1} = \frac{-50}{39 - \sqrt{2021}} = \frac{19 + \omega_1}{5},$$

$$\left.\begin{aligned} \epsilon\omega_3 &= 42\omega_3 + 25 \\ \epsilon &= 5\omega_3 + 3 \end{aligned}\right\} a = 42, b = 25, c = 5, d = 3$$

$$\chi(B_1) = 1, \chi(B_2) = \chi(\mathfrak{p}_5) = \left(\frac{-43}{5}\right) = -1, \chi(B_3) = \chi(\mathfrak{p}_5^2) = 1.$$

Thus

$$\begin{aligned} \Lambda^* = 1 &\left\{\frac{1}{12}\frac{23+22}{1} - \frac{1}{4}\right\} \\ &- 1\left\{\frac{1}{12}\frac{23+22}{5} - \sum_{k=1}^{4} \mathscr{P}_1\left(\frac{k}{5}\right)\mathscr{P}_1\left(\frac{23k}{5}\right) - \frac{1}{4}\right\} + \end{aligned}$$

$$+\left\{\frac{1}{12}\frac{42+3}{5}-\sum_{k=1}^{4}\mathscr{P}_1\left(\frac{k}{5}\right)\mathscr{P}_1\left(\frac{42k}{5}\right)-\frac{1}{4}\right\}.$$

Since $\mathscr{P}_1\left(\frac{23k}{5}\right)=-\mathscr{P}_1\left(\frac{42k}{5}\right)$, we see easily that $\Lambda^*=5/2$,

$$H=2.3.1\cdot\frac{2}{2}\cdot\frac{5}{2}=15,$$
$$\frac{2h_1h_2}{w_1w_2}=\frac{2.1.5}{2.2}=\Lambda^*.$$

The following two examples deal with the case when F is ramified (with relative discriminant $\mathfrak{f}\neq(1)$) over $\mathbf{Q}(\sqrt{D})$, (cf. Proposition 17). Now the numbers γ and u_B, v_B corresponding to the ray classes B in $\mathbf{Q}(\sqrt{D})$ have to be reckoned with.

Example 7. $D=5$, $F=\mathbf{Q}(\sqrt{5},\sqrt{\theta})$ where $\theta=(\sqrt{5}-5)/2$ and $-\theta>0$, $h=1$, $w=2$, $W=10$. The fundamental units in F and $\mathbf{Q}(\sqrt{5})$ are the same, viz. $\omega=(1+\sqrt{5})/2$ and so $r/R=1/2$. It is easy to verify that $\theta=\sqrt{5}\omega'$, **158**
$\theta'=-\sqrt{5}\omega=\theta\cdot\omega^2$, $N(\omega)=-1$.

If we set $\rho=(\sqrt{\theta}-\omega)/2$, then $\rho^{-1}=-(\sqrt{\theta}+\omega)/2=-\rho-\omega$ and so $\rho+\rho^{-1}=-\omega$. Therefore $\rho^2+\rho^{-2}=\omega^2-2=\omega-1$, so that $\rho^2+\rho^{-2}+\rho+\rho^{-1}+1=0$. Actually, since $\operatorname{Re}\rho<0$ and $\operatorname{Im}\rho>0$, we have $\rho=e^{4\pi i/5}$. As a basis of the integers in F over $\mathbf{Q}(\sqrt{5})$, we have $[1,\rho]$ and the relative discriminant $\mathfrak{f}$ of F over $\mathbf{Q}(\sqrt{5})$ is $(\theta)=(\theta')=(\sqrt{5})$. Now $\varphi(\mathfrak{f})=4$ and ω^4, $-\omega$, ω, ω^2 serve as prime residue class representatives modulo $\mathfrak{f}$ in $\mathbf{Q}(\sqrt{5})$. In fact, $\omega^4\equiv 1$, $-\omega\equiv 2$, $\omega\equiv 3$, $\omega^2\equiv 4(\mod \mathfrak{f})$. For ϵ, we take $\omega^4=(7+3\sqrt{5})/2$ and then $\epsilon>0$ as also $\epsilon\equiv 1(\mod \mathfrak{f})$. We note that $v(\mathfrak{f})=0$.

We take $\gamma=1/5$ so that $(\gamma)\mathfrak{f}=(1)/(\sqrt{5})$ has exact denominator $\mathfrak{f}$ and $\mathfrak{q}=(1)$. Further $\chi(\mathfrak{q})v(\gamma\sqrt{5})=1\times -1=-1$.

In the notation of § 5 (Chapter II), the set $\{\lambda_k\}$ consists just of 1, and $\{\mu_l\}$ consists of 1, 2ω. Therefore, since $h=1$, we may take as ray class representatives modulo $\mathfrak{f}$, the ideals $\mathfrak{p}_1=(1)$ and $\mathfrak{p}_2=(2)$ and denote the corresponding ray classes by B_1 and B_2.

For B_1, $\chi(B_1)=1$, $\beta_1=1$, $\beta_2=\omega$, $v(\beta_1)=1$, $a=5$, $b=3$, $c=3$, $d=2$, $u_{B_1}=S(\gamma)=2/5$, $v_{B_1}=S(\omega\gamma)=1/5$.

For B_2, $\chi(B_2)=-1$, necessarily; $\beta_1=2$, $\beta_2=2\omega$, $v(\beta_1)=1$, $a=5$, $b=3$, $c=3$, $d=2$, $u_{B_2}=S(2\gamma)=4/5$, $v_{B_2}=S(2\omega\gamma)=2/5$;

$$\Lambda=\left\{\left(\frac{(2/5)^2-(2/5)}{2}+\frac{1}{12}\right)\frac{5+2}{3}-\sum_{k=0}^{2}\mathscr{P}_1\left(\frac{k+2/5}{3}\right)\mathscr{P}_1\left(\frac{5(k+2/5)}{3}-\frac{1}{5}\right)\right\}-$$

$$-\left\{\left(\frac{(4/5)^2-(4/5)}{2}+\frac{1}{12}\right)\frac{5+2}{3}-\sum_{k=0}^{2}\mathscr{P}_1\left(\frac{k+4/5}{3}\right)\mathscr{P}_1\left(\frac{5(k+4/5)}{3}-\frac{2}{5}\right)\right\}$$

$$= -2/5;$$

$$H = h\cdot\frac{r}{R}\cdot\frac{W}{w}\times(-\Lambda) = 1\cdot\frac{1}{2}\cdot\frac{10}{2}\cdot\frac{2}{5} = 1.$$

We shall now show that $H = 1$, directly. Since the absolute discriminant of F is $5^2\cdot N(\mathfrak{f}) = 125$, we need only to examine the splitting in F of the ideals (2), (3), (5), (7) and (11). Since $\left(\frac{5}{p}\right) = -1$ for $p = 2,3,7$ the ideals (2), (3), (7) are prime in $\mathbf{Q}(\sqrt{5})$. Further, $\left(\frac{\theta}{\mathfrak{p}}\right) = -1$ for $\mathfrak{p} = (2), (3), (7)$ and hence, by (125), these stay prime in F too. Again, **159**

$$(11) = (4+\sqrt{5})(4-\sqrt{5}) = (1-\sqrt{\theta})(1+\sqrt{\theta})(1-\omega\sqrt{\theta})(1+\omega\sqrt{\theta})$$

$$(5) = (\sqrt{5})^2 = (\sqrt{\theta})^4.$$

Thus, all prime ideals of norm ≤ 11 in F are principal and, as a consequence, $H = 1$.

We might show that $H = 1$, also by using the following decomposition, due to Kummer, of the Dedekind zeta-function $\zeta(s, F)$, viz.

$$\zeta(s,F) = \zeta(s)L_5(s)P(s)Q(s),$$

where, for $\sigma > 1$,

$$L_5(s) = \sum_{n=1}^{\infty}\left(\frac{5}{n}\right)n^{-s},$$

$$P(s) = \sum_{n=1}^{\infty}\psi(n)n^{-s},$$

$$Q(s) = \sum_{n=1}^{\infty}\overline{\psi}(n)n^{-s},$$

and $\psi(n)$ is defined by

$$\psi(n) = \begin{cases} 0, & n \equiv 0 \pmod 5,\\ i, & n\equiv 2 \pmod 5,\\ -1, & n\equiv -1 \pmod 5,\\ -i, & n \equiv -2 \pmod 5,\\ 1, & n\equiv 1 \pmod 5. \end{cases}$$

Since $\zeta(s, F) = \zeta(s, \mathbf{Q}(\sqrt{5}))L(s, \chi)$ and $\zeta(s, \mathbf{Q}(\sqrt{5})) = \zeta(s)L_5(s)$, we have $L(s, \chi) = P(s)Q(s)$. Hence

$$P(1)Q(1) = L(1, \chi) = -\frac{\pi^2 \Lambda}{5} = \frac{2\pi^2}{25}. \qquad (*)$$

But it is easy to see that 160

$$\begin{aligned} P(1) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{5n+1} + \frac{i}{5n+2} \right), \\ &= \frac{\pi}{5} (\cot \pi/5 + i \cot 2\pi/5), \\ &= \frac{\pi}{5} \left(\frac{-i(\rho + \rho^{-1})}{\rho - \rho^{-1}} + \frac{\rho^2 + \rho^{-2}}{\rho^2 - \rho^{-2}} \right). \end{aligned}$$

Similarly

$$Q(1) = \frac{\pi}{5} \left(\frac{-i(\rho + \rho^{-1})}{\rho - \rho^{-1}} - \frac{\rho^2 + \rho^{-2}}{\rho^2 - \rho^{-2}} \right).$$

Therefore

$$\begin{aligned} P(1)Q(1) &= -\frac{\pi^2}{25} \left(\frac{\rho^2 + \rho^{-2} + 2}{\rho^2 + \rho^{-2} - 2} + \frac{\rho^4 + \rho^{-4} + 2}{\rho^4 + \rho^{-4} - 2} \right) \\ &= -\frac{\pi^2}{25} \left(\frac{\rho^2 + \rho^{-2} + 2}{\rho^2 + \rho^{-2} - 2} + \frac{\rho + \rho^{-1} + 2}{\rho + \rho^{-1} - 2} \right) \\ &= -\frac{\pi^2}{25} \left(\frac{\alpha^2(\alpha - 2) + (\alpha + 2)(\alpha^2 - 4)}{(\alpha^2 - 4)(\alpha - 2)} \right) (\alpha = \rho + \rho^{-1} = -\omega) \\ &= -\frac{\pi^2}{25} \times -\frac{10}{5} = \frac{2\pi^2}{25}, \end{aligned}$$

which confirms $(*)$ above.

Example 8. $D = 17$, $F = \mathbf{Q}(\sqrt{17}, \sqrt{\theta})$ where $\theta = (\sqrt{17} - 5)/2$, $-\theta > 0$, $h = 1$, $w = 2$, $W = 2$; $\rho = 4 + \sqrt{17} (= 2\theta + 9)$ is the fundamental unit in $\mathbf{Q}(\sqrt{17})$ as also in F and so $r/R = 1/2$.

$[1, (1 + \sqrt{\theta})/\theta']$ is a basis of the integers in F over $\mathbf{Q}(\sqrt{17})$. The relative discriminant $\mathfrak{f}$ of F over $\mathbf{Q}(\sqrt{17})$ is (θ^3); $N(\mathfrak{f}) = 8$ and $v(\mathfrak{f}) = 0$. Further $\varphi(\mathfrak{f}) = 4$ and as representatives of th prime residue classes modulo $\mathfrak{f}$ in $\mathbf{Q}(\sqrt{17})$, we take $1, -1, 3, -3$. It is easy to verify that $\rho \equiv -3$, $\rho^2 \equiv 1$, $-\rho \equiv 3$, $-\rho^2 \equiv -1($ mod $\mathfrak{f})$ and so, in every prime residue class modulo $\mathfrak{f}$, there is a unit. Also, it is 161 clear that $\epsilon = \rho^2 = 33 + 8\sqrt{17}$ is the generator of $\Gamma^*_{\mathfrak{f}}$. We take $\gamma = 1/(\theta^3\sqrt{17})$; then $(\gamma)(\sqrt{17}) = \mathfrak{f}^{-1}$. So $\mathfrak{q} = (1)$ and moreover, $N(\gamma\sqrt{17}) > 0$.

In the notation of § 5 (Chapter II) again, $\{\gamma_k\}$ consists of the single element 1 while $\{\mu_1\}$ consists of the numbers $1, -3\rho, 5\rho, -7\rho^2$. Further $h = 1$ and there are just 4 ray classes modulo $\mathfrak{f}$ which we denote by B_1, B_3, B_5 and B_7 (say) and the corresponding representatives $\mathfrak{b}_{B_1}$, $\mathfrak{b}_{B_3}$, $\mathfrak{b}_{B_5}$, $\mathfrak{b}_{B_7}$ may be chosen to be (1), (3), (5), (7) respectively. The numbers u_B, v_B corresponding to the ray classes B_1, B_3, B_5, B_7 may be denoted by $u_1, v_1, u_3, v_3, u_5, v_5, u_7, v_7$ respectively.

If $\lambda \equiv \pm 1 \pmod{\mathfrak{f}}$, then $\chi((\lambda)) = \pm 1$ according as $N(\lambda) \gtrless 0$ and hence $\chi(B_1) = 1, \chi(B_3) = \chi((-3\rho)) = -1, \chi(B_5) = \chi((5\rho)) = -1, \chi(B_7) = \chi((-7\rho^2)) = 1$. We note incidentally that, for $q = 1, 3, 5, 7$, $\chi(B_q) = \left(\frac{2}{q}\right)$.

For B_1,

$$\mathfrak{b}_{B_1} = (1), \beta_1 = 1, \beta_2 = \theta, u_1 = S(\gamma) = -\frac{23}{8},$$
$$v_1 = S(\theta\gamma) \equiv \frac{1}{4} \pmod{1}.$$

For B_3,

$$\mathfrak{b}_{B_3} = (3), \beta_1 = 3, \beta_2 = 3\theta, u_3 = S(3\gamma) = -\frac{69}{8},$$
$$v_3 = S(3\theta\gamma) \equiv \frac{3}{4} \pmod{1}.$$

For B_5,

$$\mathfrak{b}_{B_5} = (5), \beta_1 = 5, \beta_2 = 5\theta, u_5 = S(5\gamma) = -\frac{115}{8},$$
$$v_5 = S(5\theta\gamma) \equiv -v_3 \pmod{1}.$$

For B_7,

$$\mathfrak{b}_{B_7} = (7), \beta_1 = 7, \beta_2 = 7\theta, u_7 = S(7\gamma) = -\frac{161}{8},$$
$$v_7 = S(7\theta\gamma) \equiv -v_1 \pmod{1}.$$

For all the 4 classes, $a = -7$, $b = -32$, $c = 16$, $d = 73$. **162**

From (129), we now obtain

$$2H = \chi((1))v(\gamma\sqrt{17}) \sum_{q=1,3,5,7} \overline{\chi}(B_q)G(B_q)$$

i.e.

$$2H = \sum_{q=1,3,5,7} \left(\frac{2}{q}\right) \times \left\{ \mathscr{P}_2(u_q)\frac{33}{8} + \sum_{k=0}^{15} \mathscr{P}_1\left(\frac{k+u_q}{16}\right)\mathscr{P}_1\left(7\cdot\frac{k+u_q}{16}+v_q\right)\right\}.$$

It may be verified that

$$frac338 \sum_{q=1,3,5,7}\left(\frac{2}{q}\right)\mathscr{P}_2(u_q) = \frac{66}{128};$$

$$\sum_{k=0}^{15}\mathscr{P}_1\left(\frac{k+u_1}{16}\right)\mathscr{P}_1\left(7\cdot\frac{k+u_1}{16}+v_1\right) =$$

$$= \sum_{k=0}^{15}\mathscr{P}_1\left(\frac{k+u_7}{16}\right)\mathscr{P}_1\left(7\cdot\frac{k+u_7}{16}+v_7\right) = \frac{6256}{128^2};$$

$$\sum_{k=0}^{15}\mathscr{P}_1\left(\frac{k+u_3}{16}\right)\mathscr{P}_1\left(7\cdot\frac{k+u_3}{16}+v_3\right) =$$

$$= \sum_{k=0}^{15}\mathscr{P}_1\left(\frac{k+u_5}{16}\right)\mathscr{P}_1\left(7\cdot\frac{k+u_5}{16}+v_5\right) = -\frac{5904}{128^2}.$$

Thus

$$2H = \frac{66}{128} + 2\frac{6256+5904}{128^2} = 2$$

i.e. $H = 1$.

We shall now show that $H = 1$ directly. Since the absolute discriminant of F is $17^2 \times 8 = 2312 (< 49^2)$, we have to examine for mutual equivalence all integral ideals of (absolute) norm < 49 in F.

To this end, we first remark that the ideals (3), (5), (7), (11), (23), (29), (31), **163**
(37) and (41) are prime in $\mathbf{Q}(\sqrt{17})$. Further we have in $\mathbf{Q}(\sqrt{17})$ the following decomposition of other ideals, viz.

$$\begin{aligned}
&(2) = (\theta)(\theta'), (13) = (1+4\theta')(1+4\theta), (17) = (5+2\theta)(5+2\theta'),\\
&(19) = (11+2\theta)(11+2\theta'), (43) = (1+6\theta)(1+6\theta'),\\
&(47) = (3-2\theta)(3-2\theta').
\end{aligned}$$

Now, if $\mathfrak{p}$ is a prime ideal in $\mathbf{Q}(\sqrt{17})$, then

$$\chi(\mathfrak{p}) = \left(\frac{\theta}{\mathfrak{p}}\right) = \begin{cases} +1, & \text{if } \mathfrak{p} = \mathfrak{B}_1\mathfrak{B}_2 \text{ in } F, \mathfrak{B}_1 \neq \mathfrak{B}_2\\ -1, & \text{if } \mathfrak{p} \text{ is prime in } F\\ 0, & \text{if } \mathfrak{p} = \mathfrak{B}^2 \text{ in } F. \end{cases}$$

Further, if λ and μ are integers in $\mathbf{Q}(\sqrt{17})$ such that $\lambda \equiv \mu(\mod \mathfrak{f})$, then $\chi((\lambda)) = \left(\frac{\theta}{(\lambda)}\right) = \pm\left(\frac{\theta}{(\mu)}\right)$ according as $N(\lambda\mu) \gtrless 0$.

For $\mathfrak{p} = (3), (5), (11), (29), (37)$ in $\mathbf{Q}(\sqrt{17})$, $\left(\frac{\theta}{\mathfrak{p}}\right) = -1$ and so these prime ideals stay prime also in F.

$$(2) = (\theta)(\theta') = (\sqrt{\theta})^2\left(\frac{1+\sqrt{\theta}}{\theta'}\right)\left(\frac{1-\sqrt{\theta}}{\theta'}\right).$$

Now $1+4\theta' \equiv 3(\mod \mathfrak{f})$ and $1+4\theta = 1(\mod \mathfrak{f})$ so that $\chi((1+4\theta')) = -1$, while $\chi((1+4\theta)) = +1$. So $(1+4\theta')$ stays prime in F while $(1+4\theta)$ splits in F. This gives

$$(13) = (1+4\theta')(1+\theta+\sqrt{\theta})(1+\theta-\sqrt{\theta}).$$

Again $5+2\theta \equiv 1(\mod \mathfrak{f})$, $5+2\theta' \equiv -1(\mod \mathfrak{f})$ and so $\chi((5+2\theta)) = \chi((5+2\theta')) = -1$. Hence the ideal (17) does not decompose further in F.

In view of the fact that $11+2\theta' \equiv -3(\mod \mathfrak{f})$, $\chi((11+2\theta')) = -1$ and so $(11+2\theta')$ stays prime in F whereas $(11+2\theta)$ splits in F. We have in F therefore the decomposition

$$(19) = (11+2\theta')(1+(5+\theta)\sqrt{\theta})(1-(5+\theta)\sqrt{\theta}).$$

164

From $1+6\theta' \equiv -1(\mod \mathfrak{f})$, $1+6\theta \equiv -3(\mod \mathfrak{f})$ we know that $(1+6\theta)$ stays prime in F while $(1+6\theta')$ splits and we obtain the splitting of (43) in F as

$$(43) = (1+6\theta)(5+\theta+\sqrt{\theta}(4+\theta))(5+\theta-\sqrt{\theta}(4+\theta)).$$

Both the ideals $(3-2\theta)$ and $(3-2\theta')$ split in F and we have

$$\begin{aligned}(47) = {} & (1+\sqrt{\theta}(3+\theta))(1-\sqrt{\theta}(3+\theta))(3+\theta+\sqrt{\theta}(4+\theta))\times \\ & \times(3+\theta-\sqrt{\theta}(4+\theta)).\end{aligned}$$

The prime ideals (7) and (41) in $\mathbf{Q}(\sqrt{17})$ split in F as

$$(7) = (\theta+3+\sqrt{\theta})(\theta+3-\sqrt{\theta}),$$
$$(41) = (3+2\theta+2\sqrt{\theta}(5+\theta))(3+2\theta-2\sqrt{\theta}(5+\theta)).$$

Finally, from the decompositions

$$(23)(\theta') = \left(\frac{8+3\theta}{\sqrt{\theta}} + \frac{\sqrt{\theta}}{2}(\theta+\sqrt{\theta})\right)\left(\frac{(8+3\theta)+\theta(\theta-\sqrt{\theta})/2}{\sqrt{\theta}}\right),$$

$$(62)(1+4\theta) = \left(-10 - 7\theta + \frac{\theta^2}{2} + \frac{\theta\sqrt{\theta}}{2}\right)\left(-10 - 7\theta + \frac{\theta^2}{2} - \frac{\theta\sqrt{\theta}}{2}\right),$$

we can deduce that the ideals (23) and (31) split into two distinct principal prime ideals each in F, if we use the decomposition of $(1 + 4\theta)$ and (2) in F.

Thus all prime ideals of norm < 49 in F are principal and we have $H = 1$.

Bibliography

[1] R. Dedekind: Aus Briefen an Frobenius, *Gesammelte mathematische* 165
Werke, Zweiter Band, Braunschweig, (1931), 420.

[2] M. Deuring: Die Klassenkörper der komplexen Multiplikation, *Enzykl der math. Wiss.*, Band I, 2, 23, Teubner, (1958).

[3] P.G.L. Dirichlet: Recherches sur les formes quadratiques à coefficients et à indéterminées complexes, *Werke*, Bd. I, 533-618.

[4] R. Fricke: *Lehrbuch der Algebra*, Vol. 3, Braunschweig, (1928).

[5] R. Fueter: Die verallgemeinerte Kroneckersche Grenzformel und ihre Anwendung auf die Berechnung der Klassenzahl, *Rendiconti del circ. Mathematico di Palermo,* 29 (1910), 380-395.

[6] C.F. Gauss: Disquisitiones Arithmeticae, *Gesammelte Werke 1*, (1801).

[7] H. Hasse: Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper I, *Jahresbericht der D. M. V.* 35 (1926) 1-55.

[8] E. Hecke: Über die Zetafunktion beliebiger algebraischer Zahlkörper, *Werke*, 159-171.

[9] E. Hecke: Über eine neue Anwendung der Zetafunktionen auf die Arithmetik der Zahlkörper, *Werke,* 172-177.

[10] E. Hecke: Über die Kroneckersche Grenzformel für reelle quadratische Körper und ide Klassenzahl relativ-Abelscher Körper. *Werke*, 198-207.

[11] E. Hecke: Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. I, II, *Werke,* 215-234, 249-289.

[12] E. Hecke: Bestimmung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern. *Werke*. 290-312.

[13] E. Hecke: Darstellung von Klassenzahlen als Perioden von Integralen 3. Gattung aus dem Gebiet der elliptischen Modulfunktionen. *Werke*, 405-417.

[14] E. Hecke: Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik. *Werke*, 461-486.

[15] G. Herglotz: Über einen Dirichletschen Satz. *Math. Zeit*, 12 (1922), 255-261.

[16] G. Herglotz: Über die Kroneckersche Grenzformel für reelle quadratische Körper. 1, 11, *Ber. d. Sachs Akod. d. Wiss. zu Leipzig.* 75 (1923).

[17] D. Hilbert: Bericht über die Theorie der algebraischen Zahlkörper, **166**
Jahresbericht der D.M.V. (1897), 177-546. *Ges. Werke.* Bd. 1 (1932), 63-363.

[18] D. Hilbert: Über die Theorie des relativquadratischen Zahlkörpers, *Math. Ann.* 51 (1899), 1-127. Ges. Werke Bd 1 (1932), 370-482.

[19] L. Kronecker: Zur Theorie der Elliptischen Funktionen, Werke, Vol. 4, 347-495, Vol. 5, 1-132, Leipzig, (1929).

[20] E. Kummer: Mémoire sur la theorie des nombres complexes composés de racines de l'unité et de nombres entiers, *Jour. de math. pures et appliquees* XVI (1851), 377-498 (collected papers I, 363-484).

[21] H. Maass: Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* 121 (1949), 141-183.

[22] C. Meyer: *Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörpern*, Akademie Verlag, Berlin, 1957.

[23] A. Selberg: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.* 20 (1956), 47-87.

Chapter 3

Modular Functions and Algebraic Number Theory

1 Abelian functions and complex multiplications

In this section, we shall see how the complex multiplications of abelian functions lead in a natural fashion to the study of the Hilbert modular group. 167

Let $z_1, \ldots, z_n$ be n independent complex variables and let

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

Let C^n be the n-dimensional complex euclidean space and G, a domain in C^n. A complex-valued function $f(z)$ defined on G (except, perhaps, for a lower-dimensional subset of G) is *meromorphic* in G, if there exists a correspondence

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (\in G) \to \frac{p_a(z)}{q_a(z)}$$

where $p_a(z)$ and $q_a(z)$ are convergent power-series in $z_1 - a_1, \ldots z_n - a_n$ with a common domain of convergence K_a around a such that for $z \in K_a$, $f(z) = \dfrac{p_a(z)}{q_a(z)}$ whenever either $q_a(z) \neq 0$ or if $q_a(z) = 0$, the $p_a(z) \neq 0$.

Let $f(z)$ be a meromorphic function of z in C^n. A complex column $p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ is a *period* of $f(z)$, if $f(z+p) = f(z)$ for all $z \in C^n$; for example, $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is a period of $f(z)$. The periods of $f(z)$ form an additive abelian group; if we regard **168** C^n as a $2n$-dimensional real vector space, then the group of periods is, in fact, a closed vector group over the field of real numbers. Either this vector group is discrete or it contains at least one limit point in C^n. In the latter case, $f(z)$ is said to be *degenerate*; this is equivalent to the fact that $f(z)$ has infinitesimal periods and by a suitable linear transformation of the variables $z_1, \ldots, z_n$ with complex coefficients, the function $f(z)$ can be brought to depend on strictly less than n complex variables. In the former case, $f(z)$ is said to be *non-degenerate* and its period-group is a discrete vector-group i.e. a lattice in C^n. We shall consider only the case when the period-lattice has $2n$ generators linearly independent over the field of real numbers. It is easy to construct an example of a non-degenerate abelian function with $2n$ independent periods. For example, let $\mathscr{P}(w)$ be the Weierstrass' elliptic function with independent periods ω_1, ω_2. Then the function $f(z) = \mathscr{P}(z_1) \ldots \mathscr{P}(z_n)$ furnishes the required example, since it has exactly $2n$ independent periods

$$\begin{pmatrix} \omega_1 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \omega_2 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega_2 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \omega_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \omega_2 \end{pmatrix}.$$

Let, then, for a given $f(z)$ meromorphic in C^n, $p_1, \ldots, p_{2n}$ be $2n$ periods linearly independent over the field of real numbers such that any period p of $f(z)$ is of the form $\sum_{i=1}^{2n} m_i p_i$ with rational integers m_i. The n-rowed matrix $P = (p_1, \ldots, p_{2n})$ is called a **period-matrix** of $f(z)$. It is uniquely determined upto multiplication on the right by a $2n$-rowed unimodular matrix. We shall denote the period-lattice generated by $p_1, \ldots, p_{2n}$ in C^n by $\mathfrak{L}$.

A meromorphic function $f(z)$ defined in C^n and admitting as periods all the elements of $\mathfrak{L}$ is called an **abelian function for the lattice** $\mathfrak{L}$. The abelian functions for $\mathfrak{L}$ constitute a field $\mathfrak{F}_{\mathfrak{L}}$. In $\mathfrak{F}_{\mathfrak{L}}$ there exists at least one function $f(z)$ having $\mathfrak{L}$ exactly as its period-lattice. Moreover, in $\mathfrak{F}_{\mathfrak{L}}$, there exist n functions $f_1, \ldots, f_n$ which are analytically independent and hence algebraically independent over the field of complex numbers. Also, every abelian function satisfies a **169** polynomial equation of bounded degree with rational functions of $f_1, \ldots, f_n$ as coefficients. As a consequence, it can be shown that $\mathfrak{F}_{\mathfrak{L}}$ is an algebraic function field of n variables and one can find $n+1$ functions, $f_0, f_1, \ldots, f_n$ in $\mathfrak{F}_{\mathfrak{L}}$ such that every function in $\mathfrak{F}_{\mathfrak{L}}$ is a rational function of $f_0, f_1, \ldots, f_n$ with complex coefficients.

Let P be a complex matrix of n rows and $2n$ columns such that the $2n$ columns are linearly independent over the field of real numbers. The problem arises as to when P is a period-matrix of a non-degenerate abelian function or, in other words, a **Riemann matrix**. It is well known that in order that P might be a Riemann matrix, it is necessary and sufficient that there exists a rational $2n$-rowed alternate (skew-symmetric) matrix A such that

$$\begin{aligned} &\text{(i) } PAP' = 0,\\ &\text{(ii) } H = i^{-1}\overline{P}AP' > 0 \text{ (i.e. positive hermitian).} \end{aligned} \tag{133}$$

These are known as the *period relations.* If E is the n-rowed identity matrix and A is the $2n$-rowed alternate matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, then conditions (i) and (ii) were given by Riemann as precisely the conditions to be satisfied by the periods of a normalised complete system of abelian integrals of the first kind on a Riemann surface of genus n.

Setting $B = \begin{pmatrix} P \\ \overline{P} \end{pmatrix}$, the two conditions above may be written in the form

$$BAB' = \begin{pmatrix} 0 & -i\overline{H} \\ iH & 0 \end{pmatrix}, \quad H > 0. \tag{134}$$

For $n = 1$, the conditions (133) reduce to the sole condition that if $P = (\omega_1\omega_2)$, then $\omega_1\omega_2^{-1}$ should not be real. To establish the necessity and sufficiency of these conditions for $n > 1$, one has to make considerable use of theta-functions.

The problem as to when a given P is a Riemann matrix was apparently considered independently also by Weierstrass and he wished to prove to this end that every abelian function can be written as the quotient of two theta-functions. He could not, however, complete the proof of this last statement–it **170**
was Appell who gave a complete proof of the same for $n = 2$, and Poincaré, for general n.

A period matrix P which satisfies the period-relations with respect to an alternate matrix A is said to be *polarized with respect to A.* The alternate matrix A is not unique; but, in general, A is uniquely determined upto a positive rational scalar factor. Moreover, if P is polarized with respect to A, then from (134), since $H > 0$, we conclude that $|A| \neq 0$, $|B| \neq 0$.

Let P be a Riemann matrix and $\mathfrak{L}$, the associated lattice in C^n. Let $\mathfrak{L}_1$ be a sublattice of $\mathfrak{L}$, again of rank $2n$ over the field of real numbers and let P_1 be a period matrix of $\mathfrak{L}_1$. Then $P_1 = PR_1$ for a nonsingular rational integral matrix R_1. The associated function field $\mathfrak{F}_{\mathfrak{L}_1}$ is an algebraic function field of n variables, containing $\mathfrak{F}_{\mathfrak{L}}$. Since $\mathfrak{F}_{\mathfrak{L}}$ itself is an algebraic function field of n variables, we see that $\mathfrak{F}_{\mathfrak{L}_1}$ is an algebraic extension of $\mathfrak{F}_{\mathfrak{L}}$.

Conversely, let $\mathfrak{F}_{\mathfrak{L}_1}$ be a field of abelian functions, P_1 an associated Riemann matrix and $\mathfrak{L}_1$ the corresponding period-lattice in C^n. Further, let $\mathfrak{F}$ be a subfield of $\mathfrak{F}_{\mathfrak{L}_1}$ such that $\mathfrak{F}_{\mathfrak{L}_1}$ is algebraic over $\mathfrak{F}_{\mathfrak{L}}$. Then $\mathfrak{F}$ is again a field of abelian functions containing at least one non-degenerate abelian function and has a period-lattice $\mathfrak{L}$ containing $\mathfrak{L}_1$. This means that if P is a period-matrix associated with $\mathfrak{F}$, then $P_1 = PR_1$ for a nonsingular rational integral matrix R_1. We now prove

Proposition 19. *Let P_1 and P_2 be two Riemann matrices and $\mathfrak{L}_1$ and $\mathfrak{L}_2$ the associated period lattices in C^n. A necessary and sufficient condition that the elements of $\mathfrak{F}_{\mathfrak{L}_1}$ depend algebraically on those of $\mathfrak{F}_{\mathfrak{L}_2}$ and vice versa, is that $P_1 = PM$ for a nonsingular rational matrix M.*

Proof. Let $f_0, \ldots, f_n$ be generators of $\mathfrak{F}_{\mathfrak{L}_1}$ over the field of complex numbers and let elements of $\mathfrak{F}_{\mathfrak{L}_1}$ depend algebraically on those of $\mathfrak{F}_{\mathfrak{L}_2}$. This means that each f_i satisfies an irreducible polynomial equation

$$f_i^{n_i} + a_{n_i-1}^{(i)} f_i^{n_i-1} + \cdots + a_0^{(i)} = 0,$$

where $a_m^{(i)}$, $m = 0, 1, \ldots, n_i - 1$ belong to $\mathfrak{F}_{\mathfrak{L}_2}$. Moreover, it is easy to see that $a_m^{(i)}$ for $m = 0, 1, \ldots, n_i - 1$ and $i = 0, 1, \ldots, n$ lie in $\mathfrak{F}_{\mathfrak{L}_1}$. Then the field $\mathfrak{F}_{\mathfrak{L}}$ generated by $\{a_m^{(i)}\}$ over the field of complex numbers is a field of abelian **171**
functions contained in both $\mathfrak{F}_{\mathfrak{L}_1}$ and $\mathfrak{F}_{\mathfrak{L}_2}$ and hence admitting for periods, all the elements of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Since $\mathfrak{F}_{\mathfrak{L}_1}$ is algebraic over $\mathfrak{F}_{\mathfrak{L}}$, $\mathfrak{F}_{\mathfrak{L}}$ is an algebraic function field of n variables and hence $\mathfrak{F}_{\mathfrak{L}_2}$ is also algebraic over $\mathfrak{F}_{\mathfrak{L}}$. Let P be a period matrix of $\mathfrak{F}_{\mathfrak{L}}$ and $\mathfrak{L}$, the associated lattice in C^n. Then $\mathfrak{L}$ contains both $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Hence $P_1 = PR_1$ and $P_2 = PR_2$ for nonsingular rational integral matrices R_1 and R_2. Thus $P_1 = P_2M$ for a nonsingular rational matrix M. □

Conversely, if $P_1 = P_2M$ for a nonsingular rational matrix M, then there exists a Riemann matrix P such that $P_1 = PR_1$ and $P_2 = PR_2$ for nonsingular rational integral matrices R_1 and R_2 respectively. If $\mathfrak{L}$ is the period-lattice in C^n associated with P, the field $\mathfrak{F}_{\mathfrak{L}}$ of abelian functions for $\mathfrak{L}$, is contained in both $\mathfrak{F}_{\mathfrak{L}_1}$ and $\mathfrak{F}_{\mathfrak{L}_2}$. Moreover $\mathfrak{F}_{\mathfrak{L}_1}$ and $\mathfrak{F}_{\mathfrak{L}_2}$ are algebraic over $\mathfrak{F}_{\mathfrak{L}}$. Thus the condition is also sufficient.

Two lattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ in C^n, of rank $2n$ over the field of real numbers and having P_1 and P_2 for period-matrices respectively are said to be *commensurable* if there is another lattice $\mathfrak{L}$ of rank $2n$ containing both $\mathfrak{L}_1$ and $\mathfrak{L}_2$ or *equivalently*: $P_1 = P_2 M$ for a nonsingular rational matrix M. Thus in Proposition 19, the necessary and sufficient condition may also be stated that the lattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ associated with P_1 and P_2 are commensurable.

