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Analytic continuation and functional equation of $L(s,\chi)$

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We prove the analytic continuation and function equation of Dirichlet L-functions $L(s,\chi)$ imitating the argument Riemann used for proving the *analytic continuation* of $\zeta(s)$ and its *functional equation*

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,\zeta(1-s)$$

from the *integral representation*

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \int_0^\infty y^{-s/2}\,\frac{\theta(iy)-1}{2}\,\frac{dy}{y}$$

in terms of the *theta series*

$$\theta(iy) \;=\; \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}$$

whose functional equation

$$\theta(iy) \;=\; \frac{1}{\sqrt{y}}\cdot\theta\big(\frac{i}{y}\big)$$

is proven by *Poisson summation.*

The discussion of Dirichlet L-functions $L(s,\chi)$ bifurcates into two families, depending upon the *parity* of χ, that is, whether $\chi(-1)=+1$ or $\chi(-1)=-1$.

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1. *Dirichlet L-functions $L(s,\chi)$ for even Dirichlet characters*

Let χ be a *non-trivial* Dirichlet character mod $N>1$, with L-function

$$L(s,\chi) \;=\; \sum_{n\geq 1}\frac{\chi(n)}{n^s}$$

By imitation of the corresponding discussion for $\zeta(s)$, we expect to define a *theta series* θ_χ with a *functional equation* provable via Poisson summation, to exhibit an *integral representation* of $L(s,\chi)$ in terms of θ_χ, and to use this integral representation to prove the analytic continuation and see the functional equation in the symmetry of the rewritten integral representation.

The most obvious imitative approach succeeds only for *even* χ. Further, to have a symmetrical form of the functional equation, we must be able to determine the absolute value of the Gauss sum

$$\langle\chi,\psi\rangle \;=\; \sum_{b \bmod N}\chi(b)\,\overline{\psi}(b) \qquad\qquad (\text{with } \psi(b)=e^{2\pi i b/N})$$

This requires that χ be *primitive* mod N, that is, its *conductor* must be exactly N, not a proper divisor. That is, the non-zero parts of χ must *not* be well-defined modulo N' for any proper divisor of N.

[1.1] The theta series, parity of χ The obvious imitation of the theta series for ζ suggests defining

$$\theta_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}}\chi(n)\,e^{-\pi n^2 y}$$

Non-trivial χ is extended by 0 to integers having a common factor with N, so $\chi(0)=0$. By design,

$$\frac{\theta_\chi(iy)}{2} \;=\; \sum_{n\ge 1} \frac{\chi(n)+\chi(-n)}{2}\, e^{-\pi n^2 y} \;=\; \begin{cases} 0 & \text{(for } \chi \text{ odd)} \\ \sum_{n\ge 1} \chi(n)\, e^{-\pi n^2 y} & \text{(for } \chi \text{ even)} \end{cases}$$

That is, this version of θ_χ cannot be correct for *odd* χ, since it would vanish identically in that case!

[1.2] The integral representation

The integral representation is easy: for $\mathrm{Re}(s)>1$,

$$\int_0^\infty y^{\frac{s}{2}}\, \frac{\theta_\chi(iy)}{2}\, \frac{dy}{y} \;=\; \sum_{n\ge 1} \chi(n) \int_0^\infty y^{\frac{s}{2}}\, e^{-\pi n^2 y}\, \frac{dy}{y}$$

$$=\; \sum_{n\ge 1} \frac{\chi(n)}{\pi^{s/2}\, n^s} \int_0^\infty y^{\frac{s}{2}}\, e^{-y}\, \frac{dy}{y} \;=\; \pi^{-s/2}\, \Gamma(\tfrac{s}{2})\, L(s,\chi) \qquad \text{(for } \chi \text{ \textit{even})}$$

by replacing y by $y/\pi n^2$. This is the desired integral representation.

