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Special values of $L(s, \chi)$

Paul Garrett garrett@math.umn.edu <http://www.math.umn.edu/~garrett/>

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The contour-integration device applied earlier to Riemann's zeta is equally applicable to Dirichlet L -functions. It immediately proves that values of $L(s, \chi)$ at non-positive integers lie in the field generated over \mathbb{Q} by the values of χ , necessarily roots of unity. It also shows the connection to the Laurent coefficients of more general analogues of the function $1/(e^t - 1)$ which appeared in the discussion of $\zeta(s)$.

[1.1] An integral representation of $\Gamma(s) \cdot L(s, \chi)$

[1.1.1] Claim: For a Dirichlet character χ modulo $N > 1$,

$$\int_0^\infty t^s \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-tN}} \frac{dt}{t} = \Gamma(s) \cdot L(s, \chi) \quad (\text{for } \operatorname{Re}(s) > 1)$$

Proof: Expanding geometric series and using the Gamma identity $\int_0^\infty t^s e^{-ty} dt/t = y^{-s} \Gamma(s)$ for $\operatorname{Re}(s) > 1$ and $y > 0$,

$$\begin{aligned} \int_0^\infty t^s \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-tN}} \frac{dt}{t} &= \int_0^\infty t^s \sum_{j=1}^N \chi(j) e^{-tj} \cdot \left(1 + e^{-tN} + e^{-2tN} + e^{-3tN} + \dots\right) \frac{dt}{t} \\ &= \sum_{j=1}^N \chi(j) \int_0^\infty t^s \left(e^{-tj} + e^{-t(N+j)} + e^{-t(2N+j)} + e^{-t(3N+j)} + \dots\right) \frac{dt}{t} \\ &= \sum_{j=1}^N \chi(j) \cdot \Gamma(s) \cdot \left(\frac{1}{j^s} + \frac{1}{(N+j)^s} + \frac{1}{(2N+j)^s} + \frac{1}{(3N+j)^s} + \dots\right) = \Gamma(s) \cdot \sum_{n=1}^\infty \frac{\chi(n)}{n^s} = \Gamma(s) \cdot L(s, \chi) \end{aligned}$$

as asserted. ///

[1.2] **Keyhole/Hankel contour** The usual *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon > 0$, counterclockwise around a circle of radius ε at 0, back to ε on the real line, and outbound back to $+\infty$ along the real line.

The typical application is to evaluation of integrals of the form $\int_0^\infty t^s f(t) dt$ with f holomorphic on a neighborhood of $[0, +\infty)$, with $-1 < \operatorname{Re}(s)$, and with suitable convergence. In such an example, analytically continuing counterclockwise around 0 has no impact on $f(t)$, but, significantly, t^s changes by a factor $e^{2\pi is}$, since

$$t^s = (|t| \cdot e^{i\theta})^s = |t|^s \cdot e^{i\theta s} \quad (\text{and } \theta \text{ goes from } 0 \text{ to } 2\pi)$$

With whatever choice of out-bound value of t^s , the inbound version of t^s is $t^s \cdot e^{2\pi is}$. The absolute value of the integrand goes to 0 as $|t| \rightarrow 0$, so the integral over the small circle goes to 0 as $\varepsilon \rightarrow 0$, as do the integrals to and from 0, ε along the real line.

Thus, letting H_ε be the Hankel contour with circle of radius $\varepsilon > 0$, for $-1 < \operatorname{Re}(s)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} t^s \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} \frac{dt}{t} = (1 - e^{2\pi is}) \int_0^\infty t^s \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} \frac{dt}{t} = (1 - e^{2\pi is}) \cdot \Gamma(s) \cdot L(s, \chi)$$

[1.3] Evaluation of $\zeta(-n)$ Rewrite this as

$$L(s, \chi) = \frac{1}{\Gamma(s) \cdot (1 - e^{2\pi is})} \cdot \lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} t^{s-1} \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} dt$$

At $s = -n \in \{0, -1, -2, -3, -4, \dots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1 - e^{2\pi is}$ cancel, giving a finite, computable value. Second, the function t^{-n-1} is *single-valued*, so the inbound and outbound integrals of the Hankel contour *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $t^{-n-1} \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}}$ at $t = 0$.

The periodicity of $1 - e^{2\pi is}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi is} = 1 - \left(1 + \frac{2\pi is}{1!} + \frac{(2\pi is)^2}{2!} + \dots\right) = -2\pi is + \text{higher}$$

The residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\begin{aligned} L(s, \chi) &= \frac{1}{\frac{(-1)^n}{n!} \cdot (-2\pi i)} \cdot 2\pi i \cdot \text{Res}_{t=0} t^{-n-1} \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} \\ &= (-1)^{n+1} \cdot n! \cdot (n^{\text{th}} \text{ Laurent coefficient of } \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} \text{ at } t = 0) \end{aligned}$$

Those Laurent coefficients are generalized Bernoulli numbers, and lie in $\mathbb{Q}(\chi)$, the field generated over \mathbb{Q} by values of χ .

Further, these Laurent coefficients are finite linear combinations of values of χ , so the Galois group $\text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ acts compatibly on these Laurent coefficients: for $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$,

$$\left(n^{\text{th}} \text{ Laurent coefficient of } \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}}\right)^\sigma = n^{\text{th}} \text{ Laurent coefficient of } \frac{\sum_{j=1}^{N-1} \chi^\sigma(j) e^{-tj}}{1 - e^{-Nt}}$$

where χ^σ is the character $\chi^\sigma(j) = \chi(j)^\sigma$ obtained by applying σ to the values, which are roots of unity.

Thus, we have proven

[1.3.1] Theorem: $L(-n, \chi) \in \mathbb{Q}(\chi)$ for $-n = 0, -1, -2, -3, \dots$ and $L(-n, \chi)^\sigma = L(-n, \chi^\sigma)$. ///

[1.4] Vanishing of $L(-n, \chi)$ depending on parity The *parity* of χ is determined by $\chi(-1)$: for $\chi(-1) = 1$, χ is *even*, while for $\chi(-1) = -1$, χ is *odd*. For the function

$$f(t) = \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}}$$

parity refers to the comparison of $f(-t)$ to $f(t)$: for $f(-t) = f(t)$ the function would be *even*, and for $f(-t) = -f(t)$, the function is *odd*. Direct computation gives

$$f(-t) = \frac{\sum_{j=1}^{N-1} \chi(j) e^{-tj}}{1 - e^{-Nt}} = \frac{\sum_{j=1}^{N-1} \chi(j) e^{Nt-tj}}{e^{Nt} - 1} = \frac{\sum_{j=1}^{N-1} \chi(N-j) e^{tj}}{e^{Nt} - 1} = \frac{\sum_{j=1}^{N-1} \chi(-j) e^{tj}}{e^{Nt} - 1}$$

by replacing j by $N-j$. Since $\chi(-j) = \chi(-1)\chi(j)$, by adjusting the denominator, this is

$$f(-t) = -\chi(-1) \frac{\sum_{j=1}^{N-1} \chi(j) e^{tj}}{1 - e^{Nt}} = -\chi(-1) \cdot f(t)$$

That is, the parity of $f(t)$ is *opposite* the parity of χ . In particular, the *even* Laurent coefficients at $t = 0$ are 0 for χ *even*, and the *odd* Laurent coefficients at $t = 0$ are 0 for χ *odd*. We have proven

[1.4.1] Claim: $L(-n, \chi) = 0$ for $-n \leq 0$ of the same parity as χ . ///