Let $f(z)$ be a non-degenerate abelian function with period matrix P and period lattice $\mathfrak{L}$ and let m be a scalar. The function $f(mz)$ has period matrix $m^{-1}P$. If m is rational, then the period lattice associated with $m^{-1}P$ is clearly commensurable with $\mathfrak{L}$ and hence $f(mz)$ depends algebraically on the field $\mathfrak{F}_{\mathfrak{L}}$, in view of Proposition 19. This is the analogue of the well-known *multiplication theorem* for elliptic functions. We now ask for all scalars m such that if $f(z) \in \mathfrak{F}_{\mathfrak{L}}$, $f(mz)$ is algebraic over $\mathfrak{F}_{\mathfrak{L}}$. Let us first consider the case $n = 1$.

Let p_1, p_2 be two independent periods of an elliptic function $f(z)$ such that every other period p of $f(z)$ is of the form $rp_1 + sp_2$ with rational integral r and s. In order that $f(mz)$ be algebraic over the field of elliptic functions with periods p_1 and p_2, it is necessary and sufficient that

$$m(p_1 p_2) = (p_1 p_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with rational a, b, c, d. This means that m is the root of a quadratic equation **172**
with rational coefficients. More precisely, m should be an imaginary quadratic irrationality lying in the field $\mathbf{Q}(p_2 p_1^{-1})$ where $\mathbf{Q}$ is the field of rational numbers. If we require that $f(mz)$ should again be an elliptic funciton with p_1 and p_2 as periods, then a, b, c, d should be rational integers. This was the **principle of complex multiplication** for elliptic functions formulated by Abel in his work, “Recharches sur les fonctions elliptiques”. The idea of complex multiplication in its simplest form, however, appears to be contained implicitly in the work of Fagnano, concerning the doubling of an arc of the lemniscate of Bernoulli.

If $f(z)$ is a non-degenerate abelian function with period matrix P and period lattice $\mathfrak{L}$ and Q is an n-rowed complex non-singular matrix, then $f(Q^{-1}z)$ is a non-degenerate abelian function with period matrix QP. We formerly wished to find all complex numbers m such that for every $f(z) \in \mathfrak{F}_{\mathfrak{L}}$, $f(mz)$ is algebraic over $\mathfrak{F}_{\mathfrak{L}}$. We now ask, more generally, for all n-rowed complex non-singular matrices Q such that for every $f(z) \in \mathfrak{F}_{\mathfrak{L}}$, $f(Q^{-1}z)$ is algebraic over $\mathfrak{F}_{\mathfrak{L}}$. By Proposition 19, a necessary and sufficient condition to be satisfied by Q is that

$$QP = PM \tag{135}$$

for some rational non-singular $2n$-rowed matrix M. The matrix M is called a *multiplier* of P and Q, a *complex multiplication of P*. Trivial examples of multipliers are λE, where $\lambda \in \mathbf{Q}$ and $E = E^{(2n)}$ is the $2n$-rowed identity matrix.

From $QP = PM$, it follows that $\overline{QP} = \overline{PM}$, since M is rational. Hence (135) is equivalent to the condition

$$\begin{bmatrix} Q & 0 \\ 0' & Q \end{bmatrix} B = BM.$$

Since B is non-singular, we observe that for given M, Q is uniquely determined by

$$BMB^{-1} = \begin{bmatrix} Q & 0 \\ 0' & \overline{Q} \end{bmatrix} \tag{136}$$

and vice versa. The problem of determining all the complex multiplications Q 173
of a given Riemann matrix P reduces to that of finding all $2n$-rowed rational non-singular matrices M such that BMB^{-1} breaks up as in (136) for some n-rowed complex non-singular Q.

We now drop the condition that M be non-singular and call M, a multiplier still, if the matrix BMB^{-1} breaks up as in (136) or equivalently $PM = QP$ for some Q. We may denote by $\mathfrak{R}$, the set of all multipliers M of P. We see first that $\mathfrak{R}$ is a ring containing identity. For, if $M_1, M_2 \in \mathfrak{R}$, then for some Q_1 and Q_2, we have $Q_1P = PM_1$ and $Q_2P = PM_2$. Hence

$$P(M_1 + M_2) = (Q_1 + Q_2)P,$$
$$PM_1M_2 = Q_1PM_2 = Q_1Q_2P$$

and thus $M_1 + M_2$, $M_1M_2 \in \mathfrak{R}$. Further the $2n$-rowed identity matrix E lies in $\mathfrak{R}$ and serves as the identity in $\mathfrak{R}$. Since for $M \in \mathfrak{R}$ and $\lambda \in \mathbf{Q}$, λM also lies in $\mathfrak{R}$, we see that $\mathfrak{R}$ is an algebra over $\mathbf{Q}$ and in fact, of finite rank over $\mathbf{Q}$.

We now remark that if $M \in \mathfrak{R}$, $M^* = AM'A^{-1}$ also lies in $\mathfrak{R}$. This is simple to verify. Let us set $G = BAB'$, $T = \begin{pmatrix} Q & 0 \\ 0' & \overline{Q} \end{pmatrix}$; then

$$GT'G^{-1} = \begin{pmatrix} \overline{HQ}'\overline{H}^{-1} & 0 \\ 0' & HQ'H^{-1} \end{pmatrix}.$$

Further, since $B'T' = M'B'$ and $B' = A^{-1}B^{-1}G$, we have $A^{-1}B^{-1}GT' = M'A^{-1}B^{-1}G$ i.e. $(GT'G^{-1})B = B(AM'A^{-1})$. This proves that $M^* \in \mathfrak{R}$. The mapping $M \to M^*$ has the following properties, viz. if $M_1, M_2 \in \mathfrak{R}$, then

$$(M_1 + M_2)^* = A(M_1' + M_2')A^{-1} = M_1^* + M_2^*,$$
$$(M_1M_2)^* = AM_2'M_1'A^{-1} = AM_2'A^{-1}AM_1'A^{-1} = M_2^*M_1^*,$$
$$(M^*)^* = (AM'A^{-1})^* = M.$$

Thus the mapping $M \to M^*$ is an involution of $\mathfrak{R}$ and is usually called the **Rosati involution**. Further, if the matrix Q corresponds to the multiplier $M \in \mathfrak{R}$, then to M^* corresponds the complex multiplication $Q^* = \overline{HQ}'\overline{H}^{-1}$.

Thus $\mathfrak{R}$ is an involutorial algebra of finite rank over $\mathbf{Q}$. Such algebras have 174
been extensively studied by A. A. Albert who has also determined the involutorial algebras which can be realized as the rings of multipliers of Riemann matrices. We shall not go into the general theory of such algebras. We are interested only in involutorial algebras which can be realized as the ring of multipliers of a Riemann matrix P and which, in addition, are commutative rings free from divisors of zero. Such a ring of multipliers $\mathfrak{R}$ is necessarily a field and in fact an algebraic number field k of degree m say, over $\mathbf{Q}$. So $\mathfrak{R}$ gives a faithful representation of k into $\mathfrak{M}_{2n}(\mathbf{Q})$, the full matrix algebra of order $2n$ over $\mathbf{Q}$. Before we proceed further, we need to study this representation of k by $\mathfrak{R}$ more closely.

First, we have the so-called *regular representation* of k with respect to a basis $\omega_1, \ldots, \omega_m$ of k over $\mathbf{Q}$. Namely, let for $\gamma \in k$, $\gamma^{(i)}$, $i = 1, 2, \ldots, m$ denote the conjugates of γ over $\mathbf{Q}$. Then we have for $l = 1, 2, \ldots, m$

$$\gamma\omega_l = \sum_{q=1}^{m} \omega_q c_{ql}, \tag{137}$$

where $c_{ql} \in \mathbf{Q}$. If we denote by C the matrix (c_{ql}), $1 \leq q, l \leq m$ with q and l respectively as the row index and column index, then the mapping $\gamma \to C \in \mathfrak{M}_m(\mathbf{Q})$ is an isomorphism and gives the regular representation of k with respect to the basis $\omega_1, \ldots, \omega_m$ of k over $\mathbf{Q}$. If we change the basis, we get an equivalent representation. Hereafter, we shall always refer to the regular representation of k with respect to a *fixed* basis $\omega_1, \ldots, \omega_m$ of k over $\mathbf{Q}$.

Now, from (**??**), we have

$$\gamma^{(j)}\omega_l^{(j)} = \sum_{q=1}^{m} \omega_q^{(j)} c_{ql}, \; j = 1, 2, \ldots, m. \tag{137*}$$

Let $[\gamma]$ denote the m-rowed diagonal matrix $[\gamma^{(1)}, \ldots, \gamma^{(m)}]$, with $\gamma^{(1)}, \ldots, \gamma^{(m)}$ as diagonal elements. Further, let us denote by Ω, the m-rowed square matrix $(\omega_i^{(j)})$ where i and j are respectively the column index and row index. Now $(137)^*$ reads as

$$[\gamma]\Omega = \Omega C.$$

Or, since Ω is non-singular, 175

$$[\gamma] = \Omega C \Omega^{-1}. \tag{138}$$

Conversely, if $G = [\gamma_1, \ldots, \gamma_m]$ is a diagonal matrix with diagonal elements $\gamma_1, \ldots, \gamma_m$ such that $\Omega^{-1} G \Omega$ is a rational matrix C, then necessarily $\gamma_i = \gamma^{(i)}$, $i = 1, 2, \ldots, m$, for some $\gamma \in k$. In fact,

$$G = \Omega C \Omega^{-1} = \Omega C (\Omega' \Omega)^{-1} \Omega'.$$

Now, the matrix $\Omega'\Omega = (S(\omega_k \omega_l))$ is rational and hence, also, the matrix $C(\Omega'\Omega)^{-1} = (r_{kl})$ is rational. thus

$$\gamma_k = \sum_{p,q=1}^{m} \omega_p^{(k)} r_{pq} \omega_q^{(k)}$$

with $r_{pq} \in \mathbf{Q}$, for $k = 1, 2, \ldots, m$. If we set

$$\gamma = \sum_{p,q=1}^{m} \omega_p r_{pq} \omega_q$$

then we see that $\gamma_i = \gamma^{(i)}$ for $i = 1, 2, \ldots, m$. We shall make use of this remark later.

In the sequel, if $C_1, \ldots, C_r$ are square matrices then $[C_1, \ldots, C_r]$ shall stand for the direct sum of $C_1, \ldots, C_r$.

We shall now prove a theorem concerning an arbitrary faithful representation of k into $\mathfrak{M}_{2n}(\mathbf{Q})$.

Theorem 14. *Let k be an arbitrary algebraic number field of degree m over $\mathbf{Q}$ and let $\psi : \gamma \to G \in \mathfrak{M}_{2n}(\mathbf{Q})$ be faithful representation of k, of order $2n$. Then*

(i) *$2n = qm$, for a rational integer $q \geq 1$ and*

(ii) *there exists a $2n$-rowed non-singular rational matrix T independent of γ such that*

$$T^{-1} G T = [C, \ldots, C],$$

C being the image of γ under the regular representation of k.

Proof. Let $\gamma \in k$ generate k over $\mathbf{Q}$ and let $f(x) = \sum_{i=0}^{m} a_i x^i (a_m = 1)$ be the **176**
irreducible polynomial of γ over $\mathbf{Q}$. If $\psi(\gamma) = G$, we can find a complex non-singular matrix W such that $W^{-1} G W$ is in the normal form, namely

$$W^{-1} G W = [G_1, \ldots, G_h], G_i = \begin{bmatrix} \gamma_i & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \gamma_i & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & \gamma_i \end{bmatrix}$$

and $\gamma_1, \ldots, \gamma_h$ are eigenvalues of G. Using the fact that $a_0 E^{(2n)} + \sum_{i=1}^{m} a_i G^i = 0$, we see that γ_i are conjugates of γ over $\mathbf{Q}$. Again, from $a_0 E^{(2n)} + \sum_{i=1}^{m} a_i W^{-1} G^i W = 0$, we conclude that $W^{-1}GW$ is necessarily in the diagonal form. For, otherwise, even if one G_i is of the form with the elements just above the main diagonal equal to 1, then we obtain $\sum_{j=1}^{m} j a_j \gamma_i^{j-1} = 0$ contradicting the fact that γ_i has an irreducible polynomial of degree m. Thus $W^{-1}GW = [\gamma_1, \ldots, \gamma_{2n}]$ where $\gamma_1, \ldots, \gamma_{2n}$ are conjugates of γ over $\mathbf{Q}$. Moreover, since the characteristic polynomial $|xE - G|$ of G has rational coefficients and has for its zeros only those of $f(x)$, it follows that $|xE - G| = (f(x))^q$ for some rational integer q. Thus $2n = mq$, which proves (i). Moreover $\gamma_1, \gamma_2, \ldots, \gamma_{2n}$ are just the numbers $\gamma^{(1)}, \ldots, \gamma^{(m)}$ repeated q times. We may therefore suppose that

$$W^{-1}GW = [[\gamma], \ldots, [\gamma]],$$

where $[\gamma] = [\gamma^{(1)}, \ldots, \gamma^{(m)}]$. But from (138), we have

$$W^{-1}GW = V[C, \ldots, C]V^{-1}, \tag{139}$$

where $V = [\Omega, \ldots, \Omega]$.

Now, since $1, \gamma, \ldots, \gamma^{m-1}$ form a basis of k over $\mathbf{Q}$, we have, in order to prove (ii), only to find a rational matrix T such that

$$T^{-1}GT = [C, \ldots, C].$$

From (139), we see that there exists a non-singular complex matrix $U = (u_{ij})$, **177**
$1 \leq i, j \leq 2n$ such that

$$GU = U[C, \ldots, C]. \tag{140}$$

The matrix equation (140) may be regarded as a system of linear equations in u_{ij} with rational coefficients. Since there exists a solution in complex numbers for this system of linear equations, we know from the theory of linear equations that there exists at least one non-trivial solution for this system, in rational numbers. The rational solutions of this system form a non-trivial vector space of finite dimension over $\mathbf{Q}$. Let $U_1, \ldots, U_r$ be the matrices corresponding to a set of generators of this vector space. From the theory of linear equations again, we see that all complex solutions U of (140) are necessarily of the form $z_1 U_1 + \cdots + z_r U_r$ with arbitrary complex $z_1, \ldots, z_r$. Now, we know that there exists at least one complex solution U of (140) for which $|U| \neq 0$. This means that the polynomial $|z_1 U_1 + \cdots + z_r U_r|$ in $z_1, \ldots, z_r$ does not vanish identically. Hence we can find $\lambda_1, \ldots, \lambda_r \in \mathbf{Q}$ such that $|\lambda_1 U_1 + \cdots + \lambda_r U_r| \neq 0$. If we set $T = \lambda_1 U_1 + \cdots + \lambda_r U_r$, then $|T| \neq 0$ and

$$GT = T[C, \ldots, C],$$

i.e.

$$T^{-1}GT = [C, \ldots, C] = V^{-1}[[\gamma], \ldots, [\gamma]]V.$$

Our theorem is therefore proved. □

The theorem above gives us the exact form of the multipliers when the ring of multipliers is an algebraic number field. With this preliminary investigation regarding the ring of multipliers, we may now proceed to define the generalized half planes and the general modular group.

Let A be a $2n$-rowed non-singular rational alternate matrix. We denote by $\mathfrak{P}_A$, the set of all period matrices P which are polarized with respect to A. If $P \in \mathfrak{P}_A$, it is clear that, for n-rowed complex non-singular Q, $QP \in \mathfrak{P}_A$.

Now, if A_1 and A_2 are two such alternate matrices, then we can find a rational non-singular matrix S such that $A_1 = SA_2S'$. It is easy to verify that $\mathfrak{P}_{A_2}$ consists precisely of the period matrices of the form PS, for $P \in \mathfrak{P}_{A_1}$. Thus there is no loss of generality in confining ourselves to a fixed $2n$-rowed rational alternate matrix A with $|A| \neq 0$; we might take for A the matrix $\left(\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right)$ where E is the n-rowed identity matrix. **178**

Denoting by $\mathfrak{R}$, the group of n-rowed complex non-singular matrices, we know that if $P \in \mathfrak{P}_A$, then for every $Q \in \mathfrak{R}$, $QP \in \mathfrak{P}_A$. Now, we introduce an equivalence relation in $\mathfrak{P}_A$; namely, two matrices $P_1, P_2 \in \mathfrak{P}_A$ are *equivalent* if $P_1 = QP_2$, for $Q \in \mathfrak{R}$. The resulting set $\mathfrak{R}\backslash\mathfrak{P}_A$ of equivalence classes is denoted by $\mathfrak{H}_A$ and is called a **generalized half-plane.** It may be verified that $\mathfrak{H}_A$ can be identified with the space of n-rowed complex symmetric matrices $Z = X + iY$ with $Y > 0$.

Let Γ_A be the group of $2n$-rowed unimodular matrices U such that $UAU' = A$. If $P \in \mathfrak{P}_A$, then for $U \in \Gamma_A$, $PU \in \mathfrak{P}_A$. On $\mathfrak{P}_A$, the group Γ_A acts in the homogeneous manner; namely, if $P = (FG) \in \mathfrak{P}_A$ and $U = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \in \Gamma_A$ where A, B, C, D, F, G are n-rowed square matrices, then $PU = (FA+GB\ FC+GD)$. On $\mathfrak{H}_A$, Γ_A acts in the inhomogeneous way; namely, if $Z \in \mathfrak{H}_A$ and $U = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \in \Gamma_A$ then U takes Z to $(A + ZB)^{-1}(C + ZD)$. The group Γ_A is the well-known **modular group of degree** n. It has been shown by Siegel that Γ_A acts as a discontinuous group of analytic homeomorphisms of $\mathfrak{H}_A$ onto itself and that a nice fundamental domain $\mathfrak{F}_A$ can be constructed for Γ_A in $\mathfrak{H}_A$. It is known that $\mathfrak{F}_A$ is a so-called 'complex space'.

Let $\mathfrak{R}$ be the ring of multipliers of a Riemann matrix P polarized with respect to A. Corresponding to $M \in \mathfrak{R}$, there is a Q uniquely determined by (136). We denote this set of Q by $\mathfrak{D}$ and shall, hereafter, refer to a $Q \in \mathfrak{D}$ corresponding to an $M \in \mathfrak{R}$.

For the given A and $\mathfrak{R}$, we denote by $\mathfrak{P}_{A,\mathfrak{R}}$ the set of all n-rowed Riemann matrices P polarized with respect to A and having the property that for every

$M \in \mathfrak{R}$, $PM = QP$ where $Q \in \mathfrak{D}$ corresponds to $M \in \mathfrak{R}$. Clearly $\mathfrak{B}_{A,\mathfrak{R}}$ is not empty, since it contains the Riemann matrix P we started with. Moreover, let $\mathfrak{R}_{\mathfrak{R}}$ denote the group of $K \in \mathfrak{R}$ such that $QK = KQ$ for all $Q \in \mathfrak{D}$. It is obvious that if $P \in \mathfrak{B}_{A,\mathfrak{R}}$ and $K \in \mathfrak{R}_{\mathfrak{R}}$, then $KP \in \mathfrak{B}_{A,\mathfrak{R}}$. We now introduce in $\mathfrak{B}_{A,\mathfrak{R}}$ the equivalence relation that if P_1, $P_2 \in \mathfrak{B}_{A,\mathfrak{R}}$, then P_1 and P_2 are *equivalent* **179**
provided that $P_1 = KP_2$ for $K \in \mathfrak{R}_{\mathfrak{R}}$. The resulting set of equivalence classes is denoted by $\mathfrak{H}_{A,\mathfrak{R}}$ and called the *generalized half-plane corresponding to* $\mathfrak{R}$.

It is to be noted that $\mathfrak{H}_{A,\mathfrak{R}}$ may also be defined as follows: If $\mathfrak{B}^*_{A,\mathfrak{R}}$ denotes the set of all n-rowed Riemann matrices P polarized with respect to A and admitting for multipliers all elements of $\mathfrak{R}$, then we introduce in $\mathfrak{B}^*_{A,\mathfrak{R}}$ the equivalence relation that P_1, $P_2 \in \mathfrak{B}^*_{A,\mathfrak{R}}$ are equivalent if $P_1 = KP_2$ for $K \in \mathfrak{R}$. The resulting set of equivalence classes can be easily identified with $\mathfrak{H}_{A,\mathfrak{R}}$.

Let now $\Gamma_{A,\mathfrak{R}}$ be the subgroup of Γ_A, consisting of unimodular U such that $UM = MU$ for all $M \in \mathfrak{R}$. We call $\Gamma_{A,\mathfrak{R}}$ the *modular group corresponding to* $\mathfrak{R}$. Unless $\mathfrak{R}$ consists of just the trivial multipliers λE where $\lambda \in \mathbf{Q}$ and E is the $2n$-rowed identity matrix, $\Gamma_{A,\mathfrak{R}}$ is, in general, much smaller than Γ_A. Now, if $P \in \mathfrak{B}_{A,\mathfrak{R}}$ and $U \in \Gamma_{A,\mathfrak{R}}$, it is easy to see that $PU \in \mathfrak{B}_{A,\mathfrak{R}}$. The group $\Gamma_{A,\mathfrak{R}}$ acts on $\mathfrak{H}_{A,\mathfrak{R}}$ in the following way: namely, if $U \in \Gamma_{A,\mathfrak{R}}$, then U takes the class containing $P \in \mathfrak{B}_{A,\mathfrak{R}}$ into the class containing PU. It is known that in this way $\Gamma_{A,\mathfrak{R}}$ acts discontinuously on $\mathfrak{H}_{A,\mathfrak{R}}$ and one can construct a fundamental domain $\mathfrak{F}_{A,\mathfrak{R}}$ for $\Gamma_{A,\mathfrak{R}}$ in $\mathfrak{H}_{A,\mathfrak{R}}$.

The space $\mathfrak{H}_{A,\mathfrak{R}}$ is the most general half-plane and the group $\Gamma_{A,\mathfrak{R}}$ is the most general modular group of degree n. Modular groups of such general type were first considered by Siegel. The modular group of degree n over a totally real algebraic number field arose in a natural way in connection with the researches of Siegel on the analytic theory of quadratic forms over algebraic number fields. Later, the modular group of degree n over a simple involutorial algebra was discussed by Siegel fully, in his work, "Die Modulgruppe in einer einfachen involutorischen Algebra".

As indicated earlier, we are interested in finding $\mathfrak{H}_{A,\mathfrak{R}}$ and $\Gamma_{A,\mathfrak{R}}$ only when $\mathfrak{R}$ is a field i.e. an algebraic number field k of degree m, say, over $\mathbf{Q}$. We shall see presently that we are led in a natural fashion to the Hilbert modular group, when k is totally real.

First, we might make some preliminary simplification. We might suppose that if $M \in \mathfrak{R}$ corresponds to $\gamma \in k$, then M is of the form $[C, \ldots, C]$ where C is the image of γ under the regular representation. In fact, by Theorem 14, we can find a rational matrix T such that $T^{-1}MT = [C, \ldots, C]$ uniformly for **180**
all $M \in \mathfrak{R}$. Let us now set $A_1 = T^{-1}AT'^{-1}$ and let $\mathfrak{R}_1$ be the set of matrices of the form $T^{-1}MT$ for $M \in \mathfrak{R}$. Then $\mathfrak{R}_1$ gives an equivalent representation of k and all the matrices in $\mathfrak{R}_1$ are of the desired form. Moreover, $\mathfrak{B}_{A_1,\mathfrak{R}_1}$ consists

precisely of the matrices of the form PT where $P \in \mathfrak{B}_{A,\mathfrak{R}}$. Hence we might as well consider $\mathfrak{B}_{A_1,\mathfrak{R}_1}$ instead of $\mathfrak{B}_{A,\mathfrak{R}}$ and the corresponding half-plane $\mathfrak{H}_{A_1,\mathfrak{R}_1}$ instead of $\mathfrak{H}_{A,\mathfrak{R}}$. Thus, there is no loss of generality in our assumption regarding the multipliers M.

Now if $\gamma \in k$ corresponds to $M = [C, \ldots, C] \in \mathfrak{R}$ and $Q \in \mathfrak{D}$ corresponds to the multiplier M, then for $P \in \mathfrak{B}_{A,\mathfrak{R}}$ and $B = \binom{P}{\overline{P}}$, we have

$$B^{-1} \begin{pmatrix} Q & 0 \\ 0 & \overline{Q} \end{pmatrix} B = V^{-1}[[\gamma], \ldots, [\gamma]]V, \tag{141}$$

where $[\gamma] = [\gamma^{(1)}, \ldots, \gamma^{(m)}]$. Hence the eigenvalues $\gamma_1, \ldots, \gamma_n$ of Q are just conjugates of γ and on the other hand, *all* conjugates of γ occur among the eigenvalues of Q and $\overline{Q}$. Moreover, from (**??**), we see readily that we can find a non-singular matrix W uniformly for all $Q \in \mathfrak{D}$ such that

$$W^{-1}QW = [\gamma_1, \ldots, \gamma_n]. \tag{142}$$

We claim again that there is no loss of generality in assuming the complex multiplications $Q \in \mathfrak{D}$ to be already in the diagonal form as in (142). For, we could have started with the Riemann matrix $W^{-1}P$ instead of P right at the beginning and then the corresponding $\mathfrak{D}$ would have the desired property. We could then have taken the corresponding half-plane $\mathfrak{H}_{A,\mathfrak{R}}$.

We shall now study the effect of the involution in $\mathfrak{R}$ on the multiplications $Q \in \mathfrak{D}$. The involution in $\mathfrak{R}$ corresponds to an automorphism $\gamma \to \gamma^*$ in $k = \mathbf{Q}(\gamma)$. This automorphism in k may be extended to all the conjugates $k^{(i)}$ of k by prescription $(\gamma^{(i)})^* = (\gamma^*)^{(i)}$. Thus, if M, M^* are the multipliers in $\mathfrak{R}$ corresponding to γ and γ^* respectively, in k and Q, Q^* are the corresponding complex multiplications in $\mathfrak{D}$ then $Q = [\gamma_1, \ldots, \gamma_n]$ and $Q^* = [\gamma_1^*, \ldots, \gamma_n^*]$. On the other hand we know from p. 173 that

$$Q^* = \overline{HQ}'\overline{H}^{-1},$$

i.e. 181

$$Q^*\overline{H} = \overline{HQ}', \tag{143}$$

H being positive hermitian. If $h_1, \ldots, h_n$ are the diagonal elements of H, then $h_i > 0$ and moreover, from (**??**), we have

$$\gamma_k^* h_k = h_k \overline{\gamma}_k, k = 1, 2, \ldots, n,$$

i.e.

$$\gamma_k^* = \overline{\gamma}_k, k = 1, 2, \ldots, n.$$

Thus the automorphism in the conjugate fields $k^{(i)}$ is given by

$$\gamma^{(i)} \to (\gamma^{(i)})^{*} = \overline{\gamma^{(i)}}. \tag{144}$$

We now contend that k is either totally real or totally complex. For, if k has at least one real conjugate say $k^{(1)}$, then from (144), we see that the $*$-automorphism is identity on $k^{(1)}$ and hence on all conjugates of k. But then

$$\overline{\gamma^{(i)}} = (\gamma^{(i)})^{*} = \gamma^{(i)}.$$

i.e. all conjugates of k are real. Thus k is totally real and in the contrary case, k is totally complex. In the latter case, we know that the $*$-automorphism is not identity on k and has a fixed field $k_1 \in k$ such that k has degree 2 over k_1. But k_1 is necessarily totally real since the $*$-automorphism is identity on k_1 and all its conjugates. Thus, in the second case, k is an imaginary quadratic extension of a totally real field k_1 of degree $m/2$.

In the case when k is totally real, $q = 2n/m$ is necessarily even. For, if $M \in \mathfrak{R}$ corresponds to $\gamma \in k$ and if $Q \in \mathfrak{D}$ corresponds to $M \in \mathfrak{R}$, then the eigenvalues of Q are $\gamma_1, \ldots, \gamma_n$ and the eigenvalues of M are just $\gamma_1, \ldots, \gamma_n, \overline{\gamma}_1, \ldots, \overline{\gamma}_n$ i.e. $\gamma_1, \ldots, \gamma_n, \gamma_1, \ldots, \gamma_n$. But we know that the set $\gamma_1, \ldots, \gamma_n, \gamma_1, \ldots, \gamma_n$ has to run over the full set of conjugates $\gamma^{(1)}, \ldots, \gamma^{(m)}$ exactly q times. So necessarily that set $\gamma_1, \ldots, \gamma_n$ itself has to run over the complete set of conjugates $\gamma^{(1)}, \ldots, \gamma^{(m)}$ a certain number of times, i.e. in other words, q is necessarily even.

Thus when k is totally real, $n = (q/2)m$ and when k is totally complex, **182**
$n = (m/2)q$.

We are not interested in the latter case since, as it may be verified, the space $\mathfrak{H}_{A,\mathfrak{R}}$ in this case, is of complex dimension zero and we can have no useful function-theory in this space.

We shall therefore consider only the case when the ring of multipliers $\mathfrak{R}$ of the given n-rowed Riemann matrix P, gives a faithful representation of a totally real algebraic number field k of degree $m = n/(q/2)$ and moreover, we may suppose that m is as large as possible i.e. $q = 2$ and $m = n$. The other cases may be dealt with in a similar fashion.

We now prove the following

Theorem 15. *Let P be an n-rowed Riemann matrix polarized with respect to a rational alternate matrix A and let the ring of multipliers $\mathfrak{R}$ of P give a faithful representation of a totally real algebraic number field k of degree n over* $\mathbf{Q}$. Then the generalized half-plane $\mathfrak{H}_{A,\mathfrak{R}}$ *is the product of n complex half-planes and the modular group $\Gamma_{A,\mathfrak{R}}$ is of the type of the Hilbert modular group.*

Proof. (We shall retain, in the course of this proof, the simplification carried out already regarding $\mathfrak{R}$ and $\mathfrak{D}$ and shall follow our earlier notation throughout).

We know that if $k = \mathbf{Q}(\lambda)$, then, for $P \in \mathfrak{B}_{A,\mathfrak{R}}$,

$$[\lambda]PV^{-1} = PV^{-1}[[\lambda][\lambda]], \tag{145}$$

where $[\lambda] = [\lambda^{(1)}, \ldots, \lambda^{(n)}]$ and $V = [\Omega, \Omega]$.

Let us write PV^{-1} as $((r_{ij})(s_{ij}))$ where (r_{ij}) and (s_{ij}) are n-rowed square matrices. It follows from (145) then, that

$$r_{ij}\lambda^{(j)} = \lambda^{(i)}r_{ij},$$
$$s_{ij}\lambda^{(j)} = \lambda^{(i)}s_{ij}.$$

Since, for $i \neq j$, $\lambda^{(i)} \neq \lambda^{(j)}$ we see that PV^{-1} is necessarily of the form $([\xi][\eta])$ where $[\xi] = [\xi_1, \ldots, \xi_n]$ and $[\eta] = [\eta_1, \ldots, \eta_n]$. □

Moreover, since $M^* = M$, we have $MA = AM'$ for all $M \in \mathfrak{R}$. In particular, we see that

$$[[\lambda], [\lambda]]VAV' = VAV'[[\lambda], [\lambda]].$$

Splitting up VAV' and $\left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right)$ with n-rowed square matrices K, L, M, N, we see **183**
again, by the same argument as above, that

$$VAV' = \begin{pmatrix} [\mu] & [\nu] \\ [\kappa] & [\rho] \end{pmatrix}$$

where $[\mu]$, $[\nu]$, $[\kappa]$ and $[\rho]$ and n-rowed diagonal matrices. Since A is alternate, necessarily we have

$$VAV' = \begin{pmatrix} 0 & [\nu] \\ -[\nu] & 0 \end{pmatrix},$$

where $[\nu] = [\nu_1, \ldots, \nu_n]$. But now since A is rational, $\Omega^{-1}[\nu]\Omega'^{-1}$ is rational. As a consequence, $\Omega^{-1}[\nu]\Omega$ is again rational and hence by an earlier argument (see p. 174), we have $[\nu] = [\nu^{(1)}, \ldots, \nu^{(n)}]$ for a $\nu \in k$. Thus

$$A = \begin{pmatrix} 0 & \Omega^{-1}[\nu]\Omega'^{-1} \\ -\Omega^{-1}[\nu]\Omega'^{-1} & 0 \end{pmatrix}.$$

It is to be noted that $\nu \neq 0$ and further $\Omega^{-1}[\nu]\Omega'^{-1}$ is not diagonal, in general.

Let now $P \in \mathfrak{B}_{A,\mathfrak{R}}$ be the representative of a point in $\mathfrak{H}_{A,\mathfrak{R}}$. Then from the fact that $i^{-1}\overline{P}AP' > 0$, we obtain

$$i^{-1}\overline{PV}^{-1}VAV'(PV^{-1})' > 0,$$

i.e.

$$i^{-1}([\bar{\xi}][\bar{\eta}])\begin{pmatrix} 0 & [\nu] \\ -[\nu] & 0 \end{pmatrix}([\xi][\eta]) > 0,$$

i.e.

$$0 < h_k = \nu^{(k)} i^{-1}(\bar{\xi}_k \eta_k - \bar{\eta}_k \xi_k), \quad \text{for } k = 1, 2, \ldots, n.$$

If we set $\tau_j = \eta_j \xi_j^{-1}$, $j = 1, 2, \ldots, n$ and $[\tau] = [\tau_1, \ldots, \tau_n]$, then

$$[\xi]^{-1} P V^{-1} = (E^{(n)} [\tau]) \tag{146}$$

and the conditions above are just

$$\nu^{(j)} \operatorname{Im}(\tau_j) > 0, \ j = 1, 2, \ldots, n.$$

Now $[\xi]$ commutes with all the multiplications $Q \in \mathfrak{Q}$ since they are in the **184**
diagonal form. Hence we could take $[\xi]^{-1}P$ instead of P as a representative of the corresponding point in $\mathfrak{H}_{A,\mathfrak{R}}$. Thus, in view of (146), since V is independent of P, the space $\mathfrak{H}_{A,\mathfrak{R}}$ may be identified as the set of n-tuples $(z_1, \ldots, z_n)$ with complex z_i satisfying $\nu^{(i)} \operatorname{Im}(z_i) > 0$ i.e. $\mathfrak{H}_{A,\mathfrak{R}}$ is *a product of n half-planes.* If $\nu > 0$, then $\mathfrak{H}_{A,\mathfrak{R}}$ is just the product of n upper half-planes. For the general theory, we should take $\mathfrak{H}_{A,\mathfrak{R}}$ as the product of n (not necessarily upper) half-planes. The first statement in our theorem has thus been proved.

We shall now determine in explicit terms, the group $\Gamma_{A,\mathfrak{R}}$ which consists of $2n$-rowed unimodular matrices U such that $UAU' = A$ and $UM = MU$ for all $M \in \mathfrak{R}$.

Since $UM = MU$ for all $M \in \mathfrak{R}$, we have, in particular,

$$UV^{-1}[[\lambda][\lambda]]V = V^{-1}[[\lambda][\lambda]]VU,$$

i.e.

$$VUV^{-1}[[\lambda][\lambda]] = [[\lambda][\lambda]]VUV^{-1}.$$

By the same argument as above, we have necessarily

$$VUV^{-1} = \begin{pmatrix} [\delta] & [\beta] \\ [\gamma] & [\alpha] \end{pmatrix},$$

where $[\alpha]$, $[\beta]$, $[\gamma]$, $[\delta]$ are diagonal matrices. Again, since the elements of $\Omega^{-1}[\delta]\Omega$ are rational integers, it follows by an earlier remark (see p. 174) that $[\delta] = [\delta^{(1)}, \ldots, \delta^{(n)}]$ for a $\delta \in k$. Similarly, $[\beta] = [\beta^{(1)}, \ldots, \beta^{(n)}]$, $[\gamma] = [\gamma^{(1)}, \ldots, \gamma^{(n)}]$ and $[\alpha] = [\alpha^{(1)}, \ldots, \alpha^{(n)}]$ for $\alpha, \beta, \gamma \in k$. Let us denote by $\mathfrak{D}$ the order in k, consisting of numbers $\rho \in k$ for which the matrix $\Omega^{-1}[\rho^{(1)}, \ldots, \rho^{(n)}]\Omega$

is a rationla integral matrix. Then $\alpha, \beta, \gamma, \delta \in \mathfrak{D}$. The order $\mathfrak{D}$ coincides with the ring $\mathfrak{o}$ of all algebraic integers in k, when $\omega_1, \ldots, \omega_n$ is a basis of $\mathfrak{o}$.

Now, from $UAU' = A$, we have

$$\begin{pmatrix} [\delta] & [\beta] \\ [\gamma] & [\alpha] \end{pmatrix} \begin{pmatrix} 0 & [\nu] \\ -[\nu] & 0 \end{pmatrix} \begin{pmatrix} [\delta] & [\gamma] \\ [\beta] & [\alpha] \end{pmatrix} = \begin{pmatrix} 0 & [\nu] \\ -[\nu] & 0 \end{pmatrix}$$

i.e. 185

$$\nu^{(k)}(\alpha^{(k)}\delta^{(k)} - \beta^{(k)}\gamma^{(k)}) = \nu^{(k)}, k = 1, 2, \ldots, n,$$

i.e.

$$\alpha\delta - \beta\gamma = 1.$$

Thus $\Gamma_{A,\mathfrak{R}}$ consists of $2n$-rowed matrices of the form

$$V^{-1} \begin{pmatrix} [\delta] & [\beta] \\ [\gamma] & [\alpha] \end{pmatrix} V$$

where $\alpha, \beta, \gamma, \delta$ are elements of $\mathfrak{D}$ satisfying $\alpha\delta - \beta\gamma = 1$.