[1.3] Functional equation of θ_χ

We anticipate that θ_χ has a functional equation similar to that of $\theta(iy)=\sum_n e^{-\pi n^2 y}$, proven by application of the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad \text{(for nice functions } f)$$

The minor but meaningful obstacle is that in this classical guise θ_χ is *not* simply a sum of a nice function over a lattice, so Poisson summation is not immediately applicable. However, since $\chi(n)$ depends only upon n mod N, we can break the sum expressing θ_χ into a finite sum of sums-over-lattices, and then apply Poisson summation.

Specifically,

$$\theta_\chi(iy) \;=\; \sum_n \chi(n)\, e^{-\pi n^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} e^{-\pi(\ell N+b)^2 y}$$

Thus, Poisson summation is to be applied to the function

$$f(x) \;=\; e^{-\pi(xN+b)^2 y}$$

Since this f differs from the Gaussian $e^{-\pi x^2}$ merely by dilations and translation, and the Fourier transform behaves cogently with respect to such, something reasonable will come out, by changing variables:

$$\begin{aligned}
\Big(e^{-\pi(xN+b)^2 y}\Big)^{\widehat{}}(\xi) \;&=\; \int_{\mathbb{R}} e^{-\pi(xN+b)^2 y}\, e^{-2\pi\xi x}\, dx \\
&=\; \frac{1}{N}\int_{\mathbb{R}} e^{-\pi(x+b)^2 y}\, e^{-2\pi\xi x/N}\, dx && \text{(replacing } x \text{ by } x/N) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N}\int_{\mathbb{R}} e^{-\pi x^2 y}\, e^{-2\pi\xi x/N}\, dx && \text{(replacing } x \text{ by } x-b) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}}\int_{\mathbb{R}} e^{-\pi x^2}\, e^{-2\pi\xi x/N\sqrt{y}}\, dx && \text{(replacing } x \text{ by } /\sqrt{y}) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}} e^{-\pi\xi^2/N^2 y} && \text{(taking Fourier transform)}
\end{aligned}$$

Thus, by Poisson summation,

$$\theta_\chi(iy) \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} e^{-\pi(\ell N+b)^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi i\ell b/N}}{N\sqrt{y}} e^{-\pi\ell^2/N^2 y}$$

$$=\; \frac{1}{N\sqrt{y}} \sum_{\ell\in\mathbb{Z}} e^{-\pi\ell^2/N^2 y} \sum_{b \bmod N} \chi(b) e^{2\pi i\ell b/N}$$

The obvious thing would be to replace b by $b\ell^{-1}$ mod N, but the sum over ℓ does not obviously exclude ℓ with $\gcd(\ell,N) > 1$. This is the first point where the *primitivity* of χ mod N enters, to make the inner sum vanish for $\gcd(\ell,N) > 1$:

[1.3.1] Claim:

$$\sum_{b \bmod N} \chi(b)\, e^{2\pi i\ell b/N} \;=\; 0 \qquad \text{(for } \textit{primitive } \chi \text{ mod } N\text{, for } \ell \textit{ not} \text{ invertible mod } N\text{)}$$

Proof: Let $\gcd(\ell,N) = N/m > 1$. Then, replacing b by $b\cdot(1+xm$ for any x mod N/m gives

$$\sum_{b \bmod N} \chi(b)\, e^{2\pi i\ell b/N} \;=\;=\; \sum_{b \bmod N/m} \sum_{\beta \bmod m} \chi(b(1+xm))\, e^{2\pi i\ell\beta(1+xm)/N}$$

$$=\; \chi(1+xm)) \sum_{b \bmod N/m} \sum_{\beta \bmod m} \chi(b)\, e^{2\pi i\ell\beta/N}$$

This holds for all x. The primitivity of χ is exactly that $x \to \chi(1+xm)$ cannot be trivial. Thus, the sum is 0, as claimed. ///