We know how $\Gamma_{A,\mathfrak{R}}$ acts on $\mathfrak{B}_{A,\mathfrak{R}}$ namely, if $U \in \Gamma_{A,\mathfrak{R}}$, then U takes $P \in \mathfrak{B}_{A,\mathfrak{R}}$ to PU. Now we may describe the action of $\Gamma_{A,\mathfrak{R}}$ on $\mathfrak{H}_{A,\mathfrak{R}}$ as follows. If the point $(\tau_1, \ldots, \tau_n) \in \mathfrak{H}_{A,\mathfrak{R}}$ corresponds to the matrix PV^{-1} with $P \in \mathfrak{B}_{A,\mathfrak{R}}$, then the element U of $\Gamma_{A,\mathfrak{R}}$ maps $(\tau_1, \ldots, \tau_n)$ on the point in $\mathfrak{H}_{A,\mathfrak{R}}$ which corresponds to PUV^{-1}. But now

$$\begin{aligned} PUV^{-1} &= PV^{-1}VUV^{-1} \\ &= [\xi](E[\tau]) \begin{pmatrix} [\delta] & [\beta] \\ [\gamma] & [\alpha] \end{pmatrix} \\ &= [\xi]([\delta] + [\gamma][\tau][\beta] + [\alpha][\tau]). \end{aligned}$$

Hence to PUV^{-1} corresponds the point

$$\left(\frac{\alpha^{(1)}\tau_1 + \beta^{(1)}}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, \ldots, \frac{\alpha^{(n)}\tau_n + \beta^{(n)}}{\gamma^{(n)}\tau_n + \delta^{(n)}} \right)$$

in $\mathfrak{H}_{A,\mathfrak{R}}$. Thus $\Gamma_{A,\mathfrak{R}}$ is represented as the group of fractional linear transformations

$$(\tau_1, \ldots, \tau_n) \to \left(\frac{\alpha^{(1)}\tau_1 + \beta^{(1)}}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, \ldots, \frac{\alpha^{(n)}\tau_n + \beta^{(n)}}{\gamma^{(n)}\tau_n + \delta^{(n)}} \right)$$

(of $\mathfrak{H}_{A,\mathfrak{R}}$ onto itself) with $\alpha, \beta, \delta \in \mathfrak{D}$ such that $\alpha\delta - \beta\gamma = 1$. If $\omega_1, \ldots, \omega_n$ is a basis of $\mathfrak{o}$, $\Gamma_{A,\mathfrak{R}}$ is the well-known **Hilbert modular group.** In other cases, $\Gamma_{A,\mathfrak{R}}$ is seen to be commensurable with the Hilbert modular group.

Theorem 15 is therefore completely proved. **186**

We have considered above only the case $m = n$. The case when $n/m = p > 1$ can be dealt with in a similar fashion. Here $\mathfrak{H}_{A,\mathfrak{R}}$ is a product of n generalized half-planes of degree p and $\Gamma_{A,\mathfrak{R}}$ is the group of mappings

$$Z = (Z_1, \ldots, Z_n) \in \mathfrak{H}_{A,\mathfrak{R}} \to (Z_1^*, \ldots, Z_n^*) \in \mathfrak{H}_{A,\mathfrak{R}}$$

where $Z_i^* = (A^{(i)}Z_i + B^{(i)})(C^{(i)}Z_i + D^{(i)})^{-1}$ and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a modular matrix of degree p, with elements which are algebraic integers in k. The field of modular functions belonging to $\Gamma_{A,\mathfrak{R}}$ in this case has been systematically investigated by I.I. Piatetskii-Shapiro in recent years.

In the succeeding sections, we shall define the Hilbert modular function and one of the main results which we wish to prove is that the Hilbert modular functions form an algebraic function field of n variables. Our proof will run essentially on the same lines as that of K.B. Gundlach. But first we need to construct a workable fundamental domain in the space $\mathfrak{H}_{A,\mathfrak{R}}$ for the Hilbert modular group. This shall be done in the next section.

2 Fundamental domain for the Hilbert modular group

Let K be a totally real algebraic number field of degree n over $\mathbf{Q}$, the field of rational numbers and let $K^{(1)}(= K), K^{(2)}, \ldots, K^{(n)}$ be the conjugates of K. For our purposes in future, we first extend K to $\widehat{K}$ by adding an element ∞ which satisfies the following conditions, namely

$$\begin{aligned} a + \infty &= \infty, \text{ for a complex number } a \\ a \cdot \infty &= \infty, \text{ for complex } a \neq 0 \\ \infty \cdot \infty &= \infty \\ \frac{1}{0} &= \infty \\ \frac{1}{\infty} &= 0. \end{aligned}$$

(The expressions $\infty \pm \infty$, $0 \cdot \infty$ and ∞/∞ are undefined). We define the conjugates $\infty^{(i)}$ of ∞ to be ∞, again.

Corresponding to each $K^{(i)}$, $i = 1, 2, \ldots, n$, we associate a variable z_i in **187**
the extended complex plane subject to the restriction that if one z_i is ∞, all the other z_i are also ∞. We shall denote the n-tuple $(z_1, \ldots, z_n)$ by z and for $\lambda \in \widehat{K}$, we shall denote the n-tuple $(\lambda^{(1)}, \ldots, \lambda^{(n)})$ again by λ, when there is no risk of

confusion. For given $z = (z_1, \ldots, z_n)$ the *norm* $N(z)$ shall stand for $z_1, \ldots, z_n$ and the *trace* $S(z)$ for $z_1 + \cdots + z_n$. If $\lambda \in K$, $N(\lambda)$ and $S(\lambda)$ coincide with the usual norm and trace in K respectively.

Let $\mathfrak{S}$ denote the group of 2-rowed square matrices $S = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with α, β, γ, $\delta \in K$ and $\alpha\delta - \beta\gamma = 1$. Let $\mathfrak{E}$ denote the normal subgroup of $\mathfrak{S}$ consisting of $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Then, for the factor group $\mathfrak{S}/\mathfrak{E}$, we have a faithful representation as the group of mappings

$$z = (z_1, \ldots, z_n) \to z_M = (z_1^*, \ldots, z_n^*)$$

with

$$z_j^* = \frac{\alpha^{(j)} z_j + \beta^{(j)}}{\gamma^{(j)} z_j + \delta^{(j)}}$$

corresponding to each $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathfrak{S}$. We shall denote z_M as $(\alpha z + \beta)/(\gamma z + \delta)$ symbolically. Let, further, $z_j = x_j + iy_j$, $z_j^* = x_j^* + iy_j^*$. Writing $z = x + iy$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we shall denote $(y_1^*, \ldots, y_n^*)$ by y_M and $(x_1^*, \ldots, x_n^*)$ by x_M so that $z_M = x_M + iy_M$. It is easy to see that for M_1, $M_2 \in \mathfrak{S}$,

$$z_{M_1 M_2} = (z_{M_2})M_1.$$

As usual, for $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathfrak{S}$, $\infty_M = \left(\dfrac{\alpha^{(1)}}{\gamma^{(1)}}, \ldots, \dfrac{\alpha^{(n)}}{\gamma^{(n)}}\right)$.

Let $\mathfrak{M}$ be the subgroup of $\mathfrak{S}$ consisting of $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with $\alpha, \beta, \delta \in \mathfrak{o}$ (the ring of integers in K). The factor group $\mathfrak{M}/\mathfrak{E}$ is precisely the (inhomogeneous) Hilbert modular group, which we shall denote hereafeter by Γ.

One can consider more generally than $\mathfrak{M}$, the group $\mathfrak{M}_0$ of 2-rowed square matrices $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ where α, β, γ, $\delta \in \mathfrak{o}$ and $\alpha\delta - \beta\gamma = \epsilon$ with ϵ being a totally positive unit in K. Let $\mathfrak{E}_0$ be the subgroup of $\mathfrak{M}_0$ consisting of matrices of the **188**
form $\left(\begin{smallmatrix}\epsilon & 0\\ 0 & \epsilon\end{smallmatrix}\right)$ where ϵ is a unit in K. Further, let $\rho_1 (= 1), \rho_2, \ldots, \rho_\nu$ be a complete set of representatives of the group of units $\epsilon > 0$ in K modulo the subgroup of squares of units in K. Then it is clear that $\mathfrak{M}_0/\mathfrak{E}_0$ is isomorphic to the group of substitutions $z \to \rho z_M$ where $z \to z_M$ is a Hilbert modular substitution and $\rho = \rho_i$ for some i. Thus the study of $\mathfrak{M}_0/\mathfrak{E}_0$ can be reduced to that of Γ. The group Γ is, in general, "smaller than" $\mathfrak{M}_0/\mathfrak{E}_0$ and is called, therefore the *narrow* Hilbert modular group, usually.

Two elements λ, μ in $\widehat{K}$ are *equivalent* (in symbols, $\lambda \sim \mu$), if for some $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathfrak{M}$,

$$\mu = \lambda_M = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}.$$

This is a genuine equivalence relation. We shall presently show that K falls into h equivalence classes, where h is the class number of K.

Proposition 20. *There exist $\lambda_1, \ldots, \lambda_h \in \widehat{K}$ such that for any $\lambda \in \widehat{K}$, we have $\lambda \sim \lambda_i$ for some i uniquely determined by λ.*

Proof. Any $\lambda \in \widehat{K}$ is of the form ρ/σ where ρ, $\sigma \in \mathfrak{o}$. (If $\lambda = \infty$, we may take $\rho = 1$ and $\sigma = 0$). With λ, we may associate the integral ideal $\mathfrak{a} = (\rho, \sigma)$. We first see that if $\mathfrak{a}_1 = (\theta)\mathfrak{a}$ is any integral ideal in the class of $\mathfrak{a}$, then $\mathfrak{a}_1$ is of the form (ρ_1, σ_1) where ρ_1 and σ_1 are in $\mathfrak{o}$ such that $\lambda = \rho_1/\sigma_1$. This is quite obvious, since $\mathfrak{a}_1 = (\rho\theta, \sigma\theta)$ and taking $\rho_1 = \rho\theta$, $\sigma_1 = \sigma\theta$, our assertion is proved. Conversely, any integral ideal $\mathfrak{a}_1$ associated with $\lambda \in \widehat{K}$ in this way is necessarily in the same ideal-class as $\mathfrak{a}$. In fact, let λ be written in the form ρ_1/σ_1 with ρ_1, $\sigma_1 \in \mathfrak{o}$ and let $\mathfrak{a}_1 = (\rho_1, \sigma_1)$. Then we claim $\mathfrak{a}_1$ is in the same class as $\mathfrak{a}$. For, since $\lambda = \rho/\sigma = \rho_1/\sigma_1$, we have $\rho\sigma_1 = \rho_1\sigma$ and hence

$$\frac{(\rho)}{\mathfrak{a}}\frac{(\sigma_1)}{\mathfrak{a}_1} = \frac{(\rho_1)}{\mathfrak{a}_1}\frac{(\sigma)}{\mathfrak{a}}$$

Now $(\rho)/\mathfrak{a}$ and $(\sigma)/\mathfrak{a}$ as also $(\rho_1)/\mathfrak{a}_1$ and $(\sigma_1)/\mathfrak{a}_1$ are mutually coprime. Hence we have $(\rho)/\mathfrak{a} = (\rho_1)/\mathfrak{a}_1$ and similarly $(\sigma)/\mathfrak{a} = (\sigma_1)/\mathfrak{a}_1$. This means that for $a\theta \in K$, $\rho_1 = \rho\theta$ and hence $\sigma_1 = \sigma\theta$. Thus $\mathfrak{a}_1 = (\theta)\mathfrak{a}$, for $\theta \in K$. □

We choose in the h ideal classes, fixed integral ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ such that **189**
$\mathfrak{a}_i$ is of minimum norm among all the integral ideals of its class. (Perhaps, the ideal $\mathfrak{a}_i$ is not uniquely fixed in its class by this condition but there are at most finitely many possibilities for $\mathfrak{a}_i$ and we choose from these, a fixed $\mathfrak{a}_i$). It follows from the above that to a given $\lambda \in \widehat{K}$, we can make correspond an ideal $\mathfrak{a}_i$ such that $\lambda = \rho/\sigma$ and $\mathfrak{a}_i = (\rho, \sigma)$ for suitable ρ, $\sigma \in \mathfrak{o}$. Now, if $\mu = \dfrac{\alpha\lambda + \beta}{\gamma\lambda + \delta} \sim \lambda$, then to μ again corresponds the ideal $(\alpha\rho + \beta\sigma, \gamma\rho + \delta\sigma)$ which is just $\mathfrak{a}_i$ since $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is unimodular. Thus all elements of an equivalence class in $\widehat{K}$ correspond to the same ideal.

We shall now show that if the same ideal $\mathfrak{a}_i$ corresponds to λ, $\lambda^* \in \widehat{H}$, then necessarily $\lambda \sim \lambda^*$. Let, in fact, $\lambda = \rho/\sigma$, $\lambda^* = \rho^*/\sigma^*$ and $\mathfrak{a}_i = (\rho, \sigma) = (\rho^*, \sigma^*)$. It is well-known that there exist numbers ξ, η, ξ^*, η^* in $\mathfrak{a}_i^{-1}$ such that $\rho\eta - \sigma\xi = 1$ and $\rho^*\eta^* - \sigma^*\xi^* = 1$. Let $A = \left(\begin{smallmatrix}\rho & \xi\\ \sigma & \eta\end{smallmatrix}\right)$ and

$$A^* = \begin{pmatrix}\rho^* & \xi^*\\ \sigma^* & \eta^*\end{pmatrix};$$

then $A, A^* \in \mathfrak{S}$ and moreover

$$A^*A^{-1} = \begin{pmatrix}\rho^* & \xi^*\\ \sigma^* & \eta^*\end{pmatrix}\begin{pmatrix}\eta & -\xi\\ -\sigma & \rho\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

where

$$\alpha = \rho^*\eta - \xi^*\sigma, \beta = -\rho^*\xi + \xi^*\rho, \gamma = \sigma^*\eta - \eta^*\sigma, \delta = -\sigma^*\xi + \eta^*\rho$$

clearly are integers in K satisfying $\alpha\delta - \beta\gamma = 1$. Hence $A^*A^{-1} \in \mathfrak{M}$ and thus

$$\lambda^* = \frac{\rho^*}{\sigma^*} = \frac{\alpha\rho + \beta\sigma}{\gamma\rho + \delta\sigma} = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta} \sim \lambda.$$

Thus, we see that to different equivalence classes in $\widehat{H}$ correspond different ideals $\mathfrak{a}_i$. It is almost trivial to verify that to each ideal $\mathfrak{a}_i$, there corresponds an equivalence class in $\widehat{K}$. As a consequence, there are exactly h equivalence classes in $\widehat{K}$ and our proposition is proved.

We now make a convention to be followed in the sequel. We shall assume **190** that $\lambda_1 = 1/0 = \infty$, without loss of generality. Moreover, with $\lambda_i = \rho_i/\sigma_i$, we associated a *fixed* matrix, $A_i = \begin{pmatrix} \rho_i & \xi_i \\ \sigma_i & \eta_i \end{pmatrix} \in \mathfrak{S}$; let us remark that it is always possible to find, though not uniquely, numbers ξ_i, η_i in $\mathfrak{a}_i^{-1}$ such that $\rho_i\eta_i - \xi_i\sigma_i = 1$. Further, we shall suppose that $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let, for $\lambda \in \widehat{K}$, Γ_λ denote the group of Hilbert modular substitutions $z \to (\alpha z + \beta)/(\gamma z + \delta)$ such that $(\alpha\lambda + \beta)/(\gamma\lambda + \delta) = \lambda$. For our use later, we need to determine Γ_λ explicitly. It clearly suffices to find Γ_{λ_i} (for $i = 1, 2, \ldots, h$) since $\lambda = (\lambda_i)_M$ for some $M \in \Gamma$ and $\Gamma_\lambda = M\Gamma_{\lambda_i}M^{-1}$. Let then for

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{M}, \frac{\alpha\lambda_i + \beta}{\gamma\lambda_i + \delta} = \lambda_i.$$

We have $(\alpha\rho_i + \beta\sigma_i, \gamma\rho_i + \delta\sigma_i) = \mathfrak{a}_i = (\rho_i, \sigma_i)$. And since $(\alpha\rho_i + \beta\sigma_i)\sigma_i = (\gamma\rho_i + \delta\sigma_i)\rho_i$, we have

$$\frac{(\alpha\rho_i + \beta\sigma_i)}{\mathfrak{a}_i}\frac{(\sigma_i)}{\mathfrak{a}_i} = \frac{(\gamma\rho_i + \delta\sigma_i)}{\mathfrak{a}_i}\frac{(\rho_i)}{\mathfrak{a}_i}.$$

As before,

$$\frac{(\alpha\rho_i + \beta\sigma_i)}{\mathfrak{a}_i} = \frac{(\rho_i)}{\mathfrak{a}_i} \quad \text{and} \quad \frac{(\gamma\rho_i + \delta\sigma_i)}{\mathfrak{a}_i} = \frac{(\sigma_i)}{\mathfrak{a}_i}.$$

Hence, for a unit ϵ in K, we have

$$\alpha\rho_i + \beta\sigma_i = \epsilon\rho_i, \gamma\rho_i + \delta\sigma_i = \epsilon\sigma_i.$$

This means that

$$MA_i = \begin{pmatrix} \rho_i & \xi_i^* \\ \sigma_i & \eta_i^* \end{pmatrix}\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix},$$

where $\xi_i^* = (\alpha\xi_i + \beta\eta_i)\epsilon$ and $\eta_i^* = (\gamma\xi_i + \delta\eta_i)\epsilon$ lie in $\mathfrak{a}_i^{-1}$. Further since $\rho_i\eta_i - \sigma_i\xi_i = 1 = \rho_i\eta_i^* - \sigma_i\xi_i^*$, we have

$$\rho_i(\eta_i^* - \eta_i) = \sigma_i(\xi_i^* - \xi_i),$$

i.e.

$$\frac{(\rho_i)}{\mathfrak{a}_i}\frac{(\eta_i^* - \eta_i)}{\mathfrak{a}_i^{-1}} = \frac{(\sigma_i)}{\mathfrak{a}_i}\frac{(\xi_i^* - \xi_i)}{\mathfrak{a}_i^{-1}}.$$

Again, since $(\rho_i)/\mathfrak{a}_i$ is coprime to $(\sigma_i)/\mathfrak{a}_i$, we see that $(\rho_i)/\mathfrak{a}_i$ divides $\dfrac{(\xi_i^* - \xi_i)}{\mathfrak{a}_i^{-1}}$, **191**
i.e. $(\xi_i^* - \xi_i) = \mathfrak{a}_i^{-2}\mathfrak{i}(\rho_i)$ for an integral ideal $\mathfrak{i}$. In other words,

$$\xi_i^* = \xi_i + \rho_i\zeta, \zeta \in \mathfrak{a}_i^{-2}.$$

As a result, we obtain also

$$\eta_i^* = \eta_i + \sigma_i\zeta.$$

We now observe that

$$MA_i = A_i\begin{pmatrix}\epsilon & \zeta\epsilon^{-1}\\ 0 & \epsilon^{-1}\end{pmatrix}.$$

Thus Γ_{λ_i} consists precisely of the modular substitutions $z \to z_M$, where $M = A_i\left(\begin{smallmatrix}\epsilon & \zeta\\ 0 & \epsilon^{-1}\end{smallmatrix}\right)A_i^{-1} \in \mathfrak{M}$ with $\zeta \in \mathfrak{a}_i^{-2}$ and ϵ any unit in K.

Let $\mathfrak{H}_n$ denote the product of n upper half-planes, namely, the set of $z = (z_1, \ldots, z_n)$ with $z_j = x_j + iy_j$, $y_j > 0$. The Hilbert modular group Γ has a representation as a group of analytic homeomorphisms $z \to (\alpha z + \beta)/(\gamma z + \delta)$ of $\mathfrak{H}_n$ onto itself. In the following, we shall freely identify $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathfrak{M}$ with the modular substitution $z \to z_M$ in Γ and speak of M belonging to Γ, without risk of confusion.

The points $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ for $\lambda \in \widehat{K}$ lie on the boundary of $\mathfrak{H}_n$ and are called (parabolic) *cusps* of $\mathfrak{H}_n$. By Proposition 20, there exist h cusps, $\lambda_1 = (\infty, \ldots, \infty)$, $\lambda_2 = (\lambda_2^{(1)}, \ldots, \lambda_2^{(n)}), \ldots, \lambda_h = (\lambda_h^{(1)}, \ldots, \lambda_h^{(n)})$ which are not equivalent with respect to Γ and such that any other (parabolic) cusp of $\mathfrak{H}_n$ is equivalent to exactly one of them; $\lambda_1, \ldots, \lambda_h$ are called the *base cusps*.

If $V \subset C^n$, then for given $M \in \Gamma$, V_M shall denote the set of all z_M for $z \in V$. For any $z \in \mathfrak{H}_n$, Γ_z shall stand for the *isotropy group* of z in Γ, namely the group of $M \in \Gamma$, for which $z_M = z$. We shall conclude from the following proposition that Γ_z is finite, for all $z \in \mathfrak{H}_n$.

Proposition 21. *For any two compact sets B, B' in $\mathfrak{H}_n$, the number of $M \in \Gamma$ for which B_M intersects B' is finite.*

Proof. Let Λ denote the set of $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ such that B_M intersects B' i.e. given $M \in \Lambda$, there exists $w = (w_1, \ldots, w_n)$ in B such that $w_M = (w'_1, \ldots, w'_n)$ 192
$\in B'$. Let $w_j = u_j + iv_j$ and $w'_j = u'_j + iv'_j$; then $v'_j = v_j |\gamma^{(j)} w_j + \delta^{(j)}|^{-2}$. Since B and B' are compact, we see that

$$|\gamma^{(j)} w_j + \delta^{(j)}| \leq c, \ j = 1, 2, \ldots, n$$

for a constant c depending only on B and B'. But then it follows immediately that γ, δ belong to a finite set of integers in K. Let $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and let B^* and B'^* denote the images of B and B' respectively under the modular substitution $z \to z_I = -z^{-1}$. The images B^* and B'^* are again compact. Further noting that if $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, then $IMI' = M'^{-1} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$, we can show easily that $B_M \cap B' \neq \emptyset$ if and only if $B^*_{M'-1} \cap B'^* \neq \emptyset$. Now applying the same argument as above to the compact sets B^* and B'^*, we may conclude that if, for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, $(B^*)_{M'-1} \cap B'^* \neq \emptyset$, then α, β belong to a finite set of integers in K. As a result, we obtain finally that Λ is finite. ◻

Taking a point $z \in \mathfrak{H}_n$ instead of B and B', we deduce that Γ_z is finite.

We are now in a position to prove the following

Theorem 16. *The Hilbert modular group Γ acts properly discontinuously on $\mathfrak{H}_n$; in other words, for any $z \in \mathfrak{H}_n$, there exists a neighbourhood V of z such that only for finitely many $M \in \Gamma$, V_M intersects V and when $V_M \cap V \neq \emptyset$, then $M \in \Gamma_z$.*

Proof. Let $z \in \mathfrak{H}_n$ and V a neighbourhood of z such that the closure $\overline{V}$ of V in $\mathfrak{H}_n$ is compact. Taking $\overline{V}$ for B and B' in Proposition 21, we note that only for finitely many $M \in \Gamma$, say $M_1, \ldots, M_r, V_M$ intersects V. Among these M_i, let $M_1, \ldots, M_s$ be exactly those which do not belong to Γ_z. Then we can find a neighbourhood W of z such that $W_{M_i} \cap W = \emptyset$, $i = 1, 2, \ldots, s$. Now let $U = W \cap V$; then U has already the property that for $M \neq M_i (i = 1, 2, \ldots, r) U_M \cap U = \emptyset$. Further $U_{M_i} \cap U = \emptyset$, for $i = 1, 2, \ldots, s$. Thus U satisfies the requirements of the theorem. ◻

If z is not a fixed point of any M in r except the identity of in other words, if Γ_z consists only of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then there exists a neighbourhood V z such that for $M \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V_M \cap V = \emptyset$.

Let $\lambda = \rho/\sigma$ be a cusp of $\mathfrak{H}_n$ and $\mathfrak{a} = (\rho, \sigma)$ be the integral ideal among 193
$\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ which is associated with λ. Let $\alpha_1, \ldots, \alpha_n$ be a minimal basis for $\mathfrak{a}^{-2}$ and further, associated with λ, let us choose a fixed $A = \begin{pmatrix} \rho & \xi \\ \sigma & \eta \end{pmatrix} \in \mathfrak{S}$ with ξ, η lying in $\mathfrak{a}^{-1}$. Moreover, let $\epsilon_1, \ldots, \epsilon_{n-1}$ be $n - 1$ independent generators of the

group of units in K. As a first step towards constructing a fundamental domain for Γ in $\mathfrak{H}_n$, we shall introduce 'local coordinates' relative to λ, for every point z in $\mathfrak{H}_n$ and then construct a fundamental domain $\mathfrak{G}_\lambda$, for Γ_λ in $\mathfrak{H}_n$.

Let $z = (z_1, \ldots, z_n)$ be any point of $\mathfrak{H}_n$ and $z_{A^{-1}} = (z_1^*, \ldots, z_n^*)$ with $z_j^* = x_j^* + iy_j^*$. Denoting $y_{A^{-1}} = (y_1^*, \ldots, y_n^*)$ by y^*, we define the local coordinates of z relative to λ by the $2n$ quantities

$$1/\sqrt{N(y^*)}, Y_1, \ldots, Y_{n-1}, X_1, \ldots, X_n,$$

where $Y_1, \ldots, Y_{n-1}, X_1, \ldots, X_n$ are uniquely determined by the linear equations

$$\left.\begin{aligned} Y_1 \log|\epsilon_1^{(k)}| + \cdots + Y_{n-1} \log|\epsilon_{n-1}^{(k)}| &= \frac{1}{2}\log\frac{y_k^*}{\sqrt[n]{N(y^*)}}, \\ k &= 1, 2, \ldots, n-1 \\ X_1\alpha_1^{(1)} + \cdots + X_n\alpha_n^{(1)} &= x_l^*, l = 1, 2, \ldots, n. \end{aligned}\right\} \tag{147}$$

The group Γ_λ consists of all the modular substitutions of the form $z \to z_M$ where $M = ATA^{-1}$ and $T = \begin{pmatrix}\epsilon\zeta\epsilon^{-1}\\ 0\zeta\epsilon^{-1}\end{pmatrix}$ with ϵ, a unit in K and $\zeta \in \mathfrak{a}^{-2}$. The transformation $z \to z_M$ is equivalent to the transformation $z_{A^{-1}} \to z_{TA^{-1}} = \epsilon^2 z_{A^{-1}} + \zeta$. It is easily verified that Γ_λ is generated by the 'dilatations' $z_{A^{-1}} \to \epsilon_i^2 z_{A^{-1}}$, $i = 1, 2, \ldots, n-1$ and the 'translations' $z_{A^{-1}} \to z_{A^{-1}} + \alpha_j$, $j = 1, 2, \ldots, n$.

Let now $z \to z_M$ with $M = ATA^{-1}$, $T = \begin{pmatrix}\epsilon\zeta\epsilon^{-1}\\ 0\zeta\epsilon^{-1}\end{pmatrix}$ be a modular substitution in Γ_λ and let $\epsilon = \pm\epsilon_1^{k_1} \ldots \epsilon_{n-1}^{k_{n-1}}$ and $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$, where $k_1, \ldots, k_{n-1}$, $m_1, \ldots, m_n$ are rational integers. It is obvious that the first coordinate $1/\sqrt{N(y^*)}$ of z is preserved by the modular substitution $z \to z_M$ since 194
$N(y_{A^{-1}M}) = N(y_{TA^{-1}}) = N(\epsilon^2)N(y_{A^{-1}}) = N(y^*)$. The effect of the substitution $z \to z_M$ on $Y_1, \ldots, Y_{n-1}$ is given by

$$(Y_1, Y_2, \ldots, Y_{n-1}) \to (Y_1 + k_1, Y_2 + k_2, \ldots, Y_{n-1} + k_{n-1}), \tag{148}$$

as can be verified from (147). If $\epsilon^2 = 1$, then the effect of the substitution $z \to z_M$ on $X_1, \ldots, X_n$ is again just a 'translation',

$$(X_1, \ldots, X_n) \to (X_1 + m_1, \ldots, X_n + m_n). \tag{149}$$

If $\epsilon^2 \neq 1$, then the effect of the substitution $z \to z_M$ on $X_1, \ldots, X_n$ is not merely a 'translation' but an affine transformation given by

$$(X_1, \ldots, X_n) \to (X_1^* + m_1, \ldots, X_n^* + m_n),$$

where $(X_1^* X_2^* \ldots X_n^*) = (X_1 X_2 \ldots X_n)UVU^{-1}$, U denotes the n-rowed square matrix $(\alpha_i^{(j)})$ and V is the diagonal matrix $[\epsilon^{(1)^2}, \ldots, \epsilon^{(n)^2}]$.

We define a point $z \in \mathfrak{H}_n$ to be *reduced* with respect to Γ_λ, if

$$\left.\begin{aligned} -\frac{1}{2} \leq Y_i < \frac{1}{2}, i = 1, 2, \ldots, n-1, \\ -\frac{1}{2} \leq X_j < \frac{1}{2}, j = 1, 2, \ldots, n. \end{aligned}\right\} \tag{150}$$

It is, first of all, clear that for any $z \in \mathfrak{H}_n$, there exists an $M \in \Gamma_\lambda$ such that the equivalent point z_M is reduced with respect to Γ_λ. In fact, for $\epsilon = \pm\epsilon_1^{k_1} \ldots \epsilon_{n-1}^{k_{n-1}}$ and $M_1 = A\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} A^{-1}$, the effect of the substitution $z \to z_{M_1}$ is given by (**??**) and hence, by choosing $k_1, \ldots, k_{n-1}$ properly, we could suppose that the coordinates $Y_1, \ldots, Y_{n-1}$ of z_{M_1} satisfy (150). Again, since for $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ and $M_2 = A\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} A^{-1}$, the effect of the substitution $z \to z_{M_2}$ on the coordinates $X_1, \ldots, X_n$ of z_{M_1} is given by (149), we could suppose that for suitable $m_1, \ldots, m_n$, the coordinates $X_1, \ldots, X_n$ of z_{M_2, M_1} satisfy (150). It is now clear that $z_{M_2 M_1}$ is reduced with respect to Γ_λ. On the other hand, let $z = x + iy$, $w = u + iv \in \mathfrak{H}_n$ be reduced and equivalent with respect to Γ_λ and let the local coordinates of z and w relative to λ be respectively

$$1/\sqrt{N(y_{A^{-1}})}, Y_1, \ldots, Y_{n-1}, X_1, \ldots, X_n,$$
$$1/\sqrt{N(v_{A^{-1}})}, Y_1^*, \ldots, Y_{n-1}^*, X_1^*, \ldots, X_n^*.$$

Further, let $z_{A^{-1}} = \epsilon^2 w_{A^{-1}} + \zeta$. Then, in view of the fact that **195**

$$Y_i^* \equiv Y_i (\mod 1), -\frac{1}{2} \leq Y_i, Y_i^* < \frac{1}{2}, i = 1, 2, \ldots, n-1,$$

we have first $Y_i^* = Y_i$, $i = 1, 2, \ldots, n-1$ and hence $\epsilon^2 = 1$. Again, since we have

$$X_j^* \equiv X_j (\mod 1), -\frac{1}{2} \leq X_j, X_j^* < \frac{1}{2}, j = 1, 2, \ldots, n,$$

we see that $X_j^* = X_j$, $j = 1, 2, \ldots, n$ and hence $\zeta = 0$. Thus $z_{A^{-1}} = w_{A^{-1}}$, i.e. $z = w$.

Denoting by $\mathfrak{G}_\lambda$ the set of $z \in \mathfrak{H}_n$ whose local coordinates $Y_1, \ldots, Y_{n-1}, X_1, \ldots, X_n$ satisfy (150), we conclude that any $z \in \mathfrak{H}_n$ is equivalent with respect to Γ_λ, to a point of $\mathfrak{G}_\lambda$ and no two distinct points of $\mathfrak{G}_\lambda$ are equivalent with respect to Γ_λ. *Thus $\mathfrak{G}_\lambda$ is a fundamental domain for Γ_λ in $\mathfrak{h}_n$.*

For $n = 1$, the fundamental domain $\mathfrak{G}_\lambda$ is just the vertical strip $-1/2 \leq x^* < 1/2$, $y^* > 0$ with reference to the coordinate $z_{A^{-1}} = x^* + iy^*$. Going back to the coordinate z, the vertical strips $x^* = \pm 1/2$, $y^* > 0$ are mapped into semi-circles passing through λ and orthogonal to the real axis and the fundamental domain looks as in the figure.

Figure, page 195

If $z \in \mathfrak{G}_\lambda$ and if $c_1 \leq N(y_{A^{-1}}) \leq c_2$, then we claim that z lies in a compact set in $\mathfrak{H}_n$ depending on c_1, c_2 and on the choice of $\epsilon_1, \ldots, \epsilon_{n-1}$ and $\alpha_1, \ldots, \alpha_n$ in K. As a matter of fact, from (150), we know that

$$\left|\log\left(\frac{y_i^*}{\sqrt[n]{N(y^*)}}\right)\right| \leq c_3 \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$

and $|x_j^*| \leq c_4$ for $j = 1, 2, \ldots, n$. Hence we have

$$c_5 \leq \frac{y_i^*}{\sqrt[n]{N(y^*)}} \leq c_6, \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Since $c_1 \leq N(y^*) \leq c_2$, we have indeed for all $i = 1, 2, \ldots, n$, **196**

$$c_7 \leq \frac{y_i^*}{\sqrt[n]{N(y^*)}} \leq c_8.$$

As a result, we obtain for $i, j = 1, 2, \ldots, n$,

$$c_9 \leq y_i^* \leq c_{10},$$
$$|x_j^*| \leq c_4,$$

where c_9, c_{10} and c_4 depend only on c_1, c_2 and the choice of $\epsilon_1, \ldots, \epsilon_{n-1}$, $\alpha_1, \ldots, \alpha_n$ in K. Thus $z_{A^{-1}}$ and therefore z lies in a compact set in $\mathfrak{H}_n$, depending on c_1, c_2 and K.

Now, we introduce the notion of "*distance of a point* $z \in \mathfrak{H}_n$ *from a cusp* λ *of* $\mathfrak{H}_n$". We have already in $\mathfrak{H}_n$ a metric given by

$$ds^2 = \sum_{i=1}^{n} \frac{dx_i^2 + dy_i^2}{y_i^2}$$

which is non-euclidean in the case $n = 1$ and has an invariance property with respect to Γ. But since the cusps lie on the boundary of $\mathfrak{H}_n$, the distance (relative to this metric) of an inner point of $\mathfrak{H}_n$ from a cusp is infinite. Hence, this metric is not useful for our purposes.

For $z \in \mathfrak{H}_n$ and a cusp $\lambda = \rho/\sigma$ with associated $A = \begin{pmatrix} \rho & \xi \\ \sigma & \eta \end{pmatrix} \in \mathfrak{S}$, we define the *distance* $\Delta(z, \lambda)$ *of* z *from* λ by

$$\Delta(z, \lambda) = (N(y_{A^{-1}}))^{-\frac{1}{2}} = (N(y^{-1}|-\sigma z + \rho|^2))^{\frac{1}{2}}$$

$$= (N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}}.$$

For $\lambda = \infty$, $\Delta(z, \infty) = 1/\sqrt{N(y)}$; hence the larger the $N(y)$, the 'closer' is z to ∞.

Now $\Delta(z, \lambda)$ has an important *invariance property* with respect to Γ. Namely, for $M \in \Gamma$, we have

$$\Delta(z_M, \lambda_M) = \Delta(z, \lambda). \tag{151}$$

This is very easy to verify, since, by definition **197**

$$\begin{aligned}\Delta(z_M, \lambda_M) &= (N(\mathrm{Im}(z_M)_{A^{-1}M^{-1}}))^{-\frac{1}{2}}\\ &= (N(y_{A^{-1}}))^{-\frac{1}{2}} = \Delta(z, \lambda).\end{aligned}$$

Moreover, $\Delta(z, \lambda)$ does not depend on the special choice of A associated with λ. For, if $A_1 = \begin{pmatrix} \rho_i & \xi_1 \\ \sim_1 & \eta_1 \end{pmatrix} \in \mathfrak{S}$ is associated with $\lambda = \rho_1/\sigma_1$, then $A^{-1}A_1 = T = \begin{pmatrix} \epsilon & \zeta \\ 0 & \epsilon^{-1} \end{pmatrix}$ where ϵ is a unit in K. Now $N(y_{A_1^{-1}}) = N(y_{A^{-1}})N(\epsilon^{-2}) = N(y_{A^{-1}})$ and hence our assertion is proved.

Let, for a given cusp λ and $r > 0$, $\mathfrak{U}_{\lambda,r}$ denote the set of $z \in \mathfrak{H}_n$ such that $\Delta(z, \lambda) < r$. This defines a 'neighbourhood' of λ and all points $z \in \mathfrak{H}_n$ which belong to $\mathfrak{U}_{\lambda,r}$ are inner points of the same. The neighbourhoods, $\mathfrak{U}_{\lambda,r}$ for $0 < r < \infty$ cover the entire upper half-plane $\mathfrak{H}_n$. Each neighbourhood $\mathfrak{U}_{\lambda,r}$ is left invariant by a modular substitution in Γ_λ. For, by (151), if $M \in \Gamma_\lambda$, then

$$\Delta(z_M, \lambda) = \Delta(z_M, \lambda_M) = \Delta(z, \lambda)$$

and so, if $z \in \mathfrak{U}_{\lambda,r}$, then again $\Delta(z_M, \lambda) < r$ i.e. $z_M \in \mathfrak{U}_{\lambda,r}$. A fundamental domain for Γ_λ in $\mathfrak{a}_{\lambda,r}$ is obviously given by $\mathfrak{G}_\lambda \cap \mathfrak{U}_{\lambda,r}$.

We shall now prove some interesting facts concerning $\Delta(z, \lambda)$ which will be useful in constructing a fundamental domain for Γ in $\mathfrak{H}_n$.

(i) *For $z = x + iy \in \mathfrak{H}_n$, there exists a cusp λ of $\mathfrak{H}_n$ such that for all cusps μ of $\mathfrak{H}_n$, we have*

$$\Delta(z, \lambda) \leq \Delta(z, \mu). \tag{152}$$

Proof. If λ is a cusp, then $\lambda = \rho/\sigma$ for ρ, $\sigma \in \mathfrak{o}$ such that (ρ, σ) is one of the h ideals $\mathfrak{U}_1, \ldots, \mathfrak{U}_h$. Then

$$\Delta(z, \lambda) = (N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}}.$$

□

Let us consider the expression $(N((-\sigma x+\rho)^2 y^{-1}+\sigma^2 y))^{\frac{1}{2}}$ as a function of the pair of integers (ρ,σ). It remains unchanged if ρ, σ are replaced by $\rho\epsilon$, $\sigma\epsilon$ for any unit ϵ in K. We shall now show that there exists a pair of integers ρ_1, σ_1 in K such that

$$(N((-\sigma_1 x+\rho_1)^2 y^{-1}+\sigma_1^2 y))^{\frac{1}{2}} \leq (N((-\sigma x+\rho)^2 y^{-1}+\sigma^2 y))^{\frac{1}{2}} \tag{153}$$

for all pairs of integers (ρ,σ). In order to prove (153), it obviously suffices to 198
show that for given $c_1 > 0$, there are only finitely many non-associated pairs of integers (ρ,σ) such that

$$(N((-\sigma x+\rho)^2 y^{-1}+\sigma^2 y))^{\frac{1}{2}} \leq c_{11}. \tag{154}$$

Now, it is known from the theory of algebraic number fields that if $\alpha = (\alpha_1,\ldots,\alpha_n)$ is an n-tuple of real numbers with $N(\alpha)\neq 0$, then we can find a unit ϵ in K such that

$$|\alpha_i\epsilon^{(i)}| \leq c_{12}|N(\alpha)|^{1/n} \tag{155}$$

for a constant c_{12} depending only on K. In view of (154), we can suppose after multiplying ρ and σ by a suitable unit ϵ, that already we have

$$(-\sigma^{(i)}x_i+\rho^{(i)})^2 y_i^{-1}+\sigma_{y_i}^{(i)^2} \leq c_{13}, \quad i=1,2,\ldots,n,$$

for a constant c_{13} depending only on c_{11} and c_{12}. This implies that $(-\sigma^{(i)}x_i+\rho^{(i)})$ and $\sigma^{(i)}$ and as a consequence $\rho^{(i)}$ and $\sigma^{(i)}$ are bounded, for $i=1,2,\ldots,n$. Again, in this case, we know from the theory of algebraic numbers that there are only finitely many possibilities for ρ and σ. Hence (154) is true only for finitely many non-associated pairs of integers (ρ,σ). From these pairs, we choose a pair (ρ_1,σ_1) such that the value of $N((-\sigma_1 x+\rho_1)^2 y^{-1}+\sigma_1^2 y))^{\frac{1}{2}}$ is a minimum. This pair (ρ_1,σ_1) now obviously satisfies (153).

Let $(\rho_1,\sigma_1)=\mathfrak{b}$ and let $\mathfrak{b}=\mathfrak{a}_i(\theta)^{-1}$ for some $\mathfrak{a}_i$ and a $\theta\in K$. Then $\mathfrak{a}_i=(\rho_1\theta,\sigma_1\theta)$. Now, in (153), if we replace ρ, σ by $\rho_1\theta$, $\sigma_1\theta$ respectively, we get $|N(\theta)|\geq 1$. On the other hand, since $\mathfrak{a}_i$ is of minimum norm among the integral ideals of its class, $N(\mathfrak{a}_i)\leq N(\mathfrak{b})$ and therefore $|N(\theta)|\leq 1$. Thus $|N(\theta)|=1$. Let now $\rho=\rho_1\theta$, $\sigma=\sigma_1\theta$ and $\lambda=\rho/\sigma$. Then $\mathfrak{a}_i=(\rho,\sigma)$ and by definition, $\Delta(z,\lambda)=(N((-\sigma x+\rho)^2 y^{-1}+\sigma^2 y))^{\frac{1}{2}}=(N((-\sigma_1 x+\rho_1)^2 y^{-1}+\sigma_1^2 y))^{\frac{1}{2}}$. If we use (153), then we see at once that $\Delta(z,\lambda)\leq\Delta(z,\mu)$ for all cusps μ.

For given $z\in\mathfrak{H}_n$, we define

$$\Delta(z)=\int_{\lambda}\Delta(z,\lambda).$$

By (i), there exists a cusp λ such that $\Delta(z,\lambda)=\Delta(z)$. In general, λ is unique, but there are exceptional cases when the minimum is attained for more than one λ. We shall see presently that there exists a constant $d>0$, depending only on K such that if $\Delta(z)<d$, then the cusp λ for which $\Delta(z,\lambda)=\Delta(z)$ is unique. 199

(ii) *There exists $d>0$ depending only on K, such that, if for $z=x+iy\in\mathfrak{H}_n$, $\Delta(z,\lambda)<d$ and $\Delta(z,\mu)<d$, then necessarily $\lambda=\mu$.*

Proof. Let $\lambda=\rho/\sigma$ and $\mu=\rho_1/\sigma_1$ and let for a real number $d>0$,

$$\Delta(z,\lambda)=(N((-\sigma x+\rho)^2y^{-1}+\sigma^2y))^{\frac{1}{2}}<d,$$
$$\Delta(z,\mu)=(N(-\sigma_1x+\rho_1)^2y^{-1}+\sigma_1^2y)^{\frac{1}{2}}<d.$$

After multiplying ρ and σ by a suitable unit ϵ in K, we might assume in view of (155) that

$$(\sigma^{(i)}x_i-\rho^{(i)})^2y_i^{-1}+\sigma^{(i)^2}y_i<c_{12}d^{2/n},\quad i=1,2,\ldots,n.$$

Hence

$$|-\sigma^{(i)}x_i+\rho^{(i)}|y_i^{-\frac{1}{2}}<\sqrt{c_{12}}d^{1/n},$$
$$|\sigma^{(i)}|y_i^{\frac{1}{2}}<\sqrt{c_{12}}d^{1/n}.$$

Similarly we have

$$|-\sigma_1^{(i)}x_i+\rho_1^{(i)}|y_i^{-\frac{1}{2}}<\sqrt{c_{12}}d^{1/n},$$
$$|\sigma_1^{(i)}|y_i^{\frac{1}{2}}<\sqrt{c_{12}}d^{1/n}.$$

Now

$$\rho^{(i)}\sigma_1^{(i)}-\rho_1^{(i)}\sigma^{(i)}=(-\sigma^{(i)}x_i+\rho^{(i)})y_i^{-\frac{1}{2}}\sigma_1^{(i)}y_i^{\frac{1}{2}}-(-\sigma_1^{(i)}x_i+\rho_1^{(i)})y_i^{-\frac{1}{2}}\sigma^{(i)}y_i^{\frac{1}{2}}$$

and hence

$$|N(\rho\sigma_1-\sigma\rho_1)|<(2c_{12}d^{2/n})^n.$$

If we set $d=(2c_{12})^{-n/2}$, then $|N(\rho\sigma_1-\sigma\rho_1)|<1$. Since $\rho\sigma_1-\sigma\rho_1$ is an integer, it follows that $\rho\sigma_1-\sigma\rho_1=0$ i.e. $\lambda=\mu$. □ 200

Thus for $d=(2c_{12})^{-n/2}$, the conditions $\Delta(z,\lambda)<d$, $\Delta(z,\mu)<d$ for a $z\in\mathfrak{H}_n$ imply that $\lambda=\mu$. Therefore, the neighbourhoods $\mathfrak{U}_{\lambda,d}$ for the various cusps λ are mutually disjoint.

We shall now prove that for $z\in\mathfrak{H}_n$, $\Delta(z)$ is uniformly bounded in $\mathfrak{H}_n$. To this end, it suffices to prove

(iii) *There exists $c > 0$ depending only on K such that for any $z = x + iy \in \mathfrak{H}_n$, there exists a cusp λ with the property that $\Delta(z, \lambda) < c$.*

Proof. We shall prove the existence of a constant $c > 0$ depending only on K and a pair of integers (ρ, σ) not both zero such that

$$(N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}} < c.$$

Let $\omega_1, \ldots, \omega_n$ be a basis of $\mathfrak{o}$. Consider now the following system of $2n$ linear inequalities in the $2n$ variables $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, viz.

$$\left| y_k^{-\frac{1}{2}}(\omega_1^{(k)} a_1 + \cdots + \omega_n^{(k)} a_n) - x_k y_k^{-\frac{1}{2}}(\omega_1^{(k)} b_1 + \cdots + \omega_n^{(k)} b_n) \right| \leq \alpha_k$$

for $k = 1, 2, \ldots, n$ and

$$\left| y_k^{\frac{1}{2}}(\omega_1^{(k)} b_1 + \cdots + \omega_n^{(k)} b_n) \right| \leq \beta_k$$

for $k = 1, \ldots, n$. □

The determinant of this system of linear forms is $|(\omega_i^{(j)})|^2 = |\Delta|$ where Δ is the discriminant of K. By Minkowski's theorem on linear forms, this system of linear inequalities has a nontrivial solution in rational integers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ if $\alpha_1, \ldots, \alpha_n \cdot \beta_1, \ldots, \beta_n \leq |\Delta|$. Taking $\alpha_i = \beta_j = \sqrt[2n]{|\Delta|} i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, in particular, this system of inequalities has a nontrivial solution in rational integers, say, $a'_1, \ldots, a'_n$, $b'_1, \ldots, b'_n$. Let us take $\rho = a'_1 \omega_1 + \cdots + a'_n \omega_n$ and $\sigma = b'_1 \omega_1 + \cdots + b'_n \omega_n$. Then we obtain

$$\left| (-\sigma^{(i)} x_i + \rho^{(i)})^2 y_i^{-1} + \sigma^{(i)^2} y_i \right| \leq \sqrt[2n]{|\Delta|}, \quad i = 1, 2, \ldots, n.$$

Hence $(N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}} \leq 2^{n/2} |\Delta|^{\frac{1}{2}} = c$ (say). **201**

Now let $(\rho, \sigma) = \mathfrak{b}$; $\mathfrak{b} = \mathfrak{a}_i (\theta)^{-1}$ for some $\mathfrak{a}_i$ and for some $\theta \in K$. Since $\mathfrak{a}_i$ is of minimum norm among the integral ideals of its class, $|N(\theta)| \leq 1$. Further $\mathfrak{a}_i = (\rho\theta, \sigma\theta)$. Now, for the cusp $\lambda = \rho\theta/\sigma\theta = \rho/\sigma$, we have

$$\Delta(z, \lambda) = |N(\theta)| (N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}} \leq c,$$

which was what we wished to prove.

From (iii), we deduce that $\mathfrak{H}_n = \bigcup_{\lambda} \mathfrak{U}_{\lambda, c}$.

(iv) *For $z \in \mathfrak{H}_n$ and $M \in \Gamma$, $\Delta(z_M) = \Delta(z)$.*

Proof. In fact,

$$\begin{aligned}\Delta(z_M) &= \inf_{\lambda} \Delta(z_M, \lambda)\\ &= \inf_{\lambda} \Delta(z, \lambda_{M^{-1}}) \qquad \text{((by 151))}\\ &= \inf_{\lambda} \Delta(z, \lambda)\\ &= \Delta(z).\end{aligned}$$

□

We now have all the necessary material for the construction of a fundamental domain for Γ in $\mathfrak{H}_n$.

A point $z \in \mathfrak{H}_n$ is *semi-reduced* (*with respect to a* cups λ), if $\Delta(z) = \Delta(z, \lambda)$. If z is semi-reduced with respect to λ, then for all cusps μ, we have $\Delta(z, \mu) \geq \Delta(z, \lambda)$.

Let $\lambda_1 (= (\infty, \ldots, \infty))$, $\lambda_2, \ldots, \lambda_h$ be the h inequivalent base cusps of $\mathfrak{H}_n$. We denote by $\mathfrak{F}_{\lambda_i}$, the set of all $z \in \mathfrak{H}_n$ which are semi-reduced with respect to λ_i. Clearly $\mathfrak{F}_{\lambda_i} \subset \mathfrak{U}i_{\lambda_{i,c}}$, in view of (iii) above. The set $\mathfrak{F}_{\lambda_i}$ is invariant under the modular substitution $z \to z_M$, for $M \in \Gamma_{\lambda_i}$. For, by (iv) above, $\Delta(z_M) = \Delta(z)$ and further.

$$\Delta(z) = \Delta(z, \lambda_i) = \Delta(z_M, (\lambda_i)_M) = \Delta(z_M, \lambda_i).$$

Thus $\Delta(z_M) = \Delta(z_M, \lambda_i)$ and hence $z_M \in \mathfrak{F}_{\lambda_i}$, for $M \in \Gamma_{\lambda_i}$.

Let $\overline{\mathfrak{G}}_{\lambda_i}$ denote the closure in $\mathfrak{H}_n$ of the set $\mathfrak{G}_{\lambda_i}$ and $\mathfrak{q}_i = \mathfrak{F}_{\lambda_i} \cap \overline{\mathfrak{G}}_{\lambda_i}$. Then $\mathfrak{q}_i$ is explicitly defined as the set of $z \in \mathfrak{H}_n$ whose local coordinates $X_1, \ldots, X_n$, **202**
$Y_1, \ldots, Y_{n-1}$ relative to the cusp λ_i satisfy the conditions

$$-\frac{1}{2} \leq X_k \leq \frac{1}{2}, -\frac{1}{2} \leq Y_1 \leq \frac{1}{2} (k = 1, 2, \ldots, n; l = 1, 2, \ldots, n-1)$$

and further, for all cusps μ,

$$\Delta(z, \mu) \geq \Delta(z, \lambda_i).$$

Let $\mathfrak{F} = \cup_{i=1}^{h} \mathfrak{q}_i$. We see that the $\mathfrak{q}_i$ as also $\mathfrak{F}$ are closed in $\mathfrak{H}_n$.

A point $z \in \mathfrak{q}_i$ is an *inner point* of $\mathfrak{q}_i$ if in all the conditions above, strict inequality holds, viz.

$$\begin{gathered}-\frac{1}{2} < X_k < \frac{1}{2}, -\frac{1}{2} < Y_l < \frac{1}{2} (k = 1, 2, \ldots, n, l = 1, 2, \ldots, n-1)\\ \Delta(z, \mu) > \Delta(z, \lambda_i), \quad \text{for } \mu \neq \lambda_i. \qquad (150)^*\end{gathered}$$

If equality holds even in one of these conditions, then z is said to be a *boundary point* of $\mathfrak{q}_i$. The set of boundary points of $\mathfrak{q}_i$ constitute the boundary of $\mathfrak{q}_i$, which may be denoted by *Bd* $\mathfrak{q}_i$. We denote by $\mathfrak{R}_i$, the set of inner points of $\mathfrak{q}_i$.

It is clear that the $\mathfrak{q}_i$ do not overlap and intersect at most on their boundary.

A point $z \in \mathfrak{F}$ may now be called an *inner point* of $\mathfrak{F}$, if z is an inner point of some $\mathfrak{q}_i$; similarly we may define a *boundary point* of $\mathfrak{F}$ and denote the set of boundary points of $\mathfrak{F}$ by *Bd* $\mathfrak{F}$.

We say that a point $z \in \mathfrak{H}_n$ is *reduced* (with respect to Γ) if, in the first place, z is semi-reduced with respect to some one of the h cusps $\lambda_1, \ldots, \lambda_h$, say λ_i and then further $z \in \overline{\mathfrak{G}}_{\lambda_i}$. Clearly $\mathfrak{F}$ *is just the set of all* $z \in \mathfrak{H}_n$ *reduced with respect to* Γ.

Before we proceed to show that $\mathfrak{F}$ is a fundamental domain for Γ in $\mathfrak{H}_n$, we shall prove the following result concerning the inner points of $\mathfrak{F}$, namely,

The set of inner points of $\mathfrak{F}$ *is open in* $\mathfrak{H}_n$.

Proof. It is enough to show that each $\mathfrak{R}_j$ is open in $\mathfrak{H}_n$. Let, then, $z_0 = x_0 + iy_0 \in \mathfrak{R}_j$; we have to prove that there exists a neighbourhood V of z_0 in $\mathfrak{H}_n$ which is wholly contained in $\mathfrak{R}_j$. ◻

Now, for each $z = x + iy \in \mathfrak{H}_n$ and a cusp $\mu = \rho/\sigma$, we have $\Delta(z, \mu) =$ 203
$(N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{\frac{1}{2}}$. Using the fact that for each $j = 1, 2, \ldots, n$, $(-\sigma^{(j)} x_j + \rho^{(j)})^2 y^{-1} + \sigma^{(j)^2} y_j$ is a positive-definite binary quadratic form in $\sigma^{(j)}$ and $\rho^{(j)}$, we can easily show that

$$\Delta(z, \mu) \geq \alpha(N(\sigma^2 + \rho^2))^{\frac{1}{2}}$$

where $\alpha = \alpha(z)$ depends continuously on z and does not depend on μ. We can now find a sufficiently small neighbourhood W of z_0 such that for all $z \in W$,

$$\Delta(z, \mu) \geq \frac{1}{2}\alpha_0(N(\sigma^2 + \rho^2))^{\frac{1}{2}}$$

where $\alpha_0 = \alpha(z_0)$. Moreover, we can assume W so chosen that for all $z \in W$,

$$\Delta(z, \lambda_j) \leq 2\Delta(z_0, \lambda_j).$$

Thus, for all cusps $\mu = \rho/\sigma$ and $z \in W$,

$$\Delta(z, \mu) - \Delta(z, \lambda_j) \geq \frac{1}{2}\alpha_0(N(\sigma^2 + \rho^2))^{\frac{1}{2}} - 2\Delta(z_0, \lambda_j).$$

Now, employing an argument used already on p. 198, we can show that there are only finitely many non-associated pairs of integers (ρ, σ) such that

$$\frac{1}{2}\alpha_0(N(\sigma^2 + \rho^2))^{\frac{1}{2}} \leq 2\Delta(z_0, \lambda_j).$$

It is an immediate consequence that except for finitely many cusps $\mu_1, \ldots, \mu_r$, we have for $\mu \neq \lambda_j$,

$$\Delta(z, \mu) > \Delta(z, \lambda_j).$$

Now, since $\Delta(z_0, \mu) > \Delta(z_0, \lambda_j)$ for all cusps $\mu \neq \lambda_j$, we can, in view of the continuity in z of $\Delta(z, \mu_i)$ (for $i = 1, 2, \ldots, r$) find a neighbourhood U of z_0 such that

$$\Delta(z, \mu_k) > \Delta(z, \lambda_j) \quad (k = 1, 2, \ldots, r)$$

for $z \in U$ and $\mu_k \neq \lambda_j$. We then have finally for all $z \in U \cap W$ 204

$$\Delta(z, \mu) > \Delta(z, \lambda_j)$$

provided $\mu \neq \lambda_j$. Further, we could have chosen U such that for all $z \in U$, (150)* is satisfied, in addition. Thus the neighbourhood $V = U \cap W$ of z_0 satisfies our requirements and so $\mathfrak{R}_j$ is open.

It may now be seen that the closure $\overline{\mathfrak{R}}_i$ of $\mathfrak{R}_i$ is just $\mathfrak{q}_i$. In fact, let $z \in \mathfrak{q}_i$ and let $\lambda_i = \infty_{A_i}$, $z^* = z_{A_i^{-1}}$, $\mu_{A_i^{-1}} = \nu = \rho/\sigma \neq \infty$. Then we have

$$\sigma \neq 0, \frac{\Delta(z, \mu)}{\Delta(z, \lambda_i)} = \frac{\Delta(z^*, \nu)}{\Delta(z^*, \infty)} = (N((-\sigma x^* + \rho)^2 + (\sigma y^*)^2))^{\frac{1}{2}}.$$

If y^* is replaces by ty^* where t is a positive scalar factor, then the expression above is a strictly monotonic increasing function of t, whereas the coordinates $X_1, \ldots, X_n$, $Y_1, \ldots, Y_{n-1}$ remain unchanged. Thus, if $z \in \mathfrak{q}_i$ and $z_{A_i^{-1}} = z^* = x^* + iy^*$, then for $z^{(t)} = (x^* + ity^*)_{A_i}$, $t > 1$, the inequalities $\Delta(z, \mu) > \Delta(z, \lambda_i)(\mu \neq \lambda_i)$ are satisfied and moreover for a small change in the coordinates $X_1, \ldots, X_n$, $Y_1, \ldots, Y_{n-1}$, the inequalities (150)* are also satisfied. As a consequence, if $z \in \mathfrak{q}_i$, then every neighbourhood of z intersects $\mathfrak{R}_i$; in other words, $\overline{\mathfrak{R}}_i = \mathfrak{q}_i$.

From above, we have that if $z \in \mathfrak{q}_i$, then the entire curve defined by $z^{(t)} = (x^* + ity^*)_{A_i}$, $t \geq 1$ lies in $\mathfrak{q}_i$ and hence for $z \in \mathfrak{R}_i$, in particular, $z^{(t)}$ for large $t > 0$ belongs to $\mathfrak{R}_i$, Using this it may be shown that $\mathfrak{q}_i$ and similarly $\mathfrak{R}_i$ are connected.

The existence of inner points of $\mathfrak{q}_j$ is an immediate consequence of result (ii) on p. 199. For $j = 1$, it may be verified that $z = (it, \ldots, it)$, $t > 1$ is an inner point of $\mathfrak{q}_1$.

Now we proceed to prove that $\mathfrak{F}$ is a fundamental domain for Γ in $\mathfrak{H}_n$. We have first to show that

(α) *the images $\mathfrak{F}_M$ of $\mathfrak{F}$ for $M \in \Gamma$ cover $\mathfrak{H}_n$ without gaps.*

Proof. This is obvious from the very method of construction of $\mathfrak{F}$. First, for any $z \in \mathfrak{H}_n$, there exists a cusp λ such that $\Delta(z) = \Delta(z, \lambda)$. Let $\lambda = (\lambda_i)_M$ for some λ_i and $M \in \Gamma$. We have then

$$\Delta(z_{M^{-1}}) = \Delta(z) = \Delta(z, \lambda) = \Delta(z_{M^{-1}}, \lambda_i)$$

and hence $z_{M^{-1}} \in \mathfrak{F}_{\lambda_i}$. Now we can find $N \in \Gamma_{\lambda_i}$ such that $(z_{M^{-1}})_N$ is reduced 205
with respect to Γ_{λ_i} and thus $z_{NM^{-1}} \in \mathfrak{F}$. □

Next, we show that

(β) *the images $\mathfrak{F}_M$ of $\mathfrak{F}$ for $M \in \Gamma$ cover $\mathfrak{H}_n$ withtout overlaps.*

Proof. Let z_1, $z_2 \in \mathfrak{F}$ such that $z_1 = (z_2)_M$ for an $M \neq \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Γ and let $z_1 \in \mathfrak{q}_i$ and $z_2 \in \mathfrak{q}_j$. Now, since $z_1 \in \mathfrak{F}_{\lambda_i}$, we have

$$\Delta(z_1, \lambda_i) \leq \Delta(z_1, (\lambda_j)_M) = \Delta(z_2, \lambda_j).$$

Similarly

$$\Delta(z_2, \lambda_j) \leq \Delta(z_2, (\lambda_i)_{M^{-1}}) = \Delta(z_1, \lambda_i).$$

Therefore, we obtain

$$\Delta(z_1, \lambda_i) = \Delta(z_2, \lambda_j) = \Delta(z_1, (\lambda_j)_M).$$

□

Let us first suppose that $\Delta(z_1, \lambda_i) = \Delta(z_2, \lambda_j) < d$. Then, since $\Delta(z_1, \lambda_i) < d$ as also $\Delta(z_1, (\lambda_j)_M) < d$, we infer from the result (ii) on p. 199 that $\lambda_i = (\lambda_j)_M$. But λ_i and λ_j, for $i \neq j$ are not equivalent with respect to Γ. Therefore $i = j$ and $M \in \Gamma_{\lambda_i}$. Again, since both z_1 and z_2 are in $\overline{\mathfrak{G}}_{\lambda_i}$ and further $z_1 = (z_2)_M$ with $M \in \Gamma_{\lambda_i}$, we conclude that necessarily z_1 and z_2 belong to $Bd\mathfrak{F}$ and indeed their local coordinates $X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1}$ relative to λ_i satisfy at least one of the finite number of conditions $X_1 = \pm 1/2, \ldots, X_n = \pm 1/2$, $Y_1 = \pm 1/2, \ldots, Y_{n-1} = \pm 1/2$. Further M clearly belongs to a finite set $M_1, \ldots, M_r$ of elements in $\bigcup_{i=1}^{h} \Gamma_{\lambda_i}$.

We have now to deal with the case

$$d \leq \Delta(z_1, \lambda_i) \leq c \quad \text{and} \quad d \leq \Delta(z_2, \lambda_j) \leq c$$

where we may suppose that $\lambda_i \neq (\lambda_j)_M$. Let, for $i = 1, 2, \ldots, h$, B_i denote the set of $z \in \mathfrak{H}_n$ for which $d \leq \Delta(z, \lambda_i) \leq c$ and $z \in \overline{\mathfrak{G}}_{\lambda_i}$. Then, by our remark on p.

196 B_i is compact and so is $B = \bigcup_{i=1}^{\widehat{h}} B_i$. Now, both z_2 and $z_1 = (z_2)_M$ belong to the compact set B. We may then deduce from Proposition 21 that M belongs to a finite set of elements $M_{r+1}, \ldots, M_s$ in Γ, depending only on B and hence 206
only on K. Moreover z_1 satisfies

$$\Delta(z_1, \lambda_i) = \Delta(z_1, (\lambda_j)_M)$$

with $(\lambda_j)_M \neq \lambda_i$. Hence z_1 and similarly z_2 belongs to $Bd\mathfrak{F}$. As a result, arbitrarily near z_1 and z_2, there exist points z such that $\Delta(z, \lambda_i) \neq \Delta(z, (\lambda_j)_M)$ for $M = M_{r+1}, \ldots, M_s$.

Thus, finally, no two inner points of $\mathfrak{F}$ can be equivalent with respect to Γ. Further $\mathfrak{F}$ intersects only finitely many of its neighbours $\mathfrak{F}_{M_1}, \ldots, \mathfrak{F}_{M_s}$ and indeed only on its boundary. We have therefore established (ii).

From (α) and (β) above, it follows that $\mathfrak{F}$ is a fundamental domain for Γ in $\mathfrak{H}_n$. It consists of h connected 'pieces' corresponding to the h inequivalent base cusps $\lambda_1, \ldots, \lambda_h$ and is bounded by a finite number of 'manifolds' of the form

$$\Delta(z, \lambda_i) = \Delta(z, (\lambda_j)_M), i, j = 1, 2, \ldots, h, M = M_{r+1}, \ldots, M_s \tag{156}$$

and the hypersurfaces defined by

$$X_i^{(k)} = \pm\frac{1}{2}, Y_j^{(k)} = \pm\frac{1}{2}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n-1$$

where $X_1^{(k)}, \ldots, X_n^{(k)}$, $Y_1^{(k)}, \ldots, Y_{n-1}^{(k)}$ are local coordinates relative to the base cusp λ_k.

The 'manifolds' defined by (156) are seen to be generalizations of the 'isometric circles', in the sense of Ford, for a fuchsian group. In fact, if $\lambda_i = \rho_i/\sigma_i$ and $(\lambda_j)_M = \rho/\sigma$, then the condition (156) is just

$$N(|-\sigma_i z + \rho_i|) = N(|-\sigma z + \rho|).$$

If we set $n = 1$ and $\lambda_i = \infty$ or equivalently $\rho_i = 1$ and $\sigma_i = 0$, then the condition reads as

$$|-\sigma z + \rho| = 1$$

which is the familiar 'isometric circle' corresponding to the transformation $z \to \dfrac{\eta z - \xi}{-\sigma z + \rho}$ of $\mathfrak{H}_1$ onto itself.

The conditions $\Delta(z, \lambda) \geq \Delta(z, \lambda_i)$ by which $\mathfrak{F}_{\lambda_i}$ was defined, simply mean for $n = 1$ and $\lambda_i = \infty$ that $|\gamma z + \delta| \geq 1$ for all pairs of coprime rational integers 207
(γ, δ). Thus, just as the points of the well-known fundamental domain in $\mathfrak{H}_1$

for the elliptic modular group lie in the exterior of the the isometric circles $|\gamma z + \delta| = 1$ corresponding to the same group, $\mathfrak{F}_{\lambda_i}$ lies in the 'exterior' of the generalized isometric circles $\Delta(z, \lambda_i) = \Delta(z, \lambda)$.

We now prove the following important result, which will be used later.

Proposition 22. *Let $\mathfrak{F}^*$ denote the set of $z \in \mathfrak{F}$ for which $\Delta(z, \lambda_i) \geq e_i > 0$, $i = 1, 2, \ldots, h$. Then $\mathfrak{F}^*$ is compact in $\mathfrak{H}_n$.*

The *proof* is almost trivial in the light of our remark on p. 196. Let B_i denote the set of $z \in \mathfrak{G}_{\lambda_i}$ for which $e_i \leq \Delta(z, \lambda_i) \leq c$. Then B_i as also $B = \cup_{i=1}^{h} B_i$ is compact in $\mathfrak{H}_n$. Hence $\mathfrak{F}^*$ which is closed and contained in B, is again compact.

Suppose instead of $\mathfrak{H}_n$, we consider the space $\mathfrak{H}_n^*$ of $(z_1, \ldots, z_n) \in C^n$ with $\mathrm{Im}(z_1) < 0$ and $\mathrm{Im}(z_i) > 0$ for $i = 2, \ldots, n$. Then by means of the mapping $(z_1, \ldots, z_n) \to (\bar{z}_1, z_2, \ldots, z_n)$ of $\mathfrak{h}_n$ onto $\mathfrak{H}_n^*$, $\mathfrak{F}$ goes over into a fundamental domain for Γ in $\mathfrak{H}_n^*$. The general case when we take instead of $\mathfrak{H}_n$, the product of r uppper half-planes and $n - r$ lower half-planes, can be dealt with, in a similar fashion.

It was Blumenthal who first gave a method of constructing a fundamental domain for Γ in $\mathfrak{H}_n$ but his proof contained an error since he obtained a fundamental domain with just one cusp and not h cusps. This error was set right by Maass.

We remark finally that our method of constructing the fundamental domain $\mathfrak{F}$ is essentially different from the well-known method of Fricke for constructing a 'normal polygon' for a discontinuous group of analytic automorphisms of a bounded domain in the complex plane. Our method uses only the notion of "distance of a point of $\mathfrak{H}_n$ from a cusp", whereas we require a metric invariant under the group, for Fricke's method which is as follows. Let G be a bounded domain in the complex plane and Γ, a discontinuous group of analytic automorphisms of G. Further let G possess a metric $D(z_1, z_2)$ for z_1, $z_2 \in G$, having the property that $D((z_1)_M, (z_2)_M) = D(z_1, z_2)$ for $M \in \Gamma$. We choose a point z_0 which is not a fixed point of any $M \in \Gamma$, other than the identity and a fundamental domain for Γ in G is given by the set of points $z \in G$ for which

$$D(z, (z_0)_M) \geq D(z, z_0) \quad \text{for all} \quad M \in \Gamma.$$

208

Now we know that $\mathfrak{H}_n$ carries a Riemannian metric which is invariant under Γ. Suppose we use this metric and try to adapt Fricke's method of constructing a fundamental domain. For our later purposes, we require a fundamental domain whose nature near the cusps should be well known. Therefore we see in the first place that the adaptation of Fricke's method to our case is not practical in view of the fact that the distance of a point of $\mathfrak{H}_n$ from the cusps, relative to the Riemannian metric, is infinite. In the second place, it is advantageous to adapt Fricke's method only when the fundamental domain is compact, whereas we know that the fundamental domain is not compact in our case. Moreover, our method of construction of the fundamental domain uses the deep and intrinsic properties of algebraic number fields.

The following proposition is necessary for our later purposes.

Proposition 23. *For any compact set C in $\mathfrak{H}_n$, there exists a constant $b = b(C) > 0$ such that $C \cap \mathfrak{U}_{\mu,b} \neq \emptyset$ for all cusps μ.*

Proof. Since C is compact, we can find $\alpha, \beta > 0$ depending only on C such that for $z = x + iy \in C$, we have $\beta \leq N(y) \leq \alpha$. Now, for any cusp $\lambda = (\rho/\sigma)(\rho, \sigma \in \mathfrak{o})$ and $z = x + iy \in C$, it is clear that $\Delta(z, \lambda) = (N((-\sigma x + \rho)^2 y^{-1} + \sigma^2 y))^{1/2}$ satisfies

$$\Delta(z,\lambda) \geq \begin{cases} |N(\sigma)| N(y)^{\frac{1}{2}} \geq \beta^{\frac{1}{2}}, & \text{for } \sigma \neq 0 \\ |N(\rho)| N(y)^{-\frac{1}{2}} \geq \alpha^{-\frac{1}{2}}, & \text{for } \sigma = 0. \end{cases}$$

If we choose b for which $0 < b < \min(\alpha^{-\frac{1}{2}}, \beta^{\frac{1}{2}})$, then it is obvious that for *all* cusps λ, $\mathfrak{U}_{\lambda,b} \cap C = \emptyset$. □

3 Hilbert modular functions

We shall first develop here some properties of (entire) Hilbert modular forms and, in particular, show that a Hilbert modular form satisfies the regularity condition at the cusps of the fundamental domain $\mathfrak{F}$, automatically for $n \geq 2$. We shall then prove the existence of $n+1$ (entire) modular forms $f_0(z), f_1(z), \ldots, f_n(z)$ **209**
of 'large' weight k such that the functions $\dfrac{f_1(z)}{f_0(z)}, \ldots, \dfrac{f_n(z)}{f_0(z)}$ are analytically independent. Next, we shall obtain estimates for the dimensions of the space of entire modular forms of 'large' weight. Finally, we shall show that every Hilbert modular function can be expressed as the quotient of two Hilbert modular forms of the same weight and use this result to prove the *main theorem* that the Hilbert modular functions form an algebraic function field of n variables over the field of complex numbers.

Let K be a totally real algebraic number field of degree n over R and let Γ be the corresponding Hilbert modular group. Further let k be a rational integer.

A complex-valued function $f(z)$ defined in $\mathfrak{H}_n$ is called an *entire Hilbert modular form of weight k* (or *of dimension-k*) *belonging to the group* Γ, if

(1) $f(z)$ is regular in $\mathfrak{H}_n$,

(2) for every $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$, $f(z_M) N(\gamma z + \delta)^{-k} = f(z)$, (**??**)

- for every cusp $\lambda_i = \rho_i/\sigma_i$ of $\mathfrak{F}$, $f(z)N(-\sigma_i z+\rho_i)^k$ is regular at the cusp λ_i i.e. it has a Fourier expansion of the form $c(0) + \sum\limits_{\beta} c(\beta)e^{2\pi i S(\beta z_{A_i^{-1}})}$ where, in the summation, only terms corresponding to $\beta > 0$ can occur.