Returning to the functional equation for θ_χ, now we *can* make the change of variables: replace b by $b\ell^{-1}$ mod N:

$$\theta_\chi(iy) \;=\; \frac{1}{N\sqrt{y}} \sum_{\ell\in\mathbb{Z}} \chi^{-1}(\ell)\, e^{-\pi\ell^2/N^2 y} \sum_{b \bmod N} \chi(b) e^{2\pi ib/N}$$

The inner sum has become a Gauss sum $\langle\chi,\psi\rangle$ *not* depending on ℓ,

$$\langle\chi,\psi\rangle \;=\; \sum_{b \bmod N} \chi(b)\, \overline{\psi}(b) \qquad \text{(with } \psi(b) = e^{-2\pi ib/N}\text{)}$$

so we have the functional equation:

$$\theta_\chi(iy) \;=\; \frac{\langle\chi,\psi\rangle}{N\sqrt{y}} \cdot \theta_{\chi^{-1}}\big(\frac{i}{N^2 y}\big) \qquad \text{(for even } \chi \text{ primitive mod } N\text{)}$$

[1.3.2] Remark: The most striking thing is that the functional equation relates θ_χ to $\theta_{\chi^{-1}}$, rather than to θ_χ itself. That is, the functional equation related θ_χ to itself only for $\chi^{-1} = \chi$, that is, for *real-valued* χ. This is typical of a larger reality.

[1.3.3] Remark: Note that the *flip* in the argument y is $y \to 1/N^2 y$, not the earlier $y \to 1/y$. The fixed-point of this map is not 1, but $1/N$. This influences the break-up of the integral in proof of the analytic continuation and functional equation, next.

[1.4] Analytic continuation and functional equation Starting from the integral representation,

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\, L(s,\chi) \;=\; \int_0^\infty y^{\frac{s}{2}}\, \frac{\theta_\chi(iy)}{2}\, \frac{dy}{y} \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\, \frac{\theta_\chi(iy)}{2}\, \frac{dy}{y} + \int_0^{1/N} y^{\frac{s}{2}}\, \frac{\theta_\chi(iy)}{2}\, \frac{dy}{y}$$

As for $\zeta(s)$, the integral from $1/N$ to ∞ is nicely convergent for all $s \in \mathbb{C}$, and gives an entire function. The goal is to use the functional equation to convert the integral from 0 to $1/N$ to the other sort, and then symmetrize the resulting expression as much as possible. That is, replace y by $1/N^2y$ in the problemmatical integral from 0 to $1/N$, obtaining

$$\tfrac{1}{2}\int_{1/N}^\infty (1/N^2y)^{\frac{s}{2}}\,\theta_\chi\Big(\frac{i}{N^2y}\Big)\,\frac{dy}{y} \;=\; \tfrac{1}{2}\,N^{-s}\int_{1/N}^\infty y^{-\frac{s}{2}}\,\theta_\chi\Big(\frac{i}{N^2y}\Big)\,\frac{dy}{y}$$

Replacing χ by χ^{-1} in the functional equation above gives

$$\theta_{\chi^{-1}}(iy) \;=\; \frac{\langle\chi^{-1},\psi\rangle}{N\sqrt{y}}\cdot\theta_\chi\big(\frac{i}{N^2y}\big)$$

Rearranging,

$$\theta_\chi\big(\frac{i}{N^2y}\big) \;=\; \frac{N\sqrt{y}}{\langle\chi^{-1},\psi\rangle}\theta_{\chi^{-1}}(iy)$$

Substituting this gives

$$\frac{N^{1-s}}{\langle\chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

That is,

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2}\,\frac{dy}{y} \;+\; \frac{N^{1-s}}{\langle\chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

This gives the *analytic continuation*, but is not quite symmetrized. Not only is there the inescapable feature that sending s to $1-s$ requires that χ becomes χ^{-1}, but, also, the Gauss sum and the power of the conductor N appear.