It will be shown later, as a consequence of a slightly more general result, that for $n \geq 2$, conditions 1) and 2) together imply 3). However, for $n = 1$, it is clear that 1) and 2) do not imply 3) and one has necessarily to impose condition 3) also in the definition of an entire modular form.

We shall hereafter refer to entire Hilbert modular forms belonging to Γ merely as modular forms, without any risk of misunderstanding.

Now, if in (**??**), we take $-M = \left(\begin{smallmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{smallmatrix}\right)$ instead of M, we have

$$f(z_M)N(-\gamma z - \delta)^{-k} = f(z) = f(z_M)N(\gamma z + \delta)^{-k},$$

i.e.

$$f(z_M)N(\gamma z + \delta)^{-k}((-1)^{nk} - 1) = 0.$$

i.e. **210**

$$f(z)((-1)^{nk} - 1) = 0.$$

Therefore, if nk is odd, necessarily $f(z)$ is identically zero. We are not interested in this case and we shall suppose hereafter that

$$nk \equiv 0 (\mod 2).$$

Let $\lambda = \rho/\sigma$ be a cusp of $\mathfrak{H}_n$, $\mathfrak{a} = (\rho, \sigma)$ and $A = \left(\begin{smallmatrix} \rho & \xi \\ \sigma & \eta \end{smallmatrix}\right) \in \mathfrak{S}$. Further let ϑ be the different of K. We then have

Theorem 17. *Let $f(z)$ be regular in an annular neighbourhood of a cusp λ of $\mathfrak{H}_n$ defined by $z \in \mathfrak{H}_n$, $0 < r < \Delta(z, \lambda) < R$ and let for every $M \in \Gamma_\lambda$, (**??**) be satisfied by $f(z)$. Then, for $n \geq 2$, $f(z)$ is regular in $0 < \Delta(z, \lambda) < R$ and we have*

$$f(z)N(-\sigma z + \rho)^k = c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta > 0} c(\beta)e^{2\pi i S(\beta z_{A^{-1}})},$$

where the summation is over all $\beta \in \mathfrak{a}^2\vartheta^{-1}$ with $\beta > 0$ and the series converges absolutely, uniformly over any compact subset of $\mathfrak{H}_n$ with $0 < \Delta(z, \lambda) < R$.

Proof. Let $w = z_{A^{-1}}$ and $g(w) = f(w_A) = f(z)$. Consider $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = ATA^{-1} \in \Gamma_\lambda$ with $T = \left(\begin{smallmatrix} \epsilon & \zeta\epsilon^{-1} \\ 0 & \zeta\epsilon^{-1} \end{smallmatrix}\right)$ ϵ being a unit in K and $\zeta \in \mathfrak{a}^{-2}$. Then $z_M = (w_T)_A$ and $f(z_M) = g(w_T)$. Moreover, if $h(w) = N(\sigma w + \eta)^{-k}$, then it is easy to verify that

$$h(w_T) = N(\sigma w_T + \eta)^{-k}$$

$$= N(\epsilon)^{-k} N(\gamma w_A + \delta)^{-k} N(\sigma w + \eta)^{-k}$$
$$= N(\epsilon)^{-k} N(\gamma z + \delta)^{-k} h(w).$$

Let us now define $G(w) = g(w)h(w)$. Then, from $f(z_M)N(\gamma z + \delta)^{-k} = f(z)$, we obtain

$$G(\epsilon^2 w + \zeta) = G(w_T) = N(\epsilon)^k G(w)$$

In particular, we have

$$\begin{aligned} G(w + \zeta) &= G(w) \\ G(\epsilon^2 w) &= N(\epsilon)^k G(w). \end{aligned} \tag{158}$$

□ **211**

Let $\alpha_1, \ldots, \alpha_n$ be a minimal basis of $\mathfrak{a}^{-2}$. Further, let $W = (W_1, \ldots, W_n)$ where W_i are defined by

$$w_k = \sum_{i=1}^{n} \alpha_i^{(k)} W_i, \quad k = 1, 2, \ldots, n.$$

From (158), we then see that the function $H(W) = G(w)$ has period 1 in each variable W_i.

Let $w_k = u_k + iv_k$ and $W_k = X_k + iY_k$, for $k = 1, 2, \ldots, n$. Further let D be the domain in the W-space defined by $v_1 > 0, \ldots, v_n > 0$, $R^{-2} < v_1, \ldots, v_n < r^{-2}$. (For $n = 2$, v_1 and v_2 are bounded between the branches of two hyperbolas as in the figure. The mapping $w_k \to W_k$ is just an affine transformation of the w-space). It is clear from the hypothesis that the function $H(W)$ is regular in D.

Figure, page 211

Under the mapping $W \to q = (q_1, q_2, \ldots, q_n)$ with $q_j = e^{2\pi i W_j} (j = 1, 2, \ldots, n)$ the domain D goes into a Reinhardt domain D^* in the q-space and $H^*(q) = H(W)$ is regular in D^*. If now $q^* \in D^*$, we can find a Reinhardt domain V defined by $a_k < |q_k| < b_k$, $k = 1, 2, \ldots, n$ such that $q^* \in V \in D^*$. Since $H^*(q)$ is regular in V, it has a Laurent expansion $\sum\limits_{-\infty < r_1, \ldots, r_n < \infty} c_{r_1, \ldots, r_n} q_1^{r_1} \ldots q_n^{r_n}$ which **212**
converges absolutely, uniformly in every compact subset of V. In other words, for $q \in V$, we have

$$H(W) = H^*(q) = \sum_{-\infty < r_1, \ldots, r_n < \infty} c_{r_1, \ldots, r_n} e^{2\pi i (r_1 W_1 + \cdots + r_n W_n)}.$$

Since D^* is a Reinhardt domain, it is possible to continue $H^*(q)$ analytically from q^* to any point $q \in D^*$ through a finite chain of overlapping Reinhardt domains of the type of V and if, in addition, we use the fact that the Laurent expansion is unique, we see that for all $W \in D$,

$$H(W) = \sum_{-\infty < r_1, \dots, r_n < +\infty} c_{r_1, \dots, r_n} e^{2\pi i (r_1 W_1 + \cdots + r_n W_n)}, \tag{159}$$

the series converging absolutely, uniformly over every compact subset of D.

We can find $\beta_1, \dots, \beta_n$ in K such that $S(\alpha_i \beta_j) = \delta_{ij}$ (the Kronecker symbol), for $i, j = 1, 2, \dots, n$. Then $\beta_1, \dots, \beta_n$ constitute a minimal basis of $\mathfrak{a}^2 \vartheta^{-1}$ and when $r_1, \dots, r_n$ run over all rational integers from $-\infty$ to $+\infty$ independently, $\beta = r_1 \beta_1 + \cdots + r_n \beta_n$ runs over all numbers of the ideal $\mathfrak{a}^2 \vartheta^{-1}$. We shall denote $c_{r_1, \dots, r_n}$ by $c(\beta)$. Now $W_k = \sum_{i=1}^{n} \beta_k^{(i)} w_i$, $k = 1, 2, \dots, n$, as can be easily verified. We then see from (159) that when w lies in the region $v_1 > 0, \dots, v_n > 0$, $R^{-2} < v_1 \dots v_n < r^{-2}$, we have

$$G(w) = H(W) = \sum_{\beta \in \mathfrak{a}^2 \vartheta^{-1}} c(\beta) e^{2\pi i S(\beta w)}. \tag{160}$$

If ϵ is a unit in K, then the transformation $w \to \epsilon^2 w$ corresponds to an affine transformation of the W-space which takes D into itself. We therefore have from (160) that

$$G(\epsilon^2 w) = \sum_{\beta \in \mathfrak{a}^2 \vartheta^{-1}} c(\beta) e^{2\pi i S(\beta \epsilon^2 w)}.$$

On the other hand, from (158) and (160),

$$G(\epsilon^2 w) = N(\epsilon)^k \sum_{\beta \in \mathfrak{a}^2 \vartheta^{-1}} c(\beta) e^{2\pi i S(\beta w)}.$$

For $\beta \in \mathfrak{a}^2 \vartheta^{-1}$, $\beta \epsilon^2$ is also in $\mathfrak{a}^2 \vartheta^{-1}$. On comparing the Fourier coefficients of $G(\epsilon^2 w)$, we obtain **213**

$$c(\beta \epsilon^2) = N(\epsilon)^{-k} c(\beta),$$

i.e.

$$|c(\beta \epsilon^2)| = |c(\beta)|.$$

Thus

$$|c(\beta \epsilon^{2m})| = |c(\beta)| \text{ for any rational integer } m.$$

Let us consider a unit cube U in D, defined as the set of $W = (W_1, \dots, W_n)$, $W_j = X_j + i Y_j^*$ and $0 \leq X_j \leq 1$. Further let $|H(W)| \leq M$ on U. From (159), we have for $\beta = r_1 \beta_1 + \cdots + r_n \beta_n$,

$$c(\beta) = c_{r_1,\ldots,r_n} = \underset{W\in U}{\int \ldots \int} H(W)e^{-2\pi i(r_1W_1+\cdots+r_nW_n)}dX_1\ldots dX_n$$
$$= e^{2\pi S(\beta v^*)} \underset{W\in U}{\int_0^1 \ldots \int_0^1} H(W)e^{-2\pi i(r_1X_1+\cdots+r_nX_n)}dX_1\ldots dX_n,$$

i.e.

$$|c(\beta)| \leq e^{2\pi S(\beta v^*)}M,$$

where $v^* = (v_1^*, \ldots, v_n^*)$ with $v_j^* = \sum_{i=1}^{n} \alpha_i^{(j)} Y_i^* > 0$.

Taking $\beta\epsilon^{2m}$ instead of β, we have

$$|c(\beta\epsilon^{2m})| \leq e^{2\pi S(\beta\epsilon^{2m}v^*)}M,$$

i.e.

$$|c(\beta)| \leq e^{2\pi S(\beta\epsilon^{2m}v^*)}M. \tag{161}$$

Let now $\beta \in \mathfrak{a}^2\vartheta^{-1}$ with $\beta^{(i)} < 0$ for some i. We claim that $c(\beta) = 0$, necessarily. For, using the existence of $n-1(n \geq 2!)$ independent units in K we can find a unit ϵ in K such that

$$|\epsilon^{(i)}| > 1, |\epsilon^{(j)}| < 1 \text{ for } j \neq i.$$

Then, as m tends to infinity, $S(\beta\epsilon^{2m}v^*)$ tends to $-\infty$. Hence, from (161), we see **214**
that unless $\beta > 0$ or $\beta = 0$, $c(\beta) = 0$. Thus

$$G(w) = c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi iS(\beta w)},$$

i.e.

$$f(z)N(-\sigma z + \rho)^k = c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi iS(\beta z_{A^{-1}})}, \tag{162}$$

where the series on the right hand side converges absolutely, uniformly in every compact subset of the region $r < \Delta(z,\lambda) < R$.

Let now $0 < \Delta(z,\lambda) \leq r$ i.e. $r^{-2} \leq N(y_{A^{-1}}) < \infty$. We can then find a number t with $0 < t < 1$ such that $R^{-2} < N(ty_{A^{-1}}) < r^{-2}$. Hence the series $c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi iS(\beta wt)}$ converges absolutely. Now, since

$$c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi iS(\beta w)} \leq |c(0)| + \sum_{\beta} |c(\beta)||e^{2\pi iS(t\beta w)}|,$$

it converges absolutely. It is now clear that the series on the right hand side of (162) converges absolutely, uniformly over every compact subset in $\mathfrak{H}_n$ for which $0 < \Delta(z, \lambda) \leq r$ and provides the analytic continuation of $f(z)N(-\sigma z + \rho)^{+k}$ in $0 < \Delta(z, \lambda) < R$. Thus $f(z)$ is regular in the full neighbourhood $0 < \Delta(z, \lambda) < R$, of the cusp λ and moreover, in this neighbourhood,

$$f(z)N(-\sigma z + \rho)^k = c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi i S(\beta z_{A^{-1}})}. \tag{163}$$

From the theorem above, we can deduce two important corollaries.

(a) *Let V_i denote the neighbourhood $0 < \Delta(z, \lambda_i) < d_i < c$ of the cusp $\lambda_i (i = 1, 2, \ldots, h)$ of $\mathfrak{F}$ and let $\mathfrak{F}^* = \mathfrak{F} - \bigcup_{i=1}^{h} V_i$. Then a function $f(z)$ which is regular in $\mathfrak{F}^*$ and satisfies* (**??**) *for all $M \in \Gamma$ is automatically regular in the whole of $\mathfrak{F}$ and hence in the whole of $\mathfrak{h}_n$ and is an entire modular form, for $n \geq 2$.*

(b) *For $n \geq 2$, if a function $f(z)$ satisfies conditions* (1) *and* (2) *in the definition of a modular form, it satisfies condition* (3) *automatically and is* **215**
therefore regular at the cusp $\mathfrak{F}$.

The proofs of (a) and (b) are very simple, in view of Theorem 17.

The second of the two results above was first proved by Gotzky in the special case $n = 2$ and $K = \mathbf{Q}(\sqrt{5})$, the class number h being 1. In the general case, it was proved by Gundlach. An analogue of this result in the case of modular forms of degree n was established by Koecher.

In the theorem above, we might, instead of $\mathfrak{H}_n$, take, for example, the space $\mathfrak{H}_n^*$ of $(z_1, \ldots, z_n) \in C^n$ with $\mathrm{Im}(z_1) < 0$, $\mathrm{Im}(z_2) > 0, \ldots, \mathrm{Im}(z_n) > 0$. Then, under the same conditions as in the theorem above, we have for $f(z)N(-\sigma z+\rho)^k$ an expansion of the form $c(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi i S(\beta z_{A^{-1}})}$, where β runs over all numbers of $\mathfrak{a}^2\vartheta^{-1}$ for which $\beta^{(1)} < 0, \beta^{(2)} > 0, \ldots, \beta^{(n)} > 0$. For other types of such spaces again we have similar expansions for $f(z)N(-\sigma z + \rho)^k$ at the cusp ρ/σ.

So far, there was no restriction on k except that k should be integral and kn should be even. In view of the following proposition however, we shall henceforth be interested only in modular forms of weight $k > 0$.

Proposition 24. *A modular form of weight $k < 0$ vanishes identically and the only modular forms of weight* 0 *are constants.*

Proof. First, let $f(z)$ be a modular form of weight $k < 0$. Consider $\varphi(z) = N(y)^{k/2}|f(z)|$ for $z = x + iy \in \mathfrak{H}_n$. We shall prove that $\varphi(z)$ is bounded in the whole of $\mathfrak{H}_n$. Let, in fact, $\lambda_i = \rho_i/\sigma_i$ be a cusp of $\mathfrak{F}$ and V_i, the neighbourhood of λ_i defined by $0 < \Delta(z, \lambda_i) < d$. Now, since $N(y)^{k/2} = N(y_{A_i^{-1}})^{k/2}|N(-\sigma_i z + \rho_i)|^k$, we have

$$\varphi(z) = N(y_{A_i^{-1}})^{k/2}|f(z)||N(-\sigma_i z + \rho_i)|^k. \tag{164}$$

Further, from (163),

$$f(z)N(\sigma_i z + \rho_i)^k = c_i(0) + \sum_{\mathfrak{a}^2\vartheta^{-1}|\beta>0} c_i(\beta)e^{2\pi iS(\beta z_{A_i^{-1}})}$$

and clearly for $z \in V_i \cap \mathfrak{F}$, $f(z)N(-\sigma_i z + \rho_i)^k$ is bounded. Moreover, since for $z \in V_i$, $N(y_{A_i^{-1}})^{k/2} < d^{-k}$, we conclude from (**??**) that $\varphi(z)$ is bounded in 216 $\bigcup_{i=1}^{h}(V_i \cap \mathfrak{F})$. Let $\mathfrak{F}^* = \mathfrak{F} - \bigcup_{i=1}^{h}(V_i \cap \mathfrak{F})$. Since $\mathfrak{F}^*$ is compact, $\varphi(z)$ is bounded on $\mathfrak{F}^*$. Thus there exists a constant M such that $\varphi(z) \leq M$ for $z \in \mathfrak{F}$. But now for each $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, $\varphi(z_P) = \varphi(z)$, since

$$\begin{aligned} |f(z_P)|N(y_P)^{k/2} &= |f(z_P)||N(\gamma z + \delta)|^{-k}N(y)^{+k/2} = |f(z)|N(y)^{k/2} \\ &= \varphi(z). \end{aligned}$$

Thus, for all $z \in \mathfrak{h}_n$, $\varphi(z) \leq M$ and hence

$$\begin{aligned} |f(z)| &= \varphi(z)N(y)^{-k/2} \\ &\leq MN(y)^{-k/2}. \end{aligned} \tag{165}$$

Now

$$f(z) = c(0) + \sum_{\vartheta^{-1}|\beta>0} c(\beta)e^{2\pi iS(\beta z)}$$

and

$$\begin{aligned} c(\alpha) &= \int_0^1 \dots \int_0^1 f(z)e^{-2\pi iS(\alpha z)}dx_1 \dots dx_n \\ &= e^{2\pi S(\alpha y)} \int_0^1 \dots \int_0^1 f(x + iy)e^{-2\pi iS(\alpha x)}dx_1 \dots dx_n. \end{aligned}$$

Hence, from (165),

$$|c(\alpha)| \leq MN(y)^{-k/2}e^{2\pi S(\alpha y)}$$

Letting y tend to $(0, \dots, 0)$ and noting that $k < 0$, we see that $c(\alpha) = 0$ for all $\alpha \in \vartheta^{-1}$ and hence $f(z)$ is identically zero. □

Let now $\psi(z)$ be a modular form of weight 0. For each cusp λ_i of $\mathfrak{F}$, we have

$$\psi(z) = c_i(0) + \sum_{\mathfrak{a}_i^2 \vartheta^{-1} | \beta > 0} c_i(\beta) e^{2\pi i S(\beta z_{A_i^{-1}})}.$$

Consider $\chi(z) = \prod_{i=1}^{h} (\psi(z) - c_i(0))$. Let, if possible, $\chi(z)$ be not identically zero i.e. for $z_0 \in \mathfrak{F}$, let $|\chi(z_0)| = \rho \neq 0$. Since $\psi(z) - c_i(0)$ tends to zero as z tends **217**
to λ_i in $\mathfrak{F}$, we can find a neighbourhood V_i of λ_i such that for $z \in V_i \cap \mathfrak{F}$, $|\chi(z)| < (1/2)\rho$. Now $\mathfrak{F}^* = \mathfrak{F} - \bigcup_{i=1}^{h} (V_i \cap \mathfrak{F})$ is compact and $|\chi(z)|$ attains its maximum M at some point of $\mathfrak{F}^*$. We see easily that

$$M = \max_{z \in \mathfrak{F}} |\chi(z)| = \max_{z \in \mathfrak{h}_n} |\chi(z)|.$$

But his means that $\chi(z)$ attains its maximum modulus over $\mathfrak{H}_n$ at an inner point of $\mathfrak{H}_n$ and by the maximum principle, $\chi(z)$ is a constant. But now, since $\chi(z)$ tends to zero as z tends to the cusps of $\mathfrak{F}$, $\chi(z) = 0$ identically in $\mathfrak{H}_n$. This means that $\prod_{i=1}^{h} (\psi(z) - c_i(0)) = 0$. Since $\psi(z)$ is single-valued and regular in $\mathfrak{H}_n$, it follows that $\chi(z)$ is a constant.

Our object now is to give an explicit construction of modular forms of weight $k > 2$ (k being integral). Our method will involve an adaptation of the well-known idea of Poincare for constructing automorphic forms belonging to a discontinuous group Γ_0 of analytic automorphisms of the unit disc D defined by $|z| < 1$ in the complex z-plane. Now, any analytic automorphism of D is given by the mapping $z \to \dfrac{\overline{a}z + \overline{b}}{bz + a}$, where a and b are complex numbers satisfying $|a|^2 - |b|^2 = 1$. Let the elements of Γ_0 be $M_i = \begin{pmatrix} \overline{a}_i & \overline{b}_i \\ b_i & a_i \end{pmatrix}$, $i = 1, 2, \ldots$ (corresponding to the automorphisms $z \to z_{M_i} = \dfrac{\overline{a}_i z + \overline{b}_i}{b_i z + a_i}$ of D).

Let z_0 be a given point of D. Now, since Γ_0 acts discontinuously on D, we can find a neighbourhood V of z_0 such that the images V_M of V for $M \in \Gamma_0$ do not intersect V, unless $(z_0)_M = z_0$. Let n_0 be the order of the isotropy group of z_0 in Γ_0. Then we see that

$$\sum_{M \in \Gamma_0} \iint_{V_M} dx\, dy \leq n_0 \cdot \pi.$$

On the other hand, for $M = \left(\begin{smallmatrix}\bar{a} & \bar{b}\\ b & a\end{smallmatrix}\right) \in \Gamma_0$, we have

$$\iint\limits_{V_M} dx\,dy = \iint\limits_{V} |bz+a|^{-4} dx\,dy.$$

Therefore, 218

$$\sum_{M\in\Gamma_0} \iint\limits_{V} |bz+a|^{-4} dx\,dy \leq n_0 \cdot \pi. \tag{166}$$

Since $|a|^2 - |b|^2 = 1$, $a \neq 0$ and further $|b/a| < 1$. Hence, for $z \in D$,

$$1 - |z| \leq \left|\frac{bz+a}{a}\right| < 1 + |z|. \tag{167}$$

Let V be contained in the disc $|z| \leq r < 1$ and let v denote the area of V. We have, then, in view of (166) and (167),

$$v(1+r)^{-4} \sum_{M\in\Gamma_0} |a|^{-4} \leq \sum_{M\in\Gamma_0} \iint\limits_{V} |bz+a|^{-4} dx\,dy \leq n_0 \cdot \pi. \tag{168}$$

We conclude that $\sum\limits_{M\in\Gamma_0} |a|^{-4}$ is convergent. Further, from (167), we have for $z \in V$,

$$\sum_{M\in\Gamma_0} |bz+a|^{-4} < (1-r)^{-4} \sum_{M\in\Gamma_0} |a|^{-4}$$

Hence $\sum\limits_{M\in\Gamma_0} (bz+a)^{-4}$ converges absolutely, uniformly on every compact subset of V. Since z_0 is arbitrary, this is true even for every compact subset of D.

Using the same idea as above, we shall prove that actually $\sum\limits_{M\in\Gamma_0} (bz+a)^{-k}$ converges absolutely for $k > 2$. Consider

$$I_l = \iint\limits_{V} \frac{dx\,dy}{(1-|z|^2)^l};$$

the integral converges if and only if $l < 1$. Now, as before

$$\sum_{M} \iint\limits_{V_M} \frac{dx\,dy}{(1-|z|^2)^{-l}} \leq n_0 I_l.$$

But

$$\iint\limits_{V_M} \frac{dx\,dy}{(1-|z|^2)^l} = \iint\limits_{V} |bz+a|^{-4+2l} \frac{dx\,dy}{(1-|z|^2)^l} \quad \text{for } M = \begin{pmatrix}\bar{a} & \bar{b}\\ b & a\end{pmatrix} \in \Gamma_0.$$

Thus **219**

$$\sum_{M\in\Gamma_0} \iint_V |bz+a|^{2l-4} \frac{dx\,dy}{(1-|z|^2)^l} = \sum_M \iint_{V_M} \frac{dx\,dy}{(1-|z|^2)^l} \leq n_0 I_l.$$

Using the same argument as earlier, it is easily shown that $\sum_{M\in\Gamma_0}(bz+a)^{-4+2l}$ and therefore $\sum_{M\in\Gamma_0}(bz+a)^{-k}$, (for $k > 2$) converges absolutely, uniformly on every compact subset of D.

Let k be an even rational integer *greater than* 2. Then we see that $\psi(z,k) = \sum_{M\in\Gamma_0}(bz+a)^{-k}$ is analytic in D. To verify that $\psi(z,k)$ is an automorphic form belonging to Γ_0, we have only to show that

$$\text{for } N = \begin{pmatrix}\overline{p} & \overline{q}\\ q & p\end{pmatrix} \in \Gamma_0, \psi(z_N,k)(qz+p)^{-k} = \psi(z,k).$$

Now, for

$$M = \begin{pmatrix}\overline{a} & \overline{b}\\ b & a\end{pmatrix} \in \Gamma_0, \left(\frac{dz_M}{dz}\right)^{k/2} = (bz+a)^{-k}$$

and so

$$\psi(z,k) = \sum_{M\in\Gamma_0}\left(\frac{dz_M}{dz}\right)^{k/2}$$

Further

$$\begin{aligned}\psi(z_N,k) &= \sum_{M\in\Gamma_0}\left(\frac{dz_{MN}}{dz_N}\right)^{k/2}\\ &= \left(\frac{dz_N}{dz}\right)^{-k/2}\sum_{M\in\Gamma_0}\left(\frac{dz_{MN}}{dz}\right)^{k/2}\\ &= (qz+p)^k\psi(z,k),\end{aligned}$$

since MN runs over all elements of Γ_0, when M does so. Thus $\psi(z,k)$ is an automorphic form of weight k, for the group Γ_0 and is a so-called **Poincaré series** of weight k for Γ_0.

The method given above for the construction of automorphic forms for the group Γ_0 is due to Poincaré and it cannot be carried over as such to the case of the Hilbert modular group, since the euclidean volume of $\mathfrak{H}_n$ is infinite. In the following, we shall give a method of constructing Hilbert modular forms of **220** weight $k > 2$, based on a slight modification of the idea of Poincaré. We shall be interested only in the case $n \geq 2$.

Let $\lambda = \rho/\sigma$ be a cusp of $\mathfrak{H}_n$, $\mathfrak{a} = (\rho, \sigma) = \mathfrak{a}_i$ (for some i), $A = \left(\begin{smallmatrix} \rho & \xi \\ \sigma & \eta \end{smallmatrix}\right) \in \mathfrak{S}$ associated with λ and $z_{A^{-1}} = z^* = x^* + iy^*$. We had introduced in $\mathfrak{H}_n$ a system of 'local coordinates' relative to the cusp λ, namely, $N(y^*)^{-\frac{1}{2}}, Y_1, \ldots, Y_{n-1}$, $X_1, \ldots, X_n$. Defining $q = N(y^*)$, it is easily seen that the relationship between the volume element $dv = dq, dY_1, \ldots, dY_{n-1}, dX_1, dX_2, \ldots, dX_n$ and the ordinary euclidean volume element $dx_1 dy_1, \ldots, dx_n dy_n$ in $\mathfrak{H}_n$ is given by

$$dv = \frac{N(\mathfrak{a})^2}{2^{n-1} R_0 \sqrt{|d|}} |N(-\sigma z + \rho)|^{-4} dx_1 dy_1, \ldots, dx_n dy_n,$$

where R_0 and d are respectively the regulator and the discriminant of K. Let now

$$c = \frac{N(\mathfrak{a})^2}{2^{n-1} R_0 \sqrt{|d|}}$$

and let $d\omega$ be the volume element $N(y)^{-2} dx_1 dy_1, \ldots, dx_n dy_n$. Then $d\omega$ is invariant under a (modular) substitution $z \to z_M$ for $M \in \Gamma$ and moreover

$$dv = c \cdot N(y^*)^2 d\omega = c\Delta(z, \lambda)^{-4} d\omega.$$

Now, under the transformation $z \to z_M$ for $M \in \Gamma$, it may be verified that

$$dv \to c\Delta(z_M, \lambda)^{-4} d\omega = \frac{\Delta(z, \lambda)^4}{\Delta(z, \lambda_{M^{-1}})^4}, \tag{169}$$

$$N(y^*) = \Delta(z, \lambda)^{-2} \to \Delta(z_M, \lambda)^{-2} = \Delta(z, \lambda_{M^{-1}})^{-2}.$$

We can find a point $z_0 \mathfrak{U}\mathfrak{a}_{\lambda,d}$ such that z_0 is not a fixed point of any modular **221**
transformation except the identity. Now, by Theorem 16 again, we can find a neighbourhood V of z_0 contained in $\mathfrak{U}_{\lambda,d} \cap \mathfrak{G}_\lambda$ such that $V \cap V_M = \emptyset$, except for $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in Γ. Let $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $M_2, \ldots$, be a complete system of representatives of the right cosets of Γ modulo Γ_λ. For $i \neq 1$, M_i takes $\mathfrak{U}_{\lambda,d}$ into $\mathfrak{U}_{\mu,d}$ where $\mu = \lambda_{M_i} \neq \lambda$ and since $\mathfrak{U}_{\lambda,d} \cap \mathfrak{U}_{\mu,d} = \emptyset$, we see that for $i \neq 1$, V_{M_i} lies wholly outside the neighbourhood $\mathfrak{U}_{\lambda,d}$. Further $V_{M_i} \cap V_{M_j} = \emptyset$ unless $M_i = M_j$. Now, by applying suitable transformations in Γ_λ to the images V_{M_i} or parts of the same, we might suppose that $V_{M_i} \subset \mathfrak{G}_\lambda$ for all i. It is to be remarked that under this process, the sets V_{M_i} continue to have finite euclidean volume and to be mutually non-intersecting. Let now, for a suitable $d_0 > 0$, $V \subset \mathfrak{U}_{\lambda,d} - \mathfrak{U}_{\lambda,d_0}$. We then see that the mutually disjoint sets V_{M_i}, $i = 1, 2, \ldots$, are all contained in $\mathfrak{G}_\lambda - \mathfrak{U}_{\lambda,d_0}$. It is now easily verified that

$$J_s = \sum_{M_i} \int \ldots \int_{V_{M_i}} N(y^*)^{s-2} dv$$

$$\leq \int_0^{d_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \dots \int_{-\frac{1}{2}}^{\frac{1}{2}} q^{s-2} dq\, dY_1, \dots, dY_{n-1} dX_1, \dots, dX_n$$
$$= \int_0^{d_0} q^{s-2} dq$$

and J_s converges if and only if $s > 1$.

On the other hand, we obtain by (169), for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$

$$\underset{V_M}{\int \dots \int} N(y^*)^{s-2} dv = \underset{V}{\int \dots \int} \frac{|N(-\sigma z + \rho)|^4 N(y)^{s-2}}{|N(-\sigma z_M + \rho)|^{2s} |N(\gamma z + \delta)|^{2s}} dv.$$

Now, let

$$A^{-1} M_j = P_j = \begin{pmatrix} * & * \\ \gamma_j & \delta_j \end{pmatrix}, \quad j = 1, 2, \dots$$

Then P_j runs over a complete set of matrices of the form $A^{-1}M$ with $M \in \Gamma$ such that for $i \neq j$, $P_i \neq NP_j$ with $N \in A^{-1}\Gamma_\lambda A$. Equivalently, we see that (γ_j, δ_j) runs over a complete set of non-associated pairs of integers γ_j, δ_j in $\mathfrak{a}$ such that $(\gamma_j, \delta_j) = \mathfrak{a}$. Now

$$J_s = \sum_{M_i} \underset{V_M}{\int \dots \int} N(y^*)^{s-2} dv$$
$$= \sum_{\gamma_j, \delta_j}{}' \underset{V}{\int \dots \int} |N(\gamma_j z + \delta_j)|^{-2s} |N(-\sigma z + \rho)|^{2s} N(y^*)^{s-2} dv$$

where on the right hand side, the summation is over a complete set of non- 222
associated pairs of integers (γ_j, δ_j) such that $(\gamma_j, \delta_j) = \mathfrak{a}$.

Let B be a compact set in $\mathfrak{H}_n$. We can then find constants c_{13} and c_{14} depending only on B such that for all $z \in B$ and real μ and ν, we have

$$c_{13} |N(\mu j + \nu)| \leq |N(\mu z + \nu)| \leq c_{14} |N(\mu j + \nu)|. \tag{170}$$

Taking for B, the closure $\overline{V}$ and V in $\mathfrak{H}_n$, we see that for a suitable constant c_{15} depending only on V and for $s > 1$,

$$\sum_{\gamma_j, \delta_j}{}' |N(\gamma_j i + \delta_j)|^{-2s} \leq c_{15} J_s < \infty.$$

Let now z lie in an arbitrary compact set B in $\mathfrak{H}_n$. Then, in view of (170), we can find a constant c_{16} depending only on B, such that for all $z \in B$,

$$\sum_{\gamma_j, \delta_j}{}' |N(\gamma_j z + \delta_j)|^{-2s} \leq c_{16} \sum_{\gamma_j, \delta_j}{}' |N(\gamma_j i + \delta_j)|^{-2s} < \infty.$$

Thus the series $\sum'_{\gamma_j,\delta_j} N(\gamma_j z + \delta_j)^{-2s}$ converges absolutely, uniformly over every compact subset of $\mathfrak{H}_n$, for $s > 1$.

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote a full system of pairs of integers (γ, δ) satisfying $(\gamma, \delta) = \mathfrak{a}$ and not mutually associated, respectively with respect to the group of all units in K and the group of units of norm 1 in K.

Let k be a rational integer greater than 2. Corresponding to an integral ideal $\mathfrak{a}$ in K, we now define for $z \in \mathfrak{H}_n$, the function

$$G^*(z, k, \mathfrak{a}) = \sum_{(\gamma^*,\delta^*)\in\mathfrak{a}_2} N(\gamma^* z + \delta^*)^{-k}.$$

We first observe that $G^*(z, k, \mathfrak{a})$ is well-defined and is independent of the particular choice of elements of $\mathfrak{a}_2$. Further, since the group of units of norm 1 in 223
K is of index at most 2 in the group of all units in K. We deduce from the foregoing considerations that the series $\sum_{(\gamma^*,\delta^*)\in\mathfrak{a}_2} N(\gamma^* z + \delta^*)^{-k}$ converges absolutely uniformly over any compact subset of $\mathfrak{H}_n$. Hence $G^*(z, k, \mathfrak{a})$ is regular in $\mathfrak{H}_n$. Moreover, for $M = \begin{pmatrix} \rho & x \\ \mu & \nu \end{pmatrix} \in \Gamma$,

$$\begin{aligned} G^*(z_M, k, \mathfrak{a}) &= \sum_{(\gamma^*,\delta^*)\in\mathfrak{a}_2} N(\gamma^* z_M + \delta^*)^{-k} \\ &= N(\mu z + \nu)^k \sum_{(\gamma^*,\delta^*)\in\mathfrak{a}_2} N((\gamma^*\rho + \delta^*\mu)z + \gamma^* x + \delta^*\nu)^{-k}. \end{aligned}$$

Now $(\gamma^*\rho + \delta^*\mu, \gamma^* x + \delta^*\nu)$ again runs over a system $\mathfrak{a}_2$ when (γ^*, δ^*) does so. Thus

$$G^*(z_M, k, \mathfrak{a}) N(\mu z + \nu)^{-k} = G^*(z, k, \mathfrak{a}).$$

Therefore $G^*(z, k, \mathfrak{a})$ is a modular form of weight k.

Let k be odd and K contain a unit ϵ_0 of norm -1; for example, if the degree n is odd, then -1 has norm -1. If (γ, δ) runs over a system $\mathfrak{a}_1$, then (γ, δ) together with $(\epsilon_0\gamma, \epsilon_0\delta)$ runs over a system $\mathfrak{a}_2$. Thus

$$G^*(z, k, \mathfrak{a}) = (1 + N(\epsilon_0)^k) \sum_{(\gamma,\delta)\in\mathfrak{a}_1} N(\gamma z + \delta)^{-k} = 0,$$

since $N(\epsilon_0)^k + 1 = 0$. Thus the functon $G^*(z, k, \mathfrak{a})$ vanishes identically in the case when k is odd and K contains a unit of norm -1. We shall *henceforth* exclude this case from our consideration; in particular, nk should necessarily be even. We have then

$$G^*(z, k, \mathfrak{a}) = \rho G(z, k, \mathfrak{a}),$$

where, by definition

$$G(z, k, \mathfrak{a}) = \sum_{(\gamma,\delta)\in\mathfrak{a}_1} N(\gamma z + \delta)^{-k}$$

and ρ is the index of the group of units of norm 1 in K in the group of all units in K. The functions $G(z, k, \mathfrak{a})$ are again modular forms of weight k and are the so-called **Eisenstein series** for the Hilbert modular group.

If $\mathfrak{a} = (\mu)\mathfrak{a}_i$ for some $i(= 1, 2, \ldots h)$ and $\mu \in K$, then $G(z, k, \mathfrak{a}) = N(\mu)^{-k} G(z, k, \mathfrak{a}_i)$. 224
For the sake of brevity, we may denote $G(z, k, \mathfrak{a}_i)$ by $G_i(z, k)$ for $i = 1, 2, \ldots, h$.

We can obtain the Fourier expansion of $G_i(z, k)$ at the various cusps λ_j of $\mathfrak{F}$ (cf. formula (163) above) by using the following generalized 'Lipschitz formula' valid for rational integral $s > 2$ and $\tau \in \mathfrak{H}_n$, namely,

$$\sum_{\substack{\zeta \text{ integral} \\ \zeta\in K}} N(\tau + \zeta)^{-s} = \frac{\left(\frac{2\pi}{i}\right)^{ns}}{\sqrt{N(\vartheta)}(\Gamma(s))^n} \sum_{\substack{\mu\in\vartheta^{-1} \\ \mu>0}} (N(\mu))^{s-1} e^{2\pi S(\mu\tau)}. \tag{171}$$

But we shall not go into the details here.

A modular form $f(z)$ is said to *vanish at a cusp* $\lambda = \rho/\sigma$ of $\mathfrak{H}_n$, if, in the Fourier expansion of $f(z)N(-\sigma z + \rho)^k$ at the cups λ (cf. formula (163)), the constant term $c(0)$ is zero. If $f(z)$ vanishes at a cusp λ, it clearly vanishes at all the equivalent cusps.

We shall now prove the following

Proposition 25. *The Eisenstein series $G_i(z, k)(k > 2)$ vanishes at all cusps of $\mathfrak{H}_n$ except at those equivalent to λ_i.*

Proof. Let $\lambda_j = \rho_j/\sigma_j$ be a base cusp of $\mathfrak{F}$ and $A_j = \left(\begin{smallmatrix} \rho_j & \xi_j \\ \sigma_j & \eta_j \end{smallmatrix}\right)$ be the associated matrix in $\mathfrak{S}$. Then setting $w = z_{A_j^{-1}}$, we have

$$G_i(z, k)N(-\sigma_j z + \rho_j)^k = \sideset{}{'}\sum_{(\gamma,\delta)=\mathfrak{a}_i} N((\gamma\rho_j + \delta\sigma_j)w + \gamma\xi_j + \delta\eta_j)^{-k}.$$

Let us denote by c_{ij} the constant term in the Fourier expansion of $G_i(z, k)N(-\sigma_j z + \rho_j)^k$ at the cusp λ_j. If we set $w = (\sqrt{-1}t, \ldots \sqrt{-1}t)(t > 0)$, then

$$\begin{aligned} c_{ij} &= \lim_{t\to\infty} G_i(z, k)N(-\sigma_j z + \rho_j)^k \\ &= \lim_{t\to\infty} \sideset{}{'}\sum_{(\gamma,\delta)=\mathfrak{a}_i} N((\gamma\rho_j + \delta\sigma_j)\sqrt{-1}t + \gamma\xi_j + \delta\eta_j)^{-k} \end{aligned}$$

$$= \sum_{(\gamma,\delta)=\mathfrak{a}_i}' \lim_{t\to\infty} N((\gamma\rho_j + \delta\sigma_j)\sqrt{-1}t + \gamma\xi_j + \delta\eta_j)^{-k}, \tag{172}$$

since the interchange of the limit and the summation can be easily justified. 225
Now unless $\gamma\rho_j + \delta\sigma_j = 0$, the corresponding term in the series (172) is zero. If $\gamma\rho_j + \delta\sigma_j = 0$, then

$$\frac{(\gamma)}{\mathfrak{a}_i}\frac{(\rho_j)}{\mathfrak{a}_j} = \frac{(\delta)}{\mathfrak{a}_i}\frac{(\sigma_j)}{\mathfrak{a}_j}$$

and from this, it follows that necessarily $i = j$. Further, if $\gamma\rho_j + \delta\sigma_j = 0$, it can be easily shown that $\gamma\xi_j + \delta\eta_j \neq 0$. Moreover, in the series (172), there can occur only the pair (γ, δ) for which $\gamma\rho_j + \delta\sigma_j = 0$ and indeed this happens only if $- = j$. Therefore $c_{ij} = 0$, unless $i = j$. Our contention that $G_i(z, k)$ vanishes at all cusps of $\mathfrak{F}$ except at λ_i is therefore established.

We deduce that *the h Eisenstein series $G_i(z, k), i = 1, 2, \ldots, h$ are linearly independent over the field of complex numbers.* If $\sum_{i=1}^{h} \alpha_i G_i(z, k) = 0$, then letting z tend to cusp λ_j, the relation $\left(\sum_{i=1}^{h} \alpha_i G_i(z, k)\right) N(-\sigma_j z + \rho_j)^k = 0$ gives $\alpha_j c_{jj} = 0$ i.e. $\alpha_j = 0$. Thus $\alpha_i = 0$ for $i = 1, 2, \ldots, h$. □

A modular form which vanishes at all the cusps of $\mathfrak{F}$ is known as a *cusp form.*

It is now easy to prove

Proposition 26. *For a given modular form $f(z)$ of integral weight $k > 2$, we can find a linear combination $\varphi(z)$ of the Eisenstein series $G_i(z, k)$ such that $f(z) - \varphi(z)$ is a cusp form.*

Proof. If $c_j(0)$ is the constant term in the Fourier expansion of $f(z)N(-\sigma_j z + \rho_j)^k$ at cusp $\lambda_j = \rho_j/\sigma_j$ of $\mathfrak{F}$, then the function $\varphi(z) = \sum_{i=1}^{h} c_i(0)c_{ii}^{-1}G_i(z, k)$ satisfies the requirements of the proposition. □

The method outlined above for constructing modular forms yields only Eisenstein series but no cusp forms. In order to construct nontrivial cusp forms of integral weight $k > 2$, we proceed as follows.

Let $\mathfrak{H}_n^*$ denote the set of $z \in C^n$ for which $-z \in \mathfrak{H}_n$. Further let $w = u - iv \in$ 226
$\mathfrak{H}_n^*$. We define for $z = x + iy \in \mathfrak{H}_n$, integral l and integral $k > 2$, the function

$$\Phi(w, z) = \sum_{M\in\Gamma} N(\gamma z + \delta)^{-k} N(w - z_M)^{-l} \tag{173}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ runs over all elements of Γ. We shall show that the series (173) converges absolutely, uniformly for z and w lying in compact sets in $\mathfrak{H}_n$ and $\mathfrak{H}_n^*$ respectively, provided that $l > 2$, $k \geq 4$.

To this end, we choose $z_0 \in \mathfrak{R}_1$ such that z_0 is not a fixed point of any M in Γ, except $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can then find a neighbourhood V of z_0 such that the images V_M of V for $M \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Γ, do not intersect V. We may suppose moreover that for a suitable $d' > 0$, if W denotes the annular neighbourhood of the base cusp ∞ defined by $0 < d' < \Delta(z, \infty) < d$, then $V \subset \mathfrak{G}_\infty \cap W$. It will then follow from result (ii) on p. 199 that for $M \notin \Gamma_\infty$, V_M lies outside the neighbourhood $\mathfrak{U}_{\infty,d}$ and moreover for $M \in \Gamma_\infty$, $V_M \subset W$. In other words, for $M \in \Gamma$, the images V_M of V are mutually disjoint and further if $z \in \bigcup\limits_{M \in \Gamma} V_M$, then necessarily we have $\Delta(z, \infty) \geq d'$, i.e. $\bigcup\limits_{M \in \Gamma} V_M$ is contained in a region of the form $N(y) < \alpha$, for some $\alpha > 0$.

Let us now consider the integral

$$I(s, l, \alpha) = \underset{N(y) \leq \alpha}{\int \ldots \int} N(y)^{s-2} |N(w - z)|^{-l} dv,$$

where $dv = dx_1 dy_1, \ldots, dx_n dy_n$ and the integral is extended over the set in $\mathfrak{H}_n$ defined by $N(y) \leq \alpha$. If $s \geq 2$, then trivially

$$I(s, l, \alpha) \leq \alpha^{s-2} \underset{\mathfrak{H}_n}{\int \ldots \int} |N(w - z)|^{-l} dv$$

i.e.

$$\begin{aligned} I(s, l, \alpha) &\leq \alpha^{s-2} \underset{\mathfrak{H}_n}{\int \ldots \int} N((u - x)^2 + (v + y)^2)^{-l/2} dv \\ &= \alpha^{s-2} \int_0^\infty \ldots \int_0^\infty N(v + y)^{1-l} dy_1 \ldots dy_n \\ &\quad \int_{-\infty}^\infty \ldots \int_{-\infty}^\infty N(1 + x^2)^{-l/2} dx_1 \ldots dx_n \\ &< \infty, \end{aligned}$$

provided $l > 2$. Now, since the sets V_M are mutually disjoint, we have **227**

$$\sum_{M \in \Gamma} \underset{V_M}{\int \ldots \int} N(y)^{k/2-2} |N(w - z)|^{-l} dv \leq I\left(\frac{k}{2}, l, \alpha\right) < \infty,$$

if $l > 2$, $k \geq 4$. But, on the other hand, for $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$,

$$\int_{V_M} \cdots \int N(y)^{k/2-2}(N(w-z))^{-l} dv = \int_V \cdots \int N(y)^{k/2-2}(N(w-z_M))^{-l} N(\gamma z+\delta)^{-k} dv$$

Hence

$$\sum_{M\in\Gamma} \int_V \cdots \int N(y)^{k/2-2}|N(w-z_M)|^{-l}|N(\gamma z+\delta)|^{-k} dv \leq I\left(\frac{k}{2}, l, \alpha\right) < \infty. \tag{174}$$

Let B, B^* be compact sets in $\mathfrak{H}_n$ and $\mathfrak{H}_n^*$ respectively. Then, for $z \in B$ and $w \in B^*$ we can find constants c_{17} and c_{18} depending only on B and B^* and on z_0, such that

$$0 < c_{17} \leq \frac{|N(w-z_M)|^{-l}}{|N(w-(z_0)_M)^{-l}} \frac{|N(\gamma z+\delta)|^{-k}}{N(\gamma z_0+\delta)|^{-k}} \leq c_{18}. \tag{175}$$

Taking for B, the closure V of V in $\mathfrak{H}_n$, we obtain from (174) and (175) that $\sum_{M\in\Gamma} |N(w-(z_0)_M)|^{-l}|N(\gamma z_0+\delta)|^{-k}$ converges and indeed uniformly with respect 228
to $w \in B^*$. But again from (175), we have for $z \in B$ and $w \in B^*$,

$$\sum_{M\in\Gamma} |N(w-z_M)|^{-l}|N(\gamma z+\delta)|^{-k} \leq c_{18} \sum_{M\in\Gamma} |N(w-(z_0)_M)|^{-l}|N(\gamma z_0+\delta)|^{-k}.$$

Hence for $l > 2$, $k \geq 4$, the series (173) converges absolutely, uniformly when z and w lie in compact sets in $\mathfrak{H}_n$ and $\mathfrak{H}_n^*$ respectively.

We shall suppose hereafter that $l > 2$, $k \geq 4$; then we see that $\Phi(w, z)$ is regular in $\mathfrak{H}_n$ as a function of z and regular in $\mathfrak{H}_n^*$, as a function of w.

If $P = \left(\begin{smallmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{smallmatrix}\right) \in \Gamma$, then we have

$$\Phi(w, z_P) = \sum_{M\in\Gamma} N(\gamma z_P+\delta)^{-k} N(w-z_{MP})^{-l}$$

$$N(\gamma^* z+\delta^*)^k \sum_{M\in\Gamma} N((\gamma\alpha^*+\delta\gamma^*)z+\gamma\beta^*+\delta\delta^*)^{-k} N(w-z_{MP})^{-l}$$

i.e.

$$\Phi(w, z_P)N(\gamma^* z + \delta^*)^{-k} = \Phi(w, z). \tag{176}$$

Further, if $T = \left(\begin{smallmatrix} \epsilon & \zeta\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{smallmatrix}\right) \in \Gamma$ with a unit ϵ and integral ζ in K, then

$$\begin{aligned}
\Phi(w_T, \zeta) &= \sum_{M\in\Gamma} N(\gamma z + \delta)^{-k} N(\epsilon^2 w + \zeta - z_M)^{-l} \\
&= N(\epsilon)^k \sum_{M\in\Gamma} N(\epsilon\gamma z + \epsilon\delta)^{-k} N(w - z_T - 1_M)^{-l} \\
&= N(\epsilon)^k \Phi(w, z).
\end{aligned}$$

In particular, $\Phi(w + \zeta, z) = \Phi(w, z)$ for integral $\zeta \in K$. We shall now proceed to obtain the Fourier expansion of $\Phi(w, z)$.

Let Γ_t denote the subgroup of Γ consisting of $M = \left(\begin{smallmatrix} 1 & \zeta \\ 0 & 1 \end{smallmatrix}\right)$ with integral ζ in K. Then clearly

$$\Phi(w, z) = \sum_{M\in\Gamma_t\backslash\Gamma} N(\gamma z + \delta)^{-k} \sum_{\substack{\zeta \text{ integral} \\ \zeta\in K}} N(w - z_M + \zeta)^{-l}, \tag{177}$$

where, in the outer summation, M runs over a full set of representatives of the right cosets of Γ modulo Γ_t. It is now easy to see that the set $\{\pm M | M \in \Gamma_t\backslash\Gamma\}$ is in one-one correspondence with the set of all pairs of coprime integers (γ, δ) in K. To each coprime pair (γ, δ) of integers in K, let us assign a fixed $M = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$. Then **229**

$$2\Phi(w, z) = \sum_{(\gamma,\delta)=(1)} N(\gamma z + \delta)^{-k} \sum_{\substack{\zeta \text{ integral} \\ \zeta\in K}} N(w - z_M + \zeta)^{-l}.$$

The inner sum is obviously independent of the particular $M \in \Gamma$ associated with the pair (γ, δ).

In view of (**??**), we have

$$\begin{aligned}
&\sum_{\substack{\zeta \text{ integral} \\ \zeta\in K}} N(w - z_M + \zeta)^{-l} \\
&\qquad = \frac{(2\pi i)^{ln}}{\sqrt{|d|}(\Gamma(l))^n} \sum_{\lambda\in\vartheta^{-1}, \lambda>0} N(\lambda)^l e^{-2\pi i S(\lambda(w - z_M))}.
\end{aligned} \tag{178}$$

Therefore, after suitable rearrangement of the series which is certainly permissible, we obtain from (177) and (178)

$$\Phi(w, z) = \frac{(2\pi i)^{ln}}{2\sqrt{|d|}(\Gamma(l))^n} \sum_{\lambda\in\vartheta^{-1}, \lambda>0} N(\lambda)^l e^{-2\pi i S(\lambda w)} \times$$

$$\times \sum_{(\lambda,\delta)=0} e^{2\pi i S(\lambda z_M)} N(\gamma z+\delta)^{-k}.$$

We now define for $k > 2$ and $z \in \mathfrak{H}_n$,

$$f_\lambda(z,k) = \sum_{(\gamma,\delta)=0} e^{2\pi i S(\lambda z_M)} N(\gamma z+\delta)^{-k}. \tag{179}$$

Then we have

$$\Phi(w,z) = \frac{(2\pi i)^{ln}}{2\sqrt{|d|}(\Gamma(l))^n} \sum_{\lambda\in\vartheta^{-1},\lambda>0} N(\lambda)^l e^{-2\pi i S(\lambda w)} f_\lambda(z,k).$$

Now, the series (179) converges absolutely, uniformly over every compact subset of $\mathfrak{H}_n$, since the series (173) does so. Therefore $f_\lambda(z,k)$ is regular in $\mathfrak{H}_n$; further it is bounded in the fundamental domain, for $n = 1$. Moreover, for $P = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \in \Gamma$, we have from (176), **230**

$$f_\lambda(z_P,k) N(\gamma^* z+\delta^*)^{-k} - f_\lambda(z,k),$$

in view of the uniqueness of the Fourier expansion of $\Phi(w,z)$. Thus $f_\lambda(z,k)$ is a modular form of weight k and in fact, a cusp form, since it might be shown that as z tends to the cusp $\lambda_i = \rho_i/\sigma_i$ in $\mathfrak{F}$, $f_\lambda(z,k)\cdot N(-\sigma_i z+\rho_i)^k$ tends to zero. A priori, it might happen that $f_\lambda(z,k) = 0$ identically for all $\lambda \in \vartheta^{-1}$ with $\lambda > 0$. However, we shall presently see that this does not happen for large k. The series (179) are called **Poincaré series** for the Hilbert modular group. They were introduced by Poincaré in the case $n = 1$ (elliptic modular group).

Let $A_i = \begin{pmatrix} \rho_i & \xi_i \\ \sigma_i & \eta_i \end{pmatrix} \in \mathfrak{S}$ correspond to the cusp $\lambda_i = \rho_i/\sigma_i$ of $\mathfrak{F}$ and let $\mathfrak{a}_i = (\rho_i, \sigma_i)$. By considering the function

$$\Phi(w,z,A_i) = \sum_P N(\gamma z+\delta)^{-k} N(w-z_P)^{-l}$$

where $P = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$ runs over all elements of $\mathfrak{S}$ of the form $A_i^{-1}M$ with $M \in \Gamma$, we can obtain the Poincaré series

$$f_\lambda(z,k,A_i) = \sum_{(\gamma,\delta)=\mathfrak{a}_i} N(\gamma z+\delta)^{-k} e^{2\pi i S(\lambda z_P)}, \tag{180}$$

which are again cusp forms of weight k. In (180), the summation is over all pairs of integers (γ,δ) such that $(\gamma,\delta) = \mathfrak{a}_i$ and with each such pari (γ,δ) is associated a fixed $P = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \mathfrak{S}$, of the form $A_i^{-1}M$ with $M \in \Gamma$. Further $\lambda > 0$ is a number in $\mathfrak{a}_i^2\vartheta^{-1}$. It is clear that the Poincaré series $f_\lambda(z,k,A_1)$ coincide with $f_\lambda(z,k)$ introduced above. Poincaré theta-series for a discontinuous group of automorphisms of a bounded domain in C^n have been discussed by Hua.

The Fourier coefficients of the Poincaré series $f_\lambda(z,k,A_i)$ have been explicitly determined by Gundlach and they are expressible as infinite series involving Kloosterman sums and Bessel functions.

It has been shown by Maass that for fixed $i(=1,2,\ldots,h)$ the Poincaré series $f_\lambda(z,k,A_i)$ for $\lambda \in \mathfrak{a}_i^2\vartheta^{-1}$, $\lambda > 0$ together with the h Eisenstein series $G_i(z,k)$ generate the vector space of modular forms of weight $k(>2)$ over the 231
field of complex numbers. The Poincaré series $f_\lambda(z,k,A_i)$ themselves generate the subspace of all cusp forms of weight k; this is the so-called **'completeness theorem for the Poincaré series'.** The essential idea of the proof is the introduction of a scalar product (f,φ) for two modular forms $f(z)$ and $\varphi(z)$ of weight k, of which at least one is a cusp form, namely,

$$(f,\varphi) = \int \ldots \int_{\mathfrak{F}} f(z)\overline{\varphi(z)}N(y)^{k-2}dx_1dy_1,\ldots,dx_ndy_n$$

the integral being extended over a fundamental domain for the Hilbert modular group. This scalar product which is known as the "Petersson scalar product" was used by Petersson for $n=1$, in order to prove the "completeness theorem" for the Poincaré series of real weight $k \geq 2$, belonging to even more general groups than the elliptic modular group, such as the Grenzkreis groups of the first type. They are called horocyclic groups. The "completeness theorem" for the Poincaré series belonging to the principal congruence subgroups of the elliptic modular group was proved, independently of Petersson, by A. Selberg by a different method.

For $n=1$, the scalar product (f,φ) has been explicitly computed by Rankin.

As we remarked earlier, it could happen a priori that all the Poincaré series $f_\lambda(z,k)$ vanish identically. However, we shall presently see that this does not happen if k is large.

Proposition 27. *There exist $n+1$ modular forms $f_0(z)$, $f_1(z),\ldots,f_n(z)$ of large weight k such that $f_1(z)f_0^{-1}(z),\ldots,f_n(z)f_0^{-1}(z)$ are analytically independent.*

Proof. Let us choose a basis $\omega_1(=1), \omega_2,\ldots,\omega_n$ of K over Q with integral $\omega_i > 0 (i=2,\ldots,n)$; clearly $|(\omega_i^{(j)})| \neq 0$. Further let $\lambda_i = \omega_i + g$ for a rational integer $g > 0$, to be chosen later; again we have $|(\lambda_i^{(j)})| \neq 0$. Let us, moreover, denote $G_1(z,k)$, $f_{\lambda_1}(z,k),\ldots,f_{\lambda_n}(z,k)$ by $f_0(z)$, $f_1(z),\ldots,f_n(z)$ respectively. We shall show that for large k and g to be chosen suitably, the n functions $g_i(z) = f_i(z)f_0^{-1}(z)$, $i=1,2,\ldots,n$, are analytically independent. □

We know that for real $t > 1$, $z_0 = (it, \ldots, it)$ is an inner point of $\mathfrak{q}_1$. Hence 232
for all pairs of coprime integers (γ, δ) in K with $\gamma \neq 0$, we have $|N(\gamma z_0 + \delta)| > 1$. One can show easily as a consequence that $\lim\limits_{k\to\infty} G_1(z_0, k) = \sum\limits_{(\gamma,\delta)=1} \lim N(\gamma z_0 + \delta)^{-k} = 1$. Thus, for large k, $|f_0(z_0) - 1|$ can be made arbitrarily small. Similarly, by taking k to be suitably large, we can ensure that $|f_m(z_0) - \sum\limits_{\epsilon} e^{-2\pi t S(\lambda_m \epsilon^2)}|$ and hence $|g_m(z_0) - \sum\limits_{\epsilon} e^{-2\pi t S(\lambda_m \epsilon^2)}|$ for $m = 1, 2, \ldots, n$ can be made as small as we like; here, in the summation ϵ runs over all units of K. Since $g_m(z)$ and $\sum\limits_{\epsilon} e^{2\pi i S(\lambda_m \epsilon^2 z)}$ are regular in a neighborhood of z_0 we see that

$$\left| \frac{\partial g_m(z)}{\partial z_j} - \sum_{\epsilon} \frac{\partial \left(e^{2\pi i S(\lambda_m \epsilon^2 z)}\right)}{\partial z_j} \right|_{z=z_0}, m, j = 1, 2, \ldots, n,$$

can be made arbitrarily small, by choosing k large.

Now, if $\epsilon_1, \ldots, \epsilon_n$ are any n totally positive units of K, then the equation $\sum\limits_{i=1}^{n} S((\omega_i + g)\epsilon_i) = \sum\limits_{i=1}^{n} S((\omega_i + g))$ does not hold for large $g > 0$, unless we have $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n = 1$. This result can be easily proved by using the fact that if $\epsilon > 0$ is a unit of K, $S(\epsilon) > n$ unless $\epsilon = 1$. (This last-mentioned fact is itself a consquence of the well-known result that the arithmetic mean of $n(\geq 2)$ positive numbers not all equal, strictly exceeds their geometric mean). Thus, if we consider the jacobian

$$J(\lambda_1, \ldots, \lambda_m, t) = \left| \left(\sum_{\epsilon} \frac{\partial \left(e^{2\pi i S(\lambda_m \epsilon^2 z)}\right)}{\partial z_j} \right) \right|_{z=z_0}$$

as a function of $t(> 1)$, then it contains the term $(2\pi i)^n 2^n |(\lambda_m^{(j)})| e^{-2\pi t \sum\limits_{m=1}^{n} S(\lambda_M)}$ which is not cancelled by any other term. Now, since $|(\lambda_m^{(j)})| \neq 0$, it follows that for some $t > 1$, $J(\lambda_1, \ldots, \lambda_m, t) \neq 0$ and hence, by the foregoing, the jacobian $|(\partial g_m/\partial z_j)|$ does not vanish identically if k is chosen to be large enough. Thus our proposition is proved.

Incidentally, in the course of the proof above, we have shown that for large 233
k, the Poincaré series $f_\lambda(z, k)$, $\lambda \in \vartheta^{-1}$, $\lambda > 0$ cannot all vanish identically.

We might sketch here two alternative proofs of Proposition 27. The following proof is shorter and does not use the Fourier expansion of $\Phi(w, z)$ introduced earlier.

Now, by (176) and in view of the fact that (for $n = 1$) $\lim\limits_{t\to\infty} \Phi(w, it) = 0$, the function $\Phi(w, z)$ is a modular form of weight k (in z). Let now $w^{(p)} =$

$(w_1^{(p)}, \ldots, w_n^{(p)}) \in \mathfrak{H}_n^*$, $P = 1, 2, \ldots, n$ and $\Phi_P(z) = \dfrac{\Phi(w^{(p)}, z)}{G_1(z, k)}$. Then, if $z = (it, \ldots, it)$, $t > 1$, we see as before that

$$\lim_{k\to\infty} \Phi_P(z) = \sum_{\epsilon,\zeta} N(w^{(p)} - \epsilon^2 z + \zeta)^{-l},$$

where, in the summation ϵ runs over all units of K and ζ over all integers in K. If we suppose that $w_q^{(p)}$ and z_q are all purely imaginary, then for integral $\zeta \neq 0$ in K and for any unit ϵ, we have

$$\left| w_q^{(p)} - \epsilon^{(q)^2} z_q \right| < \left| w_q^{(p)} - \epsilon^{(q)^2} z_q - \zeta^{(q)} \right|, \quad p, q = 1, 2, \ldots, n.$$

If we further suppose that $iw_q^{(p)}$ and $z_q/2i$, p, $q = 1, 2, \ldots, n$ lie close to 1 and use some well-known inequalities for elementary symmetric functions of n positive real numbers, we can show that for a unit $\epsilon \neq \pm 1$,

$$|N(w^{(p)} - z)| < |N(w^{(p)} - \epsilon^2 z)|, \quad p = 1, 2, \ldots, n.$$

It now follows that

$$\lim_{l\to\infty} \lim_{k\to\infty} l^{-n} N(w^{(1)} - z)^l \ldots N(w^{(n)} - z)^l \left| \left(\frac{\partial \Phi_P(z)}{\partial z_q} \right) \right| = \left| \left(\frac{2}{w_q^{(p)} - z_q} \right) \right|$$

and for suitable choice of $w^{(1)}, \ldots, w^{(n)}$, we can ensure that

$$\left| \left(\frac{1}{w_q^{(p)} - z_q} \right) \right| \neq 0.$$

Thus, for large k and l, the n functions $\Phi_1(z), \ldots, \Phi_n(z)$ are analytically inde- **234**
pendent.

One can also prove Proposition 27 as follows. First, by the map $z = (z_1, \ldots, z_n) \to w = (w_1, \ldots, w_n)$ where $w_j = \dfrac{z_j - \theta_j}{z_j - \bar{\theta}_j}$ and $\theta = (\theta_1, \ldots, \theta_n) \in \mathfrak{F}$. One maps $\mathfrak{H}_n$ analytically homeomorphically $\mathfrak{D}$, the unit disc in C^n defined by $|w_1| < 1, \ldots, |w_n| < 1$. Now the group Γ induces a discontinuous group of analytic automorphisms of $\mathfrak{D}$. By using the well-known methods for a discontinuous group of analytic automorphisms of a bounded domain in C^n, we can find $n + 1$ Poincaré series $f_0(w)$, $f_1(w), \ldots, f_n(w)$ of weight k and belonging to G such that the n functions $\dfrac{f_1(w)}{f_0(w)}, \ldots, \dfrac{f_n(w)}{f_0(w)}$ are analytically independent.

Pulling these functions back to $\mathfrak{H}_n$ by the inverse map, we obtain the desired $n + 1$. Hilbert modular forms of weight k.

The modular forms of weight k form a vector space over the field of complex numbers. We shall see that this vector space is finite-dimensional and its dimension satisfies certain estimates in terms of k, for large k. In the following, we shall be interested only when $k > 2$.

Theorem 18. *Let $t(k)$ denote the number of linearly independent modular forms of weight k. Then for large $k(> 2)$,*

$$0 < c_{19} \leq \frac{t(k)}{k^n} \leq c_{20}, \tag{181}$$

for constants c_{19} and c_{20} depending only on K.

(*Note.* Since for $k > 2$, $t(k) \geq 1$, (181) may be upheld for all $k > 2$, by choosing c_{19} and c_{20} properly).

Proof. We shall first prove the relatively easier inequality $t(k) \geq c_{19}k^n$.

By the preceding proposition, there exist $n+1$ modular forms $f_0(z), f_1(z), \ldots, f_n(z)$ of weight j such that $f_1(z)f_0^{-1}(z), \ldots, f_n(z)f_0^{-1}(z)$ are analytically independent and therefore algebraically independent. Let us now suppose that $k > nj + 2$ and let m be the smallest positive integer such that

$$mnj + 2 \leq k \leq (m + 1)nj + 2. \tag{182}$$

□ **235**

Consider the power-products $f_1^{k_1}(z)f_2^{k_2}(z), \ldots, f_n^{k_n}(z)$, $(k_i = 0, 1, 2, \ldots, m)$. These are $(m + 1)^n$ in number. And the $(m + 1)^n$ functions $(f_0(z))^{mn-(k_1+\cdots+k_n)}$, $f_1^{k_1}(z), \ldots, f_n^{k_n}(z)(k_i = 0, 1, 2, \ldots, m)$ are modular forms of weight mnj and are linearly independent, since $f_1(z)f_0^{-1}(z), \ldots, f_n(z)f_0^{-1}(z)$ are algebraically independent. Since $k > mnj + 2$, we can obtain, by multiplying these modular forms by an Eisenstein series $G_1(z, r)$ (where r may be chosen to be $k - mnj$), $(m + 1)^n$ modular forms of weight k. Thus

$$t(k) \geq (m + 1)^n.$$

By (182), $(m + 1)^n \geq ((k - 2)/nj)^n$. Thus, for large k, we can find a constant $c_{19} > 0$ such that $t(k) \geq c_{19}k^n$.

We now proceed to prove the other inequality in (181).

Let $\mathfrak{a}$ be an ideal in K and p, a positive rational integer. Then it is known that for a constant c_{21} depending only on $\mathfrak{a}$ and K, the number of $\beta \in \mathfrak{a}$ for which $\beta > 0$ or $\beta = 0$ and $S(\beta) < p$ is at most equal to $c_{21}p^n$. Taking $\mathfrak{a}$ to be one of the h ideals ϑ^{-1}, $\mathfrak{a}_2^2\vartheta^{-1}, \ldots, \mathfrak{a}_h^2\vartheta^{-1}$, we can find a constant $c_{22} > 0$ depending only on K such that the number of $\beta \in \mathfrak{a}_i^2\vartheta^{-1}(i = 1, 2, \ldots, h)$ for which $S(\beta) < p$ and $\beta > 0$ or $\beta = 0$ is at most $c_{22}p^n$. Let $c_{23} = hc_{22}$.

Suppose $f_1(z), \ldots, f_q(z)$ are q linearly independent modular forms of weight $k(> 2)$. We shall be interested only in the case when k is so large as to allow the situation $q > c_{23}$. Let us therefore suppose that $q > c_{23}$. Let p be the largest positive rational integer such that

$$c_{23}p^n < q \leq c_{23}(p+1)^n.$$

Now each one of the modular forms $f_j(z)$, $j = 1, 2, \ldots, q$ has at the cusp $\lambda_i = \rho_i/\sigma_i$, a Fourier expansion of the form

$$f_j(z)N(-\sigma_i z + \rho_i)^k = c_j^{(i)}(0) + \sum_{\beta\in\mathfrak{a}_i^2\vartheta^{-1},\beta>0} c_j^{(i)}(\beta)e^{2\pi i S(\beta z_{A_i^{-1}})}.$$

We can find complex numbers $\mu_1, \ldots, \mu_q$ not all zero such that for $f(z) = \mu_1 f_1(z) + \cdots + \mu_q f_q(z)$, all the Fourier coefficients $c^{(j)}(\beta)$ for all $\beta \in \mathfrak{a}_j^2\vartheta^{-1}$ **236**
such that $\beta = 0$ or $\beta > 0$ and $S(\beta) < p$, in the expansion

$$f(z)N(-\sigma_j z + \rho_j)^k = c^{(j)}(0) + \sum_{\mathfrak{a}_j^2\vartheta^{-1}|\beta>0} c^{(j)}(\beta)e^{2\pi i S(\beta z_{A_j^{-1}})}, \tag{183}$$

vanish. This is possible, since it only involves solving $r(\leq c_{23}p^n)$ linear equations in $q(> c_{23}p^n)$ unknowns $\mu_1, \ldots, \mu_q$. Clearly, $f(z)$ is a cusp form of weight k.

Let $z = x + iy \in \mathfrak{F}_{\lambda_j} \cap \mathfrak{G}_{\lambda_j}$ and $y_{A_j^{-1}} = (y_1^*, \ldots, y_n^*)$. Then we know that

$$c_7 \leq \frac{y_i^*}{\sqrt[n]{N(y_{A_j^{-1}})}} \leq c_8$$

(see p. 253) and therefore as $z \in \mathfrak{F}$ tends to the cusp λ_j, each one of $y_1^*, \ldots, y_n^*$ tends to infinity. Since $c^{(j)}(0) = 0$, it follows that

$$N(y)^{k/2}|f(z)| = N(y_{A_j^{-1}})^{k/2}|N(-\sigma_j z + \rho_j)|^k|f(z)| \to 0,$$

as $z \in \mathfrak{F}$ tends to any cusp λ_j of $\mathfrak{F}$.

Let us now define for $z \in \mathfrak{H}_n$, the function $g(z) = N(y)^{k/2}|f(z)|$. This function is a non-analytic modular function i.e. for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$,

$$g(z_M) = N(y_M)^{k/2}|f(z_M)| = N(y)^{k/2}|N(\gamma z + \delta)|^{-k}|f(z_M)| = g(z).$$

By the foregoing, $g(z)$ tends to zero as z tends to the cusps of $\mathfrak{F}$ and as a consequence, if we now make use of Proposition 22, we see that $g(z)$ is bounded in $\mathfrak{F}$ and attains its maximum $\mu > 0$ at a point $z^0 = x^0 + iy^0 \in \mathfrak{F}$. But, since $g(z_M) = g(z)$ for all $M \in \Gamma$, we obtain

$$g(z) \leq g(z^0) = \mu, \tag{184}$$

for all $z \in \mathfrak{H}_n$. By multiplying $f(z)$ if necessary by a number of absolute value 1, we might suppose that $f(z^0)N(y^0)^{k/2} = \mu$.

Let us assume without loss of generality that $z^0 \in \mathfrak{F}_{\lambda_i} \cap \mathfrak{G}_{\lambda_1}$ (otherwise, if $z^0 \in \mathfrak{F}_{\lambda_j} \cap \mathfrak{G}_{\lambda_j}$, $j \neq 1$, then we have only to work with the coordinates $z_{A_j^{-1}}$ and $z^0_{A_j^{-1}}$ instead of z and z^0). Let $t_1 = u_1 + iv_1$ be a complex variable and let us denote the n-tuple $(t_1, t_1, \ldots, t_1)$ by $t = u + iv$. Then for

$$v_1 < s = \inf_{k=1,2,\ldots,n} y^0_k,$$

we see that $z = z^0 - t \in \mathfrak{H}_n$. **237**

Setting $r = e^{-2\pi i t_1}$, we now define for $|r| < e^{2\pi s}$, the function $h(r) = f(z^0 - t)e^{2\pi i p t_1} = f(z)e^{2\pi i p t_1}$. Taking $j = 1$ in (183), we obtain

$$h(r) = r^{-p} \sum_{\substack{\vartheta^{-1}|\beta>0 \\ S(\beta)\geq p}} c^{(1)}(\beta) e^{2\pi i S(\beta z^0)} r^{S(\beta)}. \tag{185}$$

The right hand side of (185) is a power series in r containing only non-negative powers of r and is absolutely convergent for $|r| < R = e^{2\pi s} > 1$. Thus $h(r)$ is regular for $|r| < R$. Let now $1 < R_1 = e^{2\pi v_1} < R$. Then, by the maximum modulus principle,

$$h(1) \leq \max_{|r|=R_1} |h(r)|,$$

i.e.

$$f(z^0) \leq \max_{|r|=R_1} |f(z^0 - t)| e^{-2\pi p v_1}$$

i.e.

$$\mu N(y^0)^{-k/2} \leq \mu N(y^0 - v)^{-k/2} e^{-2\pi p v_1},$$

in view of (184). From this, we have however

$$\begin{aligned} 2\pi p v_1 &\leq \frac{k}{2} \log N(1/(1 - v_1)/y^0) \\ &= \frac{k}{2} v_1 S(1/y^0) + \cdots \text{ terms involving higher powers of } v_1 \end{aligned}$$

i.e.

$$p \leq \frac{k}{4\pi} S(1/y^0) + \cdots \text{ terms involving } v_1 \text{ and higher powers.}$$

Now letting R_1 tend to 1 or equivalently v_1 to zero, we obtain

$$p \leq \frac{k}{4\pi} S(1/y^0).$$

From p. 196, we see that $S > c_9$ (for a constant c_9 depending only on K) 238
and $S(1/y^0) \leq n \cdot c_9$. (We have similar inequalities for $z^0_{A_j^{-1}}$, if $z^0 \in \mathfrak{F}_{\lambda_j} \cap \mathfrak{G}_{\lambda_j}$, $j \neq 1$). Thus, in any case, we can find a constant c_{24} depending only on K such that

$$p \leq c_{24} k.$$

Therefore, we see that if there exist q linearly independent modular forms of weight k, then necessarily

$$q \leq c_{23}(p+1)^n \leq c_{23}(c_{24}k+1)^n \leq c_{20}k^n$$

for large k and a suitable constant c_{20} independent of k. In particular, we have

$$t(k) \leq c_{20}k^n$$

which proves the theorem completely.

The idea used in the proof above is essentially the same as in Siegel's proof of the analogous result that the maximal number of linearly independent modular forms of degree n and weight k is at most equal to $c_{25}k^{(n/2)(n+1)}$ for a suitable constant c_{25} independent of k. (See reference (28) Siegel, p. 642).

We now proceed to consider the Hilbert modular functions.

A function $f(z)$ meromorphic in $\mathfrak{H}_n$ is called a **Hilbert modular function** (or briefly, a *modular function*), if

(i) for every $M \in \Gamma$, $f(z_M) = f(z)$;

(ii) for $n = 1$, $f(z)$ has at most a pole at the infinite cusp of the fundamental domain.

For $n \geq 2$, no condition is imposed on the behaviour of $f(z)$ at the cusps of $\mathfrak{F}$, in the definition of a modular function.

If $n = 1$, $f(z)$ is just an elliptic modular function. Let us remark, however, that for $n \geq 2$, there is no analogue of the elliptic modular invariant $j(\tau)$. For, if $f(z)$ is a Hilbert modular function regular in $\mathfrak{H}_n (n \geq 2)$ then it is automatically regular at the cusps of $\mathfrak{F}$ in view of Theorem 17 and is, in fact, a modular form of weight 0 and hence a constant.

The modular functions constitute a field which contains the field of complex numbers. Our object now is to study the structure of this field. We shall not be concerned, in the sequel, with the case $n = 1$, since, in this case, it is well-known that the elliptic modular functions form a field of rational functions generated by $j(\tau)$ over the field of complex numbers. 239

In the case $n \geq 2$, we shall see presently that the modular functions form an algebraic function field of n variables. The proof which we are going to present is mainly due to Gundlach.

A major step in the direction of this result is the following

Lemma. *For $n \geq 2$, every Hilbert modular function can be expressed as the quotient of two Hilbert modular forms of the same weight k.*

Proof. Let $\mathfrak{F}^* = \mathfrak{F} - \bigcup_{i=1}^{h} \mathfrak{U}_{\lambda_i,d}$ with $0 < d' < d$. Then we know that $\mathfrak{F}^*$ is compact, in view of Proposition 22.

Let $\varphi(z)$ be a Hilbert modular function. Suppose we can find a modular form $\chi(z)$ of weight k such that $\psi(z) = \varphi(z)\chi(z)$ is regular in $\mathfrak{F}^*$. Then, since for every $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, $\psi(z_M)N(\gamma z + \delta)^{-k} = \psi(z)$ and further since $\psi(z)$ is regular in $\mathfrak{U}_{\lambda_i,d} - \mathfrak{U}_{\lambda_i,d}$ (for $i = 1, 2, \ldots, h$), we can appeal to Corollary (a) (p. 214) and conclude that $\psi(z)$ is a modular form of weight k. Thus $\varphi(z)$ would be expressible as the quotient of the two modular forms $\psi(z)$ and $\chi(z)$ of weight k. ◻

Our object therefore is to construct a modular form $\chi(z)$ of weight k such that $\chi(z)$ annihilates the poles of $\varphi(z)$ in $\mathfrak{F}^*$ i.e. more precisely, such that $\chi(z)\varphi(z)$ is regular in $\mathfrak{F}^*$.

To every point $z^0 \in \mathfrak{F}^*$, we assign a neighbourhood $\mathfrak{R}$ in $\mathfrak{H}_n$ such that the following conditions are satisfied

(i) $\varphi(z) = \dfrac{P(z - z^0)}{Q(z - z^0)} = \dfrac{P(z_1 - z_1^0, \ldots, z_n - z_n^0)}{Q(z_1 - z_1^0, \ldots, z_n - z_n^0)}$ in $\mathfrak{R}$ where P and Q are convergent power-series in $z - z_0$ and locally coprime at every point of $\mathfrak{R}$.

(ii) By a suitable linear transformation, we can find new variables $w, v_2, \ldots, v_n$ such that by the Weierstrass' preparation theorem, we can suppose that in $\mathfrak{R}$, $Q(z_1 - z_1^0, \ldots, z_n - z_n^0) - w^g + a_i(v)w^{g-1} + \cdots + a_g(v)$ where $a_1(v), \ldots, a_g(v)$ are power-series in $v_2, \ldots, v_n$ convergent in $\mathfrak{R}$. If $\varphi(z)$ is regular in $\mathfrak{R}$, then we might take Q to be 1 in $\mathfrak{R}$. 240

(iii) Further, we might assume that $\mathfrak{R}$ is a polycylinder of the form $|w| \leq b$, $|v_2| \leq a, \ldots, |v_n| \leq a$ where a and b are so chosen that, in addition, any zero $(w, v_2, \ldots, v_n)$ of $Q(w, v_2, \ldots, v_n)$ for which $|v_2| < a, \ldots, |v_n| < a$, automatically lies in $\mathfrak{R}$. We shall denote the $(n-1)$-tuple $(v_2, \ldots, v_n)$ by v.

Let $\mathfrak{R}^* \subset \mathfrak{R}$ be the corresponding neighbourhood of $z^0 \in \mathfrak{F}^*$ defined by $|w| < b$, $|v_2| < a/2, \ldots, |v_n| < a/2$. Then the neighbourhoods $\mathfrak{R}^*$ for $z^0 \in \mathfrak{F}^*$ give an open covering of the compact set $\mathfrak{F}^*$ and we extract therefrom a finite covering of $\mathfrak{F}^*$ by neighbourhoods $\mathfrak{R}_1^*, \ldots, \mathfrak{R}_{c'}^*$ of points $z^{(1)}, \ldots, z^{(c')}$ of $\mathfrak{F}^*$. Let $\mathfrak{R}_1, \ldots, \mathfrak{R}_{c'}$ be the corresponding neighbourhoods of $z^{(1)}, \ldots, z^{(c')}$ defined above.

Let in $\mathfrak{R}_i$, $\varphi(z) = P(w, v)/Q(w, v)$ where $Q(w, v) = \sum\limits_{\nu=0}^{g_i} a_\nu^{(i)}(v) w^\nu$ and $a_\nu^{(i)}(v)$ are convergent power series in $v_2, \ldots, v_n$ in $\mathfrak{R}_i$. If $\varphi(z)$ is regular in $\mathfrak{R}_i$, then $Q(w, v) = 1$ in $\mathfrak{R}_i$ and therefore $g_i = 0$. Let us denote $\sum\limits_{i=1}^{c'} g_i$ by c_{23}. The constant c_{26} of course depends on $\varphi(z)$.

If $\mathfrak{M}$ is the set of poles of $\varphi(z)$, then $\mathfrak{M} \cap \mathfrak{R}$ is just the set of zeros of $Q(z - z^0)$ in $\mathfrak{R}$. Clearly, $\mathfrak{M}$ is invariant under the modular substitutions. Let us denote $\mathfrak{M} \cap \mathfrak{F}^*$ by $\mathfrak{N}$; since $\mathfrak{F}^*$ is compact, so is $\mathfrak{N}$.

Let $k > 2$ be a rational integer to be chosen suitably later. From Theorem 18, we know that the number $r = t(k) - h$ of linearly independent cusp forms of weight k satisfies $r > c_{27} k^n$ for a constant $c_{27} > 0$, if $k > c_{27}^*$ (a suitable constant). Let $f_1(z)$, $f_2(z), \ldots, f_r(z)$ be a complete set of linearly independent cusp forms of weight k.

Now, let $f(z) = g(w, v_2, \ldots, v_n)$ be regular in $\mathfrak{R}_i$; then, performing the division of $g(w, v_2, \ldots, v_n)$ by $Q(w, v_2, \ldots, v_n)$, we can write in $\mathfrak{R}_i$,

$$f(z) = h(w, v_2, \ldots, v_n) Q(w, v_2, \ldots, v_n) + \sum_{s=0}^{g_i - 1} b_s^{(i)}(v_2, \ldots, v_n) w^s, \tag{187}$$

where $h(w, v_2, \ldots, v_n)$ is regular in $\mathfrak{R}_i$ and $b_s^{(i)}(v_2, \ldots, v_n)$ are convergent power-series in $v_2, \ldots, v_n$.

Let l be the non-negative rational integer such that **241**

$$c_{26} l^{n-1} < c_{27} k^n \leq c_{26} (l+1)^{n-1}. \tag{188}$$

We might assume k to be so large that $l \geq 1$. We now claim we can find complex numbers $\alpha_1, \ldots, \alpha_r$ not all zero, such that for $f(z) = \alpha_1 f_1(z) + \cdots + \alpha_r f_r(z)$, the power-series $b_s^{(i)}(v_2, \ldots, v_n)$, occurring in (187) are of order at least l. in $v_2, \ldots, v_n$, for $i = 1, 2, \ldots, c'$. This is simple to establish, for it only involves solving $q(\leq c_{26} l^{n-1})$ linear equations in $r(> c_{27} k^n)$ unknowns $\alpha_1, \ldots, \alpha_r$ and by (188), $q < r$.

We shall first prove that for all sufficiently large k, the cusp form $f(z) = \sum_{i=1}^{r} \alpha_i f_i(z)$ chosen above, vanishes identically on $\mathfrak{N}$. Let, if possible, $f(z)$ be not identically zero on $\mathfrak{N}$. Then $f(z)$ attains its maximum modulus $\mu \neq 0$ at some point z^* on $\mathfrak{N}$, since $\mathfrak{N}$ is compact. We shall suppose that $f(z^*) = 1$, after normalizing $f(z)$ suitably, if necessary. Now $z^* \in \mathfrak{N}_i^*$ for some i. Let w^*, $v_2^*, \ldots, v_n^*$ be the coordinates of z^*, in terms of the new variables $w, v_2, \ldots, v_n$ in $\mathfrak{N}_i$. Let t be a complex variable. Denoting the $(n-1)$-tuple $(v_2^* t, \ldots, v_n^* t)$ by $v^* t$, we have $(w, v^* t) \in \mathfrak{N}_i$, for $|w| < b$, $|t| \leq 2$.

Figure, page 241

Let $\mathfrak{M}_i = \mathfrak{M} \cap \mathfrak{N}_i$. Consider now the analytic set defined in the polycylinder $|w| < b$, $|t| \leq 2$ by $Q(w, v^* t) = 0$. Let $\mathfrak{B}$ be an irreducible component of this analytic set, containing $(w^*, 1)$. Clearly if $(w, t) \in \mathfrak{B}$, then $(w, v^* t) \in \mathfrak{M}_i$. On $\mathfrak{B}$, therefore, we have after (187), for the function $f_0(w, v^* t) = f(z)$ that

$$f^*(w,t) = f_0(w, v^* t) = t^l \sum_{\nu=0}^{g_i - 1} t^{-1} b_\nu^{(i)}(v^* t) w^\nu.$$

Since $b_\nu^{(i)}(v_2, \ldots, v_n)$ is of order at least l in $v_2, \ldots, v_n$, it follows that $t^{-l} b_\nu^{(i)}(v^* t)$ 242
is regular in $|t| \leq 2$. Hence $t^{-l} f^*(w,t)$ is regular on $\mathfrak{B}$. Now it can be shown that if $(w_0, t_0) \in \mathfrak{B}$, then $\mathfrak{B}$ can be parametrized locally at (w_0, t_0) by

$$t - t_0 = u^q, w - w_0 = R(u),$$

where q is a positive integer and $R(u)$ is regular in u. If we note that we can apply the maximum modulus principle to the function $t^{-l} f^*(w,t)$ as a function of u, we can show that $t^{-l} f^*(w,t)$ attains its maximum modulus on $\mathfrak{B}$ at a point $(w^{(0)}, t^{(0)})$ on the boundary of $\mathfrak{B}$ and hence with $|t^{(0)}| = 2$, necessarily. Thus

$$\begin{aligned} 1 = f(z^*) = f_0(w^*, v^*) &\leq \operatorname*{Max}_{\mathfrak{B}} |t^{-l} f^*(w,t)| \\ &= 2^{-l} |f^*(w^{(0)}, t^{(0)})| \\ &\leq 2^{-l} \operatorname*{Max}_{\substack{|t| \leq 2, |w| \leq b \\ (w,t) \in \mathfrak{B}}} |f^*(w, v^* t)| \qquad (189) \\ &\leq 2^{-l} \operatorname*{Max}_{\mathfrak{M}_i} |f(z)|. \end{aligned}$$

Let $z^{(0)}$ be a point on the boundary of $\mathfrak{M}_i$ at which $\max_{z \in \mathfrak{M}_i} |f(z)|$ is attained. If $z^{(0)} \in \mathfrak{F}^*$, then $z^{(0)} \in \mathfrak{N}$ and hence $|f(z^{(0)})| \leq 1$. Hence we have $1 \leq 2^{-l}$, which gives a contradiction, since $l \geq 1$. Therefore $f(z)$ must vanish on $\mathfrak{N}$.

Suppose now $z^{(0)} \notin \mathfrak{F}^*$. Then, for some $M \in \Gamma$, we can ensure that $z_M^{(0)} \in \mathfrak{F}$.

We contend that there are only finitely many $P \in \Gamma$ for which $\mathfrak{F}_P$ intersects $\bigcup_{i=1}^{c'} \mathfrak{R}_i$. To prove this, we first apply Proposition 23 with $\bigcup_{i=1}^{c'} \mathfrak{R}_l$ with C and then for a certain $b = b(C) > 0$, we have $\mathfrak{U}_{\mu,b} \cap C = \emptyset$ for all cusps μ. It follows that for all cusps λ and for all $P \in \Gamma$, we have $(\mathfrak{U}_{\lambda,b})_P \cap C = \emptyset$. In particular, for
$\lambda = \lambda_1, \ldots, \lambda_h$, we see that $(\mathfrak{U}_{\lambda,b})_P \cap C = \emptyset$, for all $P \in \Gamma$. Now $\mathfrak{F} - \bigcup_{i=1}^{h} \mathfrak{U}_{\lambda_i,b}$ is 243
compact and we apply Proposition 21 with $\mathfrak{F} - \bigcup_{\lambda_i,b}^{h}$ for B and $\bigcup_{i=1}^{c'} \mathfrak{R}_i$ for B'. Then we see immediately that our assertion above is true.

We may now suppose that $z_M^{(0)} \in \mathfrak{F} \cap \mathfrak{U}_{\lambda_1,c}$, without loss of generality. Since $z^{(0)} \in \bigcup_{i=1}^{c'} \mathfrak{R}_i$, and since by the remark above, $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to a finite set in Γ independent of $z^{(0)}$, there exists a constant $c_{28} > 1$ such that $|N(\gamma z^{(0)} + \delta)| > c_{28}^{-1}$. Further $|f(z_M^{(0)})||N(\gamma z^{(0)} + \delta)|^{-k} = |f(z^{(0)})|$. Hence from (189), we have

$$1 \leq 2^{-l} |f(z^{(0)})| \leq 2^{-l} c_{28}^k |f(z_M^{(0)})|.$$

If $z_M^{(0)} \in \mathfrak{F}^*$, then we have $|f(z_M^{(0)})| \leq 1$ and therefore

$$1 \leq 2^{-l} c_{28}^k.$$

Suppose now $z_M^{(0)} \notin \mathfrak{F}^*$; then $z_M^{(0)} \in \mathfrak{U}_{\lambda_1,d'} \cap \mathfrak{F}$. We shall show again that $|f(z_M^{(0)})| \leq 1$.

Let $\mathfrak{M}^*$ be the connected component of $\mathfrak{M} \cap \mathfrak{U}_{\lambda_1,d'}$ which contains the point $z_M^{(0)}$. Then $f(z)$ is not locally constant on $\mathfrak{M}^*$ near $z_M^{(0)}$, because of $f(z_M^{(i)}) = 0$ and the connectedness of $\mathfrak{R}_i$.

Let $\sup_{z \in \mathfrak{M}^*} |f(z)| = \delta$. If $B = \bigcup_{T \in \Gamma_{\lambda_l}} \mathfrak{M}_T^*$ then clearly $\sup_{z \in B} |f(z)| = \delta$, again. Now, since $f(z)$ tends to zero as z tends to the cusp λ_1, there exists d'' with $0 < d'' < d'$ such that $|f(z)| < (1/2)\delta$ in $\mathfrak{U}_{\lambda_1,d''}$. Let $B_1 = B \cap (\overline{\mathfrak{U}}_{\lambda_1}, d' - \mathfrak{U}_{l_{\lambda_1}}, d'') \cap \mathfrak{F}$; then $B \cap (\overline{\mathfrak{U}}_{\lambda_1,d'} - \mathfrak{U}_{\lambda_1,d''}) = \bigcup_{T \in \Gamma_{\lambda_1}} (B_1)_T$ and further $\sup_{z \in B_1} |f(z)| = \delta$, again. Since $\overline{B}_1$ is compact, $|f(z)|$ attains its maximum δ ($< \infty$, necessarily therefore) at a point z', say, in $\overline{B}_i$. Now, there exists a sequence of points $w^{(i)} \in \mathfrak{M}_{T_i}^* \cap \mathfrak{F}$ converging to z'. Using the fact that $\mathfrak{M}$ is closed and locally connected, it can be shown that z' belongs already to $\overline{\mathfrak{M}_T^*}$ for a $T \in \Gamma_{\lambda_1}$, which is one of the T_i's above. Now $|f((z_M^{(0)})_T)| = |f(z_M^{(0)})|$ and to show that $|f(z_M^{(0)})| \leq 1$, we might as well argue with $\mathfrak{M}_T^*$ instead of $\mathfrak{M}^*$. Thus we may assume, without loss of

generality, that $|f(z)|$ attains its maximum δ on $\overline{\mathfrak{M}^*}$ at a point z'. We now claim 244
that $\Delta(z', \lambda_1) = d'$. For, if $\Delta(z', \lambda_1) < d'$, then $z' \in \mathfrak{M}^*$ and we can find a curve C in $\mathfrak{M}^*$ connecting $z_M^{(0)}$ with z'. Let z'' be the *first* point on C such that $|f(z'')| = \delta$; then $f(z)$ is not locally constant on $\mathfrak{M}^*$ near z'', and the maximum principle gives a contradiction. Thus $\Delta(z', \lambda_1) = d'$. Again, for a suitable $T \in \Gamma_{\lambda_1}$, $z'_T \in \mathfrak{F}^*$ i.e. $z'_T \in \mathfrak{N}$, as before. Thus $|f(z_M^{(0)})| \leq |f(z')| = |f(z'_R)| \leq 1$.

Thus, in all cases

$$1 \leq c_{28}^k 2^{-l},$$

i.e.

$$l \leq c_{29} k,$$

if $f(z)$ does not vanish identically on $\mathfrak{N}$. But, from this and from (188), we obtain

$$0 < c_{30} l^n < c_{27} k^n \leq c_{26} (l+1)^{n-1} \leq c_{31} l^{n-1},$$

i.e.

$$0 < c_{30} < c_{27} \left(\frac{k}{l}\right)^n \leq \frac{c_{31}}{l} \leq c_{31}.$$

From the fact that $(k/l)^n$ is bounded, we see that as k tends to infinity, l also tends to infinity. But this contradicts the inequality $l < c_{31}/c_{30}$. Thus $f(z)$ vanishes on $\mathfrak{N}$ identically, for large k.

Now taking each $\mathfrak{N}_i$, $i = 1, 2, \ldots, c'$, we observe that since $f(z)$ vanishes on the analytic set defined by $Q(z - z^0) = 0$ in $\mathfrak{N}_i$, there exists an integer n_i such that $f^{n_i}(z)$ is divisible by $Q(z - z^0)$ in $\mathfrak{N}_i$ i.e. $f^{n_i}(z)\varphi(z)$ is regular in $\mathfrak{N}_i$. (We might take $n_i = g_i$, for example). Choosing $m = c_{26}$, we see that $f^m(z)\varphi(z)$ is regular on $\mathfrak{F}^*$. Thus the modular form $f^m(z)$ of weight mk has the required property and our lemma is therefore completely proved.

We are now, in a position, to prove (for $n \geq 2$), the main theorem, viz.

Theorem 19. *The Hilbert modular functions form an algebraic function field of n variables.*

Proof. We know by Proposition 27 that there exist $n+1$ modular forms $f_0(z)$, 245
$f_1(z), \ldots, f_n(z)$ of weight j such that $\dfrac{f_1(z)}{f_0(z)}, \ldots, \dfrac{f_n(z)}{f_0(z)}$ are algebraically independent over the field of complex numbers. Since the quotient of two modular forms of the same weight is a modular function, we have n algebraically independent modular functions

$$g_1(z) = \frac{f_1(z)}{f_0(z)}, \ldots, g_n(z) = \frac{f_n(z)}{f_0(z)}.$$

□

Let Ω denote the field of Hilbert modular functions and Ω_0, the field generated by $g_1(z), \ldots, g_n(z)$ over the field of complex numbers. We shall first show that every element of Ω satisfies a polynomial equation of bounded degree, with coefficients in Ω_0.

Let $\varphi(z)$ be any modular function. By the lemma above, $\varphi(z) = \psi(z)/\chi(z)$, where $\chi(z)$ and $\psi(z)$ are two modular forms of the same weight k. Let us consider for fixed positive rational integers m and s, the power products

$$f_{k_1,\ldots,k_n,l}(z) = g_1^{k_1}(z), \ldots, g_n^{k_n}(z)\varphi^l(z), \begin{cases} k_i = 0, 1, 2, \ldots, m, \\ l = 0, 1, 2, \ldots, s. \end{cases}$$

Now $f_0^{mn}(z)\chi^s(z)f_{k_1,\ldots,k_n,l}(z)$ is a modular form of weight $w = mnj + ks$. We have indeed $(m+1)^n(s+1)$ such modular forms of weight w. If we can choose m and s such that

$$(m+1)^n(s+1) > c_{20}w^n (\leq t(w)), \tag{190}$$

then we get a nontrivial linear relation between these modular forms and hence between the $(m+1)^n(s+1)$ functions $f_{k_1,\ldots,k_n,l}(z)$. But this linear relation cannot be totally devoid of terms involving powers of $\varphi(z)$, since otherwise, this will contradict the algebraic independence of $g_1(z), \ldots, g_n(z)$. Thus $\varphi(z)$ satisfies a polynomial equation of degree at most s and with coefficients in Ω_0. Since Ω is a separable algebraic extension of Ω_0 and since every element of Ω satisfies a polynomial equation of degree at most s over Ω_0, it follows that Ω is a finite algebraic extension of Ω_0 i.e. an algebraic function field of n variables.

In order to complete the proof of the theorem, we have only to find rational integers $\underline{m}$ and $\underline{s}$ which satisfy (190), s being bounded. This is very simple, for **246**
when m tends to infinity, $\dfrac{w^n}{(m+1)^n}$ tends to $(jn)^n$ so that if we choose m large enough and $s > c_{20}j^n n^n$, then (190) will be fulfilled. Our theorem is therefore completely proved.

By adopting methods similar to the above, Siegel has recently proved that (for $n \geq 2$) every modular function of degree n (i.e. a complex-valued function meromorphic in the Siegel half-plane and invariant under the modular transformations) can be expressed as the quotient of two "entire" modular forms of degree n. (See reference (32), Siegel). It had been already shown by Siegel that the modular functions of degree n which are quotients of "entire" modular forms, form an algebraic function field of $n(n+1)/2$ variables.

Bibliography

[1] A. A. Albert: A solution of the principal problem in the theory of Riemann matrices, *Ann. of Math.*, 35 (1934), 500-515.

[2] O. Blumenthal: Über Modulfunktionen von mehreren Veränderlichen, *Math. Ann.*, 56 (1903), 509-548; ibid 58 (1904), 497-527.

[3] L. R. Ford: *Automorphic functions*, Chelsea, New York (1951).

[4] F. Götzky: Über eine zahlentheoretische Anwendung von Modulfunktionen zweier Veränderlicher, *Math. Ann.* 100 (1928), 411-437.

[5] K. B. Gundlach: Über die Darstellung der ganzen Spitzenformen zu den Idealstufen der Hilbertschen Modulgruppe und die Abschätzung ihrer Fourierkoeffizienten, *Acta. Math.* 92 (1954), 309-345.

[6] ——: Poincaresche und Eisensteinsche Reihen Zur Hilbertschen Modulgruppe, *Math. Zeit.* 64 (1956), 339-352.

[7] ——: Modulfunktionen zur Hilbertschen Modulgruppe und ihre Darstellung als Quotienten ganzer Modulformen, *Arch. Math.* 7 (1956), 333-338.

[8] ——: Über den Rang der Schar der ganzen automorphen Formen zu hyperabelschen Transformationsgruppen in zwei Variablen, *Gött. Nachr.* Nr. 3. (1958).

[9] ——: Quotientenraum und meromorphe Funktionen zur Hilbertschen **247**
Modulgruppe, *Gött. Nachr.* Nr. 3, (1960).

[10] E. Hecke: Grundlagen einer Theorie der Integralgruppen und der Integralperioden bei den Normalteilern der Modulgruppe, *Math. Werke*, Göttingen, (1959), 731-772.

[11] L. Hua: On the theory of Fuchsian functions of several variables, *Ann. of Math.*, 47 (1946), 167-191.

[12] H. D. Kloosterman: Theorie der Eisensteinschen Reihen von mehreren Veränderlichen, *Abh. Math. Sem Hamb. Univ.* 6 (1928), 163-188.

[13] M. Moecher: Zur Theorie der Modulformen *n*-ten Grades I, *Math. Zeit.* 59 (1954), 399-416.

[14] S. Lefschetz: On certain numerical invariants of algebraic varieties with applications to Abelian varieties, *Trans. Amer. Math. Soc.*, 22 (1921), 327-482.

[15] H. Maass: Über Gruppen von hyperabelschen Transformationen, *Sitz-Ber. der Heidelb. Akad. der Wiss., math.-nat.* K1. 2 Abh. (1940).

[16] —–: Zur Theorie der automorphen Funktionen von n Veränderlichen, *Math. Ann.*, 117 (1940), 538-578.

[17] ——: Theorie der Poincaréschen Riehen zu den hyperbolischen Fixpunktsystemem der Hilbertschen Modulgruppe, *Math. Ann.* 118 (1941), 518-543.

[18] H. Petersson: Einheitliche Begründung der Vollständigkeitssätze für die Poincaréschen Reihen von reeler Dimension bei beliebigen Grenzkreisgruppen von erster *Art, Abh. Math. Sem. Hamb. Univ.* 14 (1941), 22-60.

[19] ——: Über die Berechnung der Skalarprodukte ganzer Modulformen, Comment. Math. Helv. 22 (1949), 168-199.

[20] E. Picard: Sur les fonctions hyperabéliennes, *J. de Liouville*, Ser. IV, 1 (1885).

[21] H. Poincaré: Sur les fonctions fuchsiennes, *Oeuvres*, II, 169-257.

[22] ——: Fonctions modularies et fonctions fuchsiennes, *Oeuvres*, II, 592-618.

[23] R. A. Rankin: The scalar product of modular forms, *Proc. Lond. Math. Soc.* 3, II (1952), 198-217.

[24] C. Rosati: Sulle matrici di Riemann, *Rendiconti del Circ. Mathematico di* **248**
Palermo, 53 (1929), 79-134.

[25] A. Selberg: Beweis eines Darstellungssatzes aus der Theorie der ganzen Modulformen, *Arch. for Math.* Naturvid. 44, Nr. 3 (1940).

[26] I. I. Piatetskii-Shapiro: Singular modular functions, *Izv. Akad. Nauk SSSR, Ser. Mat.* 20 (1956), 53-98.

[27] C. L. Siegel: Über die analytische Theorie der quadratischen Formen, III, *Ann. of Math.* 38 (1937), 213-292.

[28] ——: Einführung in die Theorie der Modulfunktionen n-ten Grades, *Math. Ann.* 116 (1939), 617-657.

[29] ——: Die Modulgruppe in einer einfachen involutorischen Algebra, *Festschrift Akad. Wiss.* Göttingen, 1951.

[30] ——: *Automorphe Funktionen in mehreren Variablen,* Göttingen, (Function theory in 3 Volumes) (1954-55).

[31] ——: Zur Vorgeschichte des Eulerschen Additons-theorem, Sammelband Leonhard Euler, *Akad. Verlag,* Berlin, (1959), 315-317.

[32] ——: Über die algebraische Abhängigkeit von Modulfunktionen n-ten Grades, *Gött. Nacher.* Nr. 12, (1960).

[33] H. Weyl: On generalized Riemann matrices, *Ann. of Math.* 35 (1934), 714-729.

Appendix

Evaluation of Zeta Functions for Integral Values of Arguments

By Carl Ludwig Siegel

Eulerpresumably enjoyed the summation of the series of reciprocals of **249**
squares of natural numbers in the same way as Leibnitz did the determination of the sum of the series which was later named after him. In the nineteenth century, generalizations of these results to the L-series introduced by Dirichlet were studied; these were important for the class number formulae of cyclotomic fields; corresponding questions for the L-series of imaginary quadratic fields got elucidated by Kronecker's limit formulae.

In one of his most beautiful papers, Hecke has obtained an analogous result for the L-series of real quadratic fields, which led him to the assertion that the value of the zeta function of every ideal class of a real quadratic number field of arbitrary discriminant d is, for the values $s = 2, 4, 6, \ldots,$ of the variable, equal to the product of $\pi^{2s}\sqrt{d}$ and a rational number. He further gave a brief indication of two possible methods of proof for his assertion. The first of these has recently been carried out in different ways [1]. The second method of proof leads, after suitable modifications to the generalization, published by Klingen [2], to arbitrary totally real algebraic number fields. This second method is indeed quite clear in its underlying general ideas but, however, in the form available till now, it admits of no effective determination of the said rational number. By means of a theorem on elliptic modular forms, which in itself seems not uninteresting, we shall, in the sequel, present a proof by a constructive procedure meeting the usual requirement of Number Theory.

1 Elliptic Modular Forms

Let z be a complex variable in the upper half plane and

$$F_k(z) = \frac{1}{2}\sum_{l,m}{}'(lz+m)^{-k}(k = 4, 6, 8, \ldots), \tag{1}$$

where l, m run over all pairs of rational integers except 0, 0. If we set **250**

$$\sum_{t|n} l^g = \sigma_g(n) \quad (g = 0, 1, 2, \ldots), e^{2\pi iz} = q$$

then for this Eisenstein series we have the Fourier expansion

$$F_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{2}$$

which, in view of

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2\cdot k!}, \tag{3}$$

can also be expressed by the formulae

$$F_k(z) = \zeta(k)G_k(z), G_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n(k = 4, 6, \ldots). \tag{4}$$

In particular, here, we have the Bernoulli numbers

$$B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{14} = \frac{7}{6};$$

and hence

$$\begin{aligned} G_4 &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, G_6 = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n \\ G_8 &= 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, G_{10} = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n \\ G_{14} &= 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n \end{aligned} \tag{5}$$

with rational integral coefficients and constant term 1. We set further

$$G_0 = 1.$$

Let h be a non-negative even rational integer. If we denote the dimensionof the linear space $\mathfrak{M}_h$ of elliptic modular forms of weight h by r_h, where for fixed h we write, for brevity, also r for r_h, then it is well known that 251

$$r_h = \left[\frac{h}{12}\right] + 1(h \not\equiv (\mod 12)), r_h = \left[\frac{h}{12}\right](h \equiv 2(\mod 12)).$$

We have, the particular,

$$\begin{gathered} G_4^2 = G_8, G_4G_6 = G_{10}, G_4^2G_6 = G_{14} \\ G_lG_{14-l} = G_{14}(l = h - 12r_h + 12 = 0, 4, 6, 8, 10, 14) \end{gathered} \tag{6}$$

and for the modular form

$$\Delta = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$$

of weight 12,

$$1728\Delta = G_4^3 - G_6^2.$$

If

$$j(z) = G_4^3\Delta^{-1} = q^{-1} + \cdots \tag{7}$$

denotes the absolute invariant, then

$$\Delta^2\frac{dj}{dz} = 3G_4^2\frac{dG_4}{dz}\Delta - G_4^3\frac{d\Delta}{dz} = \frac{1}{1728}G_4^2G_6\left(2G_4\frac{dG_6}{dz} - 3G_6\frac{dG_4}{dz}\right),$$

and the expression in the brackets yields a modular form of weight 12 and indeed a cusp form which can therefore differ from Δ at most by a constant factor. Comparing the coefficients of q in the Fourier expansions, we get

$$\frac{dj}{d\log q} = -G_{14}\Delta^{-1}. \tag{8}$$

Let hereafter, $h > 2$ and hence $r_h > 0$. The r isobaric power-products $G_4^aG_6^b$ where the exponents a, b run over all non-negative rational integer solutions of

$$4a + 6b = h,$$

form a basis of $\mathfrak{M}_h$. It follows from this that, for every function M in $\mathfrak{M}_h$, 252
$MG^{-1}_{h-12r+12}$ always belongs to $\mathfrak{M}_{12r-12}$. Since Δ^{r-1} is a modular form of weight $12r - 12$, not vanishing anywhere in the interior of the upper half-plane,

$$MG^{-1}_{h-12r+12}\Delta^{1-r} = W \tag{9}$$

is an entire modular function and hence a polynomial in j with constant coefficients.

Let

$$T_h = G_{12r-h+2}\Delta^{-r} \tag{10}$$

with the Fourier expansion

$$T_h = C_{hr}q^{-r} + \cdots + C_{h1}q^{-1} + C_{h0} + \cdots \tag{11}$$

and first coefficient $C_{hr} = 1$. Since

$$\Delta^{-1} = q^{-1}\prod_{n=1}^{\infty}(1 + q^n + q^{2n} + \cdots)^{24}, \tag{12}$$

all the Fourier coefficients of T_h turn out to be rational integers.

Theorem 1. *Let*

$$M = a_0 + a_1 q + a_2 q^2 + \cdots$$

be the Fourier series of a modular form M of weight h. Then

$$C_{h0}a_0 + C_{h1}a_1 + \cdots + C_{hr}a_r = 0$$

Proof. For $l = 0, 1, 2, \ldots$

$$j^l \frac{dj}{dz} = \frac{1}{l+1}\frac{dj^{l+1}}{dz}$$

and hence, by (**??**), it has a Fourier series without constant term. Since the function W defined by (9) is a polynomial in j, the product $W\frac{dj}{dz}$ is, similarly, a Fourier series without constant term. □

Becauseof (6) and (8), we have **253**

$$-\frac{1}{2\pi i}W\frac{dj}{dz} = MG^{-1}_{h-12r+12}\Delta^{1-r}G_{14}\Delta^{-1} = MG_{12r-h+2}\Delta^{-r} = MT_h$$

from which the theorem follows on substituting the series (11) and (**??**) for T_h and M respectively.

Let us put for brevity,

$$c_{h0} = c_h.$$

It is important, for the entire sequel, to show that c_h does not vanish.

Theorem 2. *We have*

$$c_h \neq 0.$$

Proof. First, let $h \equiv 2(\mod 4)$, so that $h \equiv 2t(\mod 12)$ with $t = 1, 3, 5$. Then correspondingly $12r = h - 2, h + 6, h + 2$; hence $12r - h + 2 = 0, 8, 4$ and

$$G_{12r-h+2} = G_0, G_4^2, G_4.$$

Since by (5), G_4 has all its Fourier coefficients positive and the same holds for Δ^{-5} as a consequence of (12), we conclude from (10) that all the coefficients in the expansion (11) are positive. Therefore in the present case, the integers c_{h0}, $c_{h1}, \ldots, c_{hr}$ are all positive and, in particular, $c_h = c_{h0} > 0$ i.e. $c_h \neq 0$. □

Let now $h \equiv 0(\mod 4)$ so that $h \equiv 4t(\mod 12)$ with $t = 0, 1, 2$ whence $12r = h - 4t + 12$, $h - 12r + 12 = 4t$ and

$$G_{h-12r+12} = G_{4t} = G_4^t.$$

Further we have now

$$\begin{aligned} T_h &= -G_{12r+h+2}\Delta^{1-r}G_{14}^{-1}\frac{dj}{d\log q} = -G_4^{-t}\Delta^{1-r}\frac{dj}{d\log q} \\ &= \frac{3}{t-3}\Delta^{1-r-t/3}\frac{dj^{1-t/3}}{d\log q} \\ &= \frac{3}{t-3}\frac{d(G_4^{3-t}\Delta^{-r})}{d\log q} + \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{d\Delta^{-r}}{d\log a}; \end{aligned}$$

hence C_{h0} is also the constant term in the Fourier expansion of the function **254**

$$V_h = \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{d\Delta^{-r}}{d\log q}.$$

In view of the assumption $h > 2$, we have $3r + t - 3 > 0$. The series for G_4^{3-t} begins with 1 and has again all its coefficients positive. Further, by (12), the coefficients of the negative powers $q^{-r}, \ldots, q^{-1}$ of q in the derivative of Δ^{-r} with respect to $\log q$ are all negative while the constant term is absent. Hence the constant term in V_h is negative and $c_h = c_{h0} < 0$ i.e. $c_h \neq 0$. The proof is thus complete.

A most important consequence of the two theorems is the fact that, for every modular form M of weight h, the constant term a_0 in its Fourier expansion is determined by the r Fourier coefficients $a_1, \ldots, a_r$ that follow, namely by the formula

$$a_0 c_h^{-1}(c_{h1}a_1 + \cdots + c_{hr}a_r). \tag{14}$$

If, in particular, $a_1, \ldots, a_r$ are rational integers, then a_0 itself is rational and the denominator of a_0 divides c_h.

As a further remarkable result, we note that the vanishing of a_0 follows from the vanishing of the Fourier coefficients $a_1, \ldots, a_r$ and hence the vanishing of the modular form M itself, since then the entire modular function W defined in (9) has a zero at $q = 0$.

From (10) and (11), it follows that the numbers $c_{hl}(l = 0, 1, \ldots, r)$ are the coefficients of q^{-l} in the product of $G_{12r-h+2}$ with Δ^{-r} as is indeed seen by direct calculation, which can be slightly simplified by using the formula

$$\prod_{m=1}^{\infty}(1-q^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

Now $c_{hr} = 1$ always. Also we easily obtain the values

$$c_4 = -240 = -2^4 \cdot 3 \cdot 5; \qquad c_6 = 504 = 2^3 \cdot 3^2 \cdot 7;$$
$$c_8 = -480 = -2^5 \cdot 3 \cdot 5; \qquad c_{10} = 264 = 2^3 \cdot 3 \cdot 11;$$
$$c_{12} = -196560 = -2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13, \qquad c_{12,1} = 24 = 2^3 \cdot 3;$$

255

$$c_{14} = 24 = 2^3 \cdot 3;$$

$$c_{16} = -146880 = -2^6 \cdot 3^3 \cdot 5 \cdot 17, \qquad c_{16,1} = -216 = -2^3 \cdot 3^3;$$
$$c_{18} = 86184 = 2^3 \cdot 3^4 \cdot 7 \cdot 19, \qquad c_{18,1} = 528 = 2^4 \cdot 3 \cdot 11;$$
$$c_{20} = -39600 = -2^4 \cdot 3^2 \cdot 5^2 \cdot 11, \qquad c_{20,1} = -456 = -2^3 \cdot 3 \cdot 19;$$
$$c_{22} = 14904 = 2^3 \cdot 3^4 \cdot 23, \qquad c_{22,1} = 288 = 2^5 \cdot 3^2;$$
$$c_{24} = -52416000 = -2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13,$$
$$c_{24,1} = -195660 = -2^2 \cdot 3^2 \cdot 5 \cdot 1087, \qquad c_{24,2} = 48 = 2^4 \cdot 3;$$

$$c_{26} = 1224 = 2^3 \cdot 3^2 \cdot 17, \qquad c_{26,1} = 48 = 2^4 \cdot 3.$$

From this table, we see for example, with $h = 12$ and $h = 24$, that the r numbers $c_{hl}(l = 0, \ldots, r-1)$ need not all have the same sign, if h is a multiple of 4. Such a change of sign always occurs, moreover, for $h > 248$ since, in particular, for this h, $c_{h,r-1} > 0$ whereas $C_{h0} < 0$. Further, since the number $c_{h,r-1}$ is 0 for $h = 214$ and $h = 248$, the assertion of Theorem 2 does not hold, without exception, for all $c_{hl}(l = 1, 2, \ldots, r-1)$.

For number-theoretic applications of theorem 1, the prime-factors of c_h are of interest. We get some information regarding these prime factors if in the statement of Theorem 1 we consider the Fourier coefficients of $G_h(h = 4, 6, \ldots)$ given by (4) instead of the modular form M. Then we have from (14),

$$\frac{c_h B_h}{2h} = \sum_{l=1}^{r} c_{hl}\sigma_{h-1}(l), \sigma_{h-1}(l) = \sum_{t|l} t^{b-1}(h = 4, 6, \ldots) \tag{15}$$

Since the first sum here is an integer, c_h must be divisible by the denominator of $\frac{B_h}{2h}$; in any case, the denominator of the Bernoulli numberB_h is a divisor **256**
of c_h. By the Staudt-Clausen theorem, however, this denominator is equal to the product of those primes p, for which $p-1$ divides h, Thus, for example, for $h = 24$, one sees the reason for the presence of the prime divisors 2, 3, 5, 7, 13 of c_{24}. On the other hand, for $h = 26$, the factor 17 of c_{26} cannot be explained in this simple manner. By (15), we have however explicitly

$$\begin{aligned} \frac{2 \cdot 3^2 \cdot 17 \cdot B_{26}}{13} - \frac{c_{26}B_{26}}{2 \cdot 26} &= 2^4 \cdot 3 \cdot 1^{25} + (1^{25} + 2^{25}) = 33554481 \\ &= 3 \cdot 17 \cdot 657931 \end{aligned}$$

where thus the factor 17 appears on the right side also and we obtain

$$B_{26} = \frac{13 \cdot 657931}{2 \cdot 3}$$

agreeing with the known value.

2 Modular Forms of Hecke

Let K be a totally real algebraic number field of degree g and discriminant d and let further k, be a natural number which is, inaddition, even if there exists in K a unit of norm -1. We denote the norm and trace by N and S. Let $\mathfrak{u}$ be an ideal, $\mathfrak{d}$ the different of K and $\mathfrak{u}^* = (\mathfrak{u}\mathfrak{d})^{-1}$, the complementary ideal to $\mathfrak{u}$.

We assign to every one of the g fields conjugate to K, one of the variables $z^{(1)}, \ldots, z^{(g)}$ in the upper half-plane and form with them the Eisenstein series

$$F_k(\mathfrak{u}, z) = N(\mathfrak{u}^k) \sum_{\mathfrak{u}|(\lambda,\mu)}{}' N((\lambda z + \mu)^{-k})$$

where λ, μ run over a complete system of pairs of numbers in the ideal $\mathfrak{u}$, different from 0, 0 and not differing from one another by a factor which is a unit. The series converges absolutely for $k > 2$. For the cases $k = 1, 2$, in order to

obtain convergence, the general term of the series is multiplied, as usual, by the factor $N(|\lambda z+\mu|^{-s})$, where the real part of the complex variable s is greater than $2-k$ and $F_k(\mathfrak{u},z)$ is defined as thevalue of the analytic continuation at $s=0$. **257**
A comparison with (1) shows that $F_k(\mathfrak{u},z)$ gives for $g=1$ and $k=4,6,\ldots$ the function denoted there as $F_k(z)$. Let hereafter $g>1$.

The generalization of (2) gives for the Fourier expansion of $F_k(\mathfrak{u},z)$, the formula

$$F_k(\mathfrak{u},z)=\zeta(\mathfrak{u},k)+\left(\frac{(2\pi i)^k}{(k-1)!}\right)^g d^{1/2-k}\sum_{\mathfrak{d}^{-1}|\nu>0}\sigma_{k-1}(\mathfrak{u},\nu)e^{2\pi i S(\nu z)}, \qquad (16)$$

where ν runs over all totally positive numbers in $\mathfrak{d}^{-1}$ and

$$\begin{aligned}\zeta(\mathfrak{u},k)&=N(\mathfrak{u}^k)\sum_{\mathfrak{u}|(\mu)}N(\mu^{-k}),\\ \sigma_{k-1}(\mathfrak{u},\nu)&=\sum_{\mathfrak{d}^{-1}|(\alpha)\mathfrak{u}|\nu}\operatorname{sign}(N(\alpha^k))N((\alpha)\mathfrak{u}\mathfrak{d})^{k-1}.\end{aligned} \qquad (17)$$

Here the summation is over principal ideals (μ), (α) under the conditions given. For even k, (17) can be put in the form

$$\sigma_{k-1}(\mathfrak{u},\nu)=\sum_{\mathfrak{u}\mathfrak{d}\sim\mathfrak{t}|(\nu)\mathfrak{d}}N(\mathfrak{t}^{k-1})(k=2,4,\ldots) \qquad (18)$$

where $\mathfrak{t}$ runs over all integral ideals in the class of $\mathfrak{u}\mathfrak{d}$ dividing $(\nu)\mathfrak{d}$. As a departure from the case $g=1$, formula (**??**) is valid also for $k=2$, while for $k=1$, an additional term is to be tagged on. For this reason, it is provisionally assumed that $k>1$.

The function $F_k(\mathfrak{u},z)$ is a modular form of Hecke of weight k, since under the substitutions

$$z\to\frac{\alpha z+\beta}{\gamma z+\mathfrak{d}}$$

of the Hilbert modular group, it takes the required factor $N((\gamma z+\mathfrak{d})^k)$. If we set all the g independent variables $z^{(1)},\ldots,z^{(g)}$ equal to the same value z in the upper half-plane, then form $F_k(\mathfrak{u},z)$ one obtains an elliptic modular form, which we denote by $\Phi_k(\mathfrak{u},z)$. Its weight is clearly $h=kg$. Let its Fourier expansion in powers of $q=e^{2\pi iz}$ be

$$\Phi_k(\mathfrak{u},z)=\sum_{n=0}^{\infty}b_nq^n$$

wherenow the Fourier coefficients $b_0, b_1, \ldots$ are determined, in view of (**??**) by **258**
the formulae

$$b_0 = \zeta(\mathfrak{u}, k), b_n = \left(\frac{(2\pi i)^k}{(k-1)!}\right)^g d^{1/2-k} \sum_{S(\nu)=n} \sigma_{k-1}(\mathfrak{u}, \nu)(n = 1, 2, \ldots) \tag{19}$$

Here ν runs over all totally positive numbers with trace n in the ideal $\mathfrak{d}^{-1}$ and $\sigma_{k-1}(\mathfrak{u}, \nu)$ is defined by (17). For brevity, let us put

$$\sum_{S(\nu)=n} \sigma_{k-1}(\mathfrak{u}, \nu) = s_n(\mathfrak{u}, k)(n = 1, 2, \ldots); \tag{20}$$

this is a rational integer. Explicitly,

$$s_n(\mathfrak{u}, k) = \sum_{\substack{\mathfrak{u}^*|(\alpha),\mathfrak{u}|\beta \\ \alpha\beta>0, S(\alpha\beta)=n}} \operatorname{sign}(N(\alpha^k))N((\alpha)\mathfrak{u}\mathfrak{d})^{k-1} \tag{21}$$

where the summation is over the principal ideals (α) and the numbers β in the field, subject to the given conditions.

On applying (14) with $M = \Phi_k(\mathfrak{u}, z)$, it now follows

$$((k-1)!)^g (2\pi i)^{-h} d^{k-1/2} \zeta(\mathfrak{u}, k) = -c_h^{-1} \sum_{l=1}^{r} c_h S_l(\mathfrak{u}, k); \tag{22}$$

here we have to take $h = kg$ and $r = r_h$. We note further that $N(-1) = (-1)^g$ and therefore h is always an even number. These formulae permit the calculation of $\zeta(\mathfrak{u}, k)$, if the ideal $\mathfrak{u}$ and the natural number k are given. They show, in particular, that the value

$$\pi^{-kg} d^{k-1/2} \zeta(\mathfrak{u}, k) = \psi(\mathfrak{u}, k)$$

is rational and more precisely, its denominator divides $((k-1)!)^g c_h$. It is to be stressed here that c_h depends only on $h = kg$ but not on the other properties of the field K.

In the special case $g = 2$ and thus for a real quadratic field, it was already known that the number on the left side of (22) is rational and that its denominator divides $4((2k)!)^2 v^{2k}$ where however v still depends on K, being given by
the smallest positive solution $y = v$ of Pell's equation$x^2 - dy^2 = 4$. On the **259**
other hand, with the help of a computing machine, Lang has listed the values of $\psi(\mathfrak{u}, 2)$ in 50 different examples for $k = g = 2$ and remarked, in addition, that in every one of the cases considered, only a divisor of 15 appears as the denominator. It is now easy to explain this phenomenon since $-2^{-4}c_4 = 15$.

For every fixed totally real algebraic number field K, we can, following (3), look upon the positive rational numbers $\psi(\mathfrak{u}, k)(k = 2, 4, \ldots)$ as the analogues of the values $2^{k-1}(k!)^{-1}|B_k|$ formed with the Bernoulli numbers. Unfortunately, however, one cannot expect that for this generalisation, corresponding recursion-formulae and divisibility theorems will exist as for the Bernoulli numbers themselves. For, they are connected with the properties of the function $(e^{2\pi z} - 1)^{-1}$ whose partial fraction decomposition also leads to (3). Indeed, similar to (**??**) we have even the simpler formule

$$\sum_{\mathfrak{u}|\mu} N((z+\mu)^{-k}) = \left(\frac{(2\pi i)^k}{(k-1)!}\right)^g d^{-1/2} N(\mathfrak{u}^{-1}) \sum_{\mathfrak{u}^*|\nu>0} N(\nu^{k-1}) e^{2\pi i S(\nu z)} (k = 2, 3, \ldots);$$

but in contrast with the case of one variable, the analytic function appearing here for $g > 1$ does not have, any more, important properties such as the addition theorem and the differential equation enjoyed by the exponential function. Further by an important result of Hecke, it is always singular for all real values of any one of the g variables, so that one cannot as in the case $g = 1$ obtain the numbers $\zeta(\mathfrak{u}, k)$ by expansion in powers of z. We get these numbers by going to the modular forms $F_k(\mathfrak{u}, z)$ of Hecke, which in spite of their complicated construction are really connected with one another by algebraic relations.

To the result (22) we may add a remark which relates to a lower bound for d discovered by Minkowski. The left side of (22) is clearly different from 0 for even k, since then $\zeta(\mathfrak{u}, k)$ is positive. Consequently among the r sums $s_l(\mathfrak{u}, k)$ given for $l = 1, 2, \ldots, r$ by (21), at least one must be non-empty and hence there always exist two numbers α, β in K which satisfy the four conditions

$$\mathfrak{u}^*|\alpha, \mathfrak{u}|\beta, \alpha\beta > 0, S(\alpha\beta) \leq r$$

with $r = r_h = r_{kg}$. For this assertion, it is optimal to choose $k = 2$ and hence 260
$h = 2g$; accordingly

$$r = \left[\frac{g}{6}\right] + 1 (g \not\equiv 1 (\mod 6)), r = \left[\frac{g}{6}\right] (g \equiv 1 (\mod 6))$$

so that in every case

$$r \leq \frac{g}{6} + 1. \tag{23}$$

From the decompositions

$$(\alpha) = \mathfrak{u}^*\mathfrak{t}, (\beta) = \mathfrak{u}\mathfrak{b}$$

with integral ideals $\mathfrak{t}$, $\mathfrak{b}$, it follows that there exist two integral ideals $\mathfrak{t}$ and $\mathfrak{b}$ (in every ideal class and its complementary class), such that

$$\mathfrak{t}\mathfrak{b}\mathfrak{d}^{-1} = (\nu), \nu > 0, S(\nu) \leq r \tag{24}$$

We therefore have

$$
\begin{aligned}
d^{-1}N(\mathfrak{t}\mathfrak{b}) = N(\mathfrak{t}\mathfrak{b}\mathfrak{d}^{-1}) = N(\nu) \leq (g^{-1}S(\nu))^g \leq \left(\frac{r}{g}\right)^g \\
N(\mathfrak{t}\mathfrak{b}) \leq \left(\frac{r}{g}\right)^g d
\end{aligned}
\tag{25}
$$

and hence

$$\mathrm{Min}(N(\mathfrak{t}), N(\mathfrak{b})) \leq \left(\frac{r}{g}\right)^{g/2} \sqrt{d}$$

Since especially for every real quadratic field, the ideal $\mathfrak{d} = (\sqrt{d})$ belongs to the principal class, the classes of $\mathfrak{t}$ and $\mathfrak{b}$ are inverses of each other because of the first formula (24) and hence the conjugate ideal $\mathfrak{b}' = (N(\mathfrak{b}))\mathfrak{b}^{-1}$ is equivalent to $\mathfrak{t}$; further, $g = 2$ and so $r = 1$. Since $N(\mathfrak{b}) = N(\mathfrak{b}')$, there exists, in every ideal class of a real quadratic field, an integral ideal of norm $\leq \frac{1}{2}\sqrt{d}$ always.

For arbitrary totally real fields of degree $g \geq 2$, we always have by (25)

$$1 \leq N(\mathfrak{t}) \leq \left(\frac{r}{g}\right)^g d,$$

from which, by (23) the inequality **261**

$$d \geq \left(\frac{g}{r}\right)^g \geq 6^g \left(1 + \frac{6}{g}\right)^{-g} > 6^g e^{-6} \tag{26}$$

follows. Since in the cases $g = 2, 3, 4, 5, 7$ the number $r = 1$, we have then

$$d \geq g^g$$

and this is a new result for field degrees 5 and 7. Since $e^2 > 6$, the estimate

$$d \geq \left(\frac{g^g}{g!}\right)^2$$

found by Minkowski is better than (**??**) for all sufficiently large g, but not for a few small g to which set the given values 5 and 7 belong. For $g = 4$, it is well-known that 725 is the smallest discriminant of a totally real field while

$$4^4 = 256, \frac{4^8}{(4!)^2} = 113 + \frac{7}{9}.$$

The special case $g = 2$ yields also a connection with the reduction theory of indefinite quadratic forms due to Gauss. Namely, let $[\kappa, \lambda]$ be a basis of the ideal $\mathfrak{u}$ and hence

$$\frac{\lambda}{\kappa} = \frac{b + \sqrt{d}}{2a}, d = b^2 - 4ac, N(\mathfrak{u})\sqrt{d} = |\kappa\lambda' - \lambda\kappa'|$$

with rational integers

$$a = N(\kappa)N(\mathfrak{u}^{-1}), b = S(\kappa\lambda')N(\mathfrak{u}^{-1}), c = N(\lambda)N(\mathfrak{u}^{-1}).$$

Then the conjugate ideal is $\mathfrak{u}' = (\kappa', \lambda')$ and $(\kappa\lambda' - \lambda\kappa')^{-1}[\lambda', -\kappa']$ gives the basis of $\mathfrak{u}^* = (\mathfrak{u}\mathfrak{d})^{-1}$ complementary to $[\kappa, \lambda]$. If we now put

$$\alpha = \frac{\kappa' x_2 + \lambda' y_2}{\kappa\lambda' - \lambda\kappa'}, \beta = \kappa x_1 + \lambda y_1$$

withrational integers x_1, y_1, x_2, y_2, then 262

$$\begin{aligned}\alpha\beta &= \frac{2\kappa\kappa' x_1 x_2 + (\kappa\lambda' + \lambda\kappa')(x_1 y_2 + x_2 y_1) + 2\lambda\lambda' y_1 y_2}{2(\kappa\lambda' - \lambda\kappa')} + \frac{x_1 y_2 - x_2 y_1}{2} \\ &= \frac{2a x_1 x_2 + b(x_1 y_2 + x_2 y_1) + 2c y_1 y_2}{2\,\mathrm{sign}(\kappa\lambda' - \lambda\kappa')\sqrt{d}} + \frac{x_1 y_2 - x_2 y_1}{2}\end{aligned}$$

and the two conditions

$$S(\alpha\beta) = 1, \alpha\beta > 0$$

go over into

$$x_1 y_2 - x_2 y_1 = 1, |2a x_1 x_2 + b(x_1 y_2 + x_2 y_1) + 2c y_1 y_2| < \sqrt{d}.$$

But this precisely means that the binary quadratic form $au^2 + buv + cv^2$ of discriminant d goes over, under the linear substitution $u \to x_1 u + x_2 v, v \to y_1 u + y_2 v$ of determinant 1, into a properly equivalent form whose middle coefficient is smaller than $\sqrt{d}$ in absolute value. This means again that in the transformed form both the outer coefficients are of opposite signs and this is just the essence of reduction theory.

We have still to consider the case $k = 1$ excluded hitherto. In this case [3], in contrast to (**??**), the constant term is

$$b_0 = \zeta(\mathfrak{u}, 1) + \zeta(\mathfrak{u}^*, 1)$$

where we have to define

$$\zeta(\mathfrak{u}, 1) = N(\mathfrak{u}) \lim_{s \to 0} \sum_{\mathfrak{u}|(\mu)} N(\mu^{-1})N(|\mu|^{-s})$$

and also change (22) correspondingly. This way we do not obtain the individual numbers $\zeta(\mathfrak{u}, 1)$ and $\zeta(\mathfrak{u}^*, 1)$ but only their sum. By the way, this is always 0 if either $g = 2$ or $\mathfrak{u}^2\mathfrak{d}$ is a principal ideal (γ) and $N(\gamma) < 0$.

3 Examples

In conclusion, we give three simple examples for evaluating $\zeta(\mathfrak{u}, k)$ using(22), **263**
where the result can also be checked in another way.

With $k = g = 2$, we have $h = 4$, $r = 1$ and

$$N(\mathfrak{u}^2)\sum_{\mathfrak{u}|(\mu)} N(\mu^{-2}) = -\frac{(2\pi)^4}{c_4 d^{3/2}} s_1(\mathfrak{u}, 2) = \frac{\pi^4}{15d\sqrt{d}} s_1(\mathfrak{u}, 2);$$

here $s_n(\mathfrak{u}, k)$ for $k = 2, 4, \ldots$ and $n = 1, 2, \ldots$ is given by (18) and (20). If we take, in particular, $d = 4.79$, $\mathfrak{u} = \mathfrak{b} = (1)$, then $\mathfrak{d} = (2\delta)$, $v = \dfrac{a+\delta}{2\delta}$ with $\delta = \sqrt{79}$ and $a = 0, \pm1, \ldots, \pm8$, $-N(a + \delta) = 79$, $2 \cdot 3 \cdot 13$, $3 \cdot 5^2$, $2 \cdot 5 \cdot 7$, $3^2 \cdot 7$, $2 \cdot 3^3$, 43, $2 \cdot 3 \cdot 5$, $3 \cdot 5$. We now note that $2 = N(9 + \delta)$, while the cube of the prime ideal $\mathfrak{p} = (3, 1 + \delta)$ gives a principal ideal. For the determination of the principal ideal divisors (τ) of $(a+\delta)$, we have to consider besides the trivial decomposition $(a + \delta) = (1) \cdot (a + \delta)$, only

$$(a + \delta) = (9 + \delta)\left(\frac{a+\delta}{9+\delta}\right)$$

with odd a; hence, for

$$\sigma_1(\mathfrak{v}, v) = \mathfrak{v}(a) = \sum_{\mathfrak{v}|(\tau)|(a+\delta)} N((\tau)),$$

we have the values

$$\begin{aligned} \sigma(0) &= 1 + 79 = 80, & \sigma(\pm1) &= 1 + 78 + 2 + 39 = 120,\\ \sigma(\pm2) &= 1 + 75 = 76, & \sigma(\pm3) &= 1 + 70 + 2 + 35 = 108,\\ \sigma(\pm4) &= 1 + 63 = 64, & \sigma(\pm5) &= 1 + 54 + 2 + 27 = 84,\\ \sigma(\pm6) &= 1 + 43 = 44, & \sigma(\pm7) &= 1 + 30 + 2 + 15 = 48, \end{aligned}$$

$$\sigma(\pm8) = 1 + 15 = 16.$$

Accordingly, in the present case,

$$s_1(\mathfrak{v}, 2) = 80 + 2(120 + 76 + 108 + 64 + 84 + 44 + 48 + 16) = 1200$$

$$\sum_{\mathfrak{o}|(\mu)} N(\mu^{-2}) = \frac{10\pi^4}{79\sqrt{79}}$$

inagreement with the value found by me elsewhere. For $\mathfrak{u} = \mathfrak{p} = (3, 1+\delta)$, we obtain correspondingly **264**

$$\begin{aligned} s_1(\mathfrak{p}, 2) = 0 &+ (3+6+13+26) + (3+5+15+25)+ \\ &+ (5+10+7+14) + (3+7+21+9)+ \\ &+ (3+6+9+18) + 0 + (3+6+5+10) + (3+5) = 240 \\ N(\mathfrak{p}^2) &\sum_{\mathfrak{p}|(\mu)} N(\mu^{-2}) = \frac{2\pi^4}{79\sqrt{79}} \end{aligned}$$

and the same for $\mathfrak{u} = \mathfrak{p}' = (3, 1-\delta)$. Since the 3 ideal classes of the field are represented by 1, $\mathfrak{p}$, $\mathfrak{p}'$, we have therefore for the zeta function $\zeta_K(s)$ of the field,

$$\zeta_K(2) = \frac{\pi^4}{79\sqrt{79}}(10+2+2) = \frac{14\pi^4}{79\sqrt{79}},$$

while from the law of decomposition, we get, likewise, on using the formula

$$\sum_{n=1} \left(\frac{d}{n}\right) n^{-k} = -\frac{(2\pi i)^k}{1\cdot k!\sqrt{d}} \sum_{l=1}^{d} \left(\frac{d}{l}\right) p_k\left(\frac{l}{d}\right),$$

$$P_k(x) = \sum_{l=0}^{k} \binom{k}{l} B_l x^{k-l} (k = 2, 4, \ldots)$$

that

$$\begin{aligned} \zeta_K(2) &= \zeta(2) \sum_{n=1}^{\infty} \left(\frac{316}{n}\right) n^{-2} \\ &= \frac{\pi^2}{6} \frac{(2\pi)^2}{2\cdot 2!\sqrt{316}} \sum_{l=1}^{316} \left(\frac{316}{l}\right) P_2\left(\frac{l}{316}\right) \\ &= \frac{\pi^4}{12\sqrt{79}} \frac{1}{2} \sum_{l=1}^{39} \left(\frac{79}{2l-1}\right)\left(1 - \frac{2l-1}{79}\right) = \frac{14\pi^4}{79\sqrt{79}}. \end{aligned}$$

We now take $k = 12$, $g = 2$; then $h = 24$, $r = 3$ and

$$N(\mathfrak{u}^{12}) \sum_{\mathfrak{u}|(\mu)} N(\mu^{-12})$$

265

$$= -\frac{(2\pi)^{24}}{(11!)^2 c_{24} d^{23/2}}(c_{24,1}s_1(\mathfrak{u},12) + c_{24,2}s_2(\mathfrak{u},12) + s_3(\mathfrak{u},12))$$
$$= \frac{\pi^{24}}{2\cdot 3^{10}\cdot 5^7\cdot 7^3\cdot 11^2\cdot 13 d^{11}\sqrt{d}}(-195660 s_1(\mathfrak{u},12) + 48 s_2(\mathfrak{u},12) + s_3(\mathfrak{u},12)).$$

In the special case $d = 5$, there exists only one class $\mathfrak{d} = (\sqrt{5})$ and the totally positive numbers ν of the ideal $\mathfrak{d}^{-1}$ with trace 1, 2, 3 are given by $\frac{\pm 1 + \sqrt{5}}{2\sqrt{5}}$, $\frac{a + \sqrt{5}}{\sqrt{5}}(a = 0, \pm 1, \pm 2)$, $\frac{a + 3\sqrt{5}}{2\sqrt{5}}(a + \pm 1, \pm 3, \pm 5)$. Since $\frac{\pm 1 + \sqrt{5}}{2}$, $\pm 2 + \sqrt{5}$ are units and also $\sqrt{5}$, $\pm 1 + \sqrt{5}$, $\frac{\pm 1 + 3\sqrt{5}}{2}$, $\frac{\pm 3 + 3\sqrt{5}}{2}$, $\frac{\pm 5 + 3\sqrt{5}}{2}$ are indecomposable, we have

$$\begin{aligned} s_1(\mathfrak{o},12) &= 2\cdot 1^{11} = 2,\ s_2(\mathfrak{b},12) = 1^{11} + 5^{11} + 2(1^{11} + 4^{11}) + 2\cdot 1^{11} \\ &= 57216738, \\ s_3(\mathfrak{o},12) &= 2(1^{11} + 11^{11}) + 2(1^{11} + 9^{11}) + 2(1^{11} + 5^{11}) = 633483116696 \end{aligned}$$

and thus

$$\sum_{\mathfrak{o}|(\mu)|} N(\mu^{-12}) = \frac{2^4\cdot 691\cdot 110921\pi^{24}}{3^{10}\cdot 5^{16}\cdot 7^3\cdot 11^2\cdot 13\sqrt{5}}$$

On the other hand, in the present case too, we have

$$\begin{aligned} \sum_{\mathfrak{o}|(\mu)} N(\mu^{-12}) &= \zeta_K(12) \\ &= \zeta(12)\sum_{n=1}^{\infty}\left(\frac{5}{n}\right)n^{-12} = \frac{(2\pi)^{24}B_{12}}{(2\cdot 12!)^2\sqrt{5}}\sum_{l=1}^{4}\left(\frac{5}{l}\right)P_{12}\left(\frac{l}{5}\right) \end{aligned}$$

with the same result, where this time the factor 691 comes in through the numerator of B_{12}.

Finallylet $k = 2$, $g = 3$ so that $h = 6$, $r = 1$ and **266**

$$N(\mathfrak{u}^2)\sum_{\mathfrak{u}|(\mu)} N(\mu^{-2}) = \frac{(2\pi)^6}{c_6 d^{3/2}}s_1(\mathfrak{u},2) = \frac{8\pi^6}{63d\sqrt{d}}s_1(\mathfrak{u},2).$$

In particular, let $d = 49$ and hence K, the totally real cubic subfield of the field of 7^{th} roots of unity. It is generated by

$$\rho = 2 - \epsilon - \epsilon^{-1} = (1-\epsilon)(1-\epsilon^{-1})$$

with $\epsilon = e^{2\pi i/7}$ and has class number 1, as is also evident from (25). Since $[1, \rho, \rho^2]$ is a basis of $\mathfrak{o}$ and $N(\rho) = 7$, we have $(\rho^3) = 7$ and so $\mathfrak{d} = (\rho^2)$. We have hence to determine for

$$v = x + y\rho^{-1} + z\rho^{-2},$$

the solutions of $S(v) = 1$, $v > 0$ with rational integers x, y, z. A short computation gives

$$\rho^3 = 7(\rho - 1)^2, S(\rho) = 7, S(\rho^2) = 21, S(\rho^{-1}) = 2, S(\rho^{-2}) = 2,$$
$$S(\rho^{-3}) = 2 + \frac{3}{7},$$
$$S(v) = 3x + 2y + 2z = 1$$

By the substitution $x = 1 + 2u$ with integral u, it follows that $y = -1 - y - 3u$ and the condition $v > 0$, and consequently $\rho^2 v > 0$, goes over into

$$(2\rho^2 - 3\rho)u + (1 - \rho)z + \rho^2 - \rho > 0. \tag{27}$$

If we set again

$$\epsilon^k + \epsilon^{-k} = 2\cos\frac{2k\pi}{7} = \lambda_k \quad (k = 1, 2, 3),$$

then

$$1 = 2\cos\frac{\pi}{3} < \lambda_1 < 2\cos\frac{\pi}{4} = \sqrt{2} < \frac{3}{2},$$
$$-\frac{1}{2} < -\frac{\pi}{7} < -2\sin\frac{\pi}{14} = \lambda_2 < 2\cos\frac{\pi}{2} = 0$$
$$-2 = 2\cos\pi < \lambda_3 = -2\cos\frac{\pi}{7} < -2\cos\frac{\pi}{6} = -\sqrt{3} < -\frac{3}{2}.$$

267

With this, we obtain from (27) by a simple modification, the three conditions

$$z + \lambda_1 - 2 > (\lambda_2 + 5)u, z + \lambda_2 - 2 < (\lambda_3 + 5)u,$$
$$z + \lambda_3 - 2 < (\lambda_1 + 5)u$$

whence, it follows, in particular, that

$$\lambda_3 < u < \lambda_1$$

Therefore, for u, only the values 0, ± 1 have to be considered. Of these however, $u = 1$ leads to the contradiction

$$5 < \lambda_2 - \lambda_1 + 7 < z < \lambda_3 - \lambda_2 + 7 < 6.$$

For $u = 0$, we have

$$2 - \lambda_1 < z < 2 - \lambda_2 < 2 - \lambda_3,$$

and hence either $z = 1$, $y = -2$, $x = 1$ or $z = 2$, $y = -3$, $x = 1$ and for $u = -1$,

$$\lambda_3 - 2 < z < \lambda_2 - 2 < \lambda_1 - 2$$

and hence $z = -3$, $y = 5$, $x = -1$. The first of the three solutions formed above, gives

$$\rho^2 v = (\rho - 1)^2, v = \frac{\rho}{7}.$$

and the other two v must therefore be conjugate to this, as we can also check directly. In all the three cases, $\rho^2 v$ proves to be a unit and hence $(v)\mathfrak{d} = \mathfrak{o}$, $\mathfrak{t} = \mathfrak{o}$, $s_1(\mathfrak{o}, 2) = 3$ and

$$\sum_{\mathfrak{o}|(\mu)} N(\mu^{-2}) = \frac{8\pi^6 \cdot 3}{63 \cdot 49 \cdot \sqrt{49}} = \frac{2^3 \cdot \pi^6}{3 \cdot 7^4}.$$

In order to verify this result in another way, we note from the law of decomposition for the cyclotomic field, that

$$\sum_{\mathfrak{o}|(\mu)} N(\mu^{-2}) = \zeta_K(2) = (1 - 7^{-2})^{-1} \prod_{l=0}^{2} (w_1 + \eta^l w_2 + \eta^{-l} w_3)$$
$$= \frac{7^2}{2^4 \cdot 3} (w_1^3 + w_2^3 + w_3^3 - 3 w_1 w_2 w_3)$$

with 268

$$w_k = \sum_{n=-\infty}^{\infty} (7n + k)^{-2} (k = 1, 2, 3), \eta = e^{2\pi i/3}.$$

But now

$$w_k = \left(\frac{\pi}{7 \sin \frac{k}{7}} \right)^2 = \left(\frac{2\pi}{7} \right)^2 (2 - \epsilon^k - \epsilon^{-k})^{-1} = \left(\frac{2\pi}{7} \right)^2 \rho^{-1}$$

$$w_1^3 + w_2^3 + w_3^3 = \left(\frac{2\pi}{7} \right)^6 S(\rho^{-3}) = \left(\frac{2\pi}{7} \right)^6 \left(2 + \frac{3}{7} \right)$$

$$w_1 w_2 w_3 = \left(\frac{2\pi}{7}\right)^6 N(\rho^{-1}) = \left(\frac{2\pi}{7}\right)^6 \cdot \frac{1}{7}$$

and indeed we have again

$$\sum_{\mathfrak{o}|(\mu)} N(\mu^{-2}) = \frac{7^2}{2^4 \cdot 3} \left(\frac{2\pi}{7}\right)^6 \cdot 2 = \frac{2^3 \cdot \pi^6}{3 \cdot 7^4}$$

Bibliography

[1] MEYER, C: Über die Bildung von elementar-arithmetischen Klasseninvarianten in reell-quadratischen Zahlkörpern *Berichte. math. Forsch-Ist. Oberwalfach* Z, Mannheim 1966, 165-215.

LANG, H: Über eine Gattung elementar-arithmetischer Klasseninvarianten reell-quadratischer Zahkkórper. Inaug Diss. Köln. 1967 VII u. 84 S.

SIEGEL, C. L.: Bernoullische Polynome und quadratische Zahlkörper. *Nachr. Akad. Wiss. Gottigen, Math-Phys.* K1. 1968, 7-38.

BARNER, K: Über die Werte der Ringklassen-L-Funktionen reell-quadratischer Zahlkörper an natürlichen Argumentstellen. *J. Numbr. Th.* 1. 28-64 (1969).

[2] KLINGEN, H: Über die Werte der Dedekindschen Zetafunktionen, *Math. Ann.* 145, 265-272 (1962).

[3] GUNDLACH, K. B: Poincarésche und Eisensteinsche Reihen zur Hilbertschen Modulgruppe. *Math. Zeitschr.* 64, 339-352 (1956); insbes 350-351.