Recall that for χ *primitive*,

$$|\langle\chi,\psi\rangle| \;=\; \sqrt{N}$$

Since we do not pretend to know the *argument* of the Gauss sum, only its *size*, let

$$\varepsilon(\chi) \;=\; \frac{\sqrt{N}}{\langle\chi^{-1},\psi\rangle}$$

Then $|\varepsilon(\chi)| = 1$ and

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big(y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi)N^{\frac{1}{2}-s}y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Ignoring $\varepsilon(\chi)$ for a moment, we *can* symmetrize the powers of the conductor N, by multiplying through by $N^{s/2}$, giving

$$N^{s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

The seeming asymmetry in $\varepsilon(\chi)$ is an illusion: since χ is *even*,

$$\overline{\langle\chi^{-1},\psi\rangle} \;=\; \langle\chi,\overline{\psi}\rangle \;=\; \sum_b \chi(b)\overline{\psi}(-b) \;=\; \chi(-1)\sum_b \chi(b)\overline{\psi}(b) \;=\; \chi(-1)\langle\chi,\psi\rangle \;=\; \langle\chi,\psi\rangle \qquad (\text{for } \chi \text{ even})$$

That is,

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; \frac{N}{\langle\chi^{-1},\psi\rangle\cdot\langle\chi,\psi\rangle} \;=\; \frac{N}{\overline{\langle\chi,\psi\rangle}\cdot\langle\chi,\psi\rangle} \;=\; 1$$

Thus, there is the symmetry

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad\qquad \text{(for } \chi \text{ even)}$$

Thus, we have the *functional equation*

$$N^{\frac{s}{2}}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^{\infty}\Big((Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\int_{1/N}^{\infty}\Big(\varepsilon(\chi^{-1}(Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; (Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{1-s}{2}}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad\qquad \text{(for } \chi \text{ even)}$$

In summary,

$$N^{\frac{s}{2}}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{1-s}{2}}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad\qquad \text{(for } \chi \text{ even)}$$

where

$$\varepsilon(\chi) \;=\; \frac{\sqrt{N}}{\langle\chi^{-1},\psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1, \qquad \varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad\qquad (\chi \text{ even})$$

[1.4.1] Remark: The appearance of the conductor N is inescapable, as is the appearance of $\varepsilon(\chi)$.

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2. *Dirichlet L-functions $L(s,\chi)$ for odd Dirichlet characters*

Now accommodations are made for *odd* χ, *primitive* mod N.

[2.1] The theta series, parity of χ Again, the obvious imitation of the theta series for ζ suggests defining

$$\theta_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}} \chi(n)\,e^{-\pi n^2 y} \qquad\qquad \text{(bad: vanishes identically for odd } \chi!)$$

but for *odd* χ this fails, being identically 0 due to cancellation of the $\pm n$ summands. The Gaussian $e^{-\pi x^2}$ must be modified to be *odd*: the function $xe^{-\pi x^2}$ will do. Thus, let [1]

$$\widetilde{\theta}_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}} \chi(n)\,n\sqrt{y}\,e^{-\pi n^2 y} \qquad\qquad \text{(for odd } \chi)$$

[2.2] The integral representation The integral representation is easy, assuming that we keep track of the little shifts created by the alterations in $\widetilde{\theta}$: for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2}\,\frac{dy}{y} \;=\; \sum_{n\geq 1}\chi(n)\int_0^\infty y^{\frac{s}{2}}\,n\sqrt{y}\,e^{-\pi n^2 y}\,\frac{dy}{y}$$

$$=\; \sum_{n\geq 1}\frac{\chi(n)\,n}{\pi^{(s+1)/2}\,n^{s+1}}\int_0^\infty y^{\frac{s+1}{2}}\,e^{-y}\,\frac{dy}{y} \;=\; \pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi)$$

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[1] Given the existing literature, it is important to understand that this is *not* the standard classical-style normalization of the corresponding theta function, whence the modifying tilde. Specifically, the usual classical theta series would omit the $\sqrt{y}$. Of course, such variations can be compensated-for in companion normalizations. The choice here is motivated by somewhat longer-range considerations.

by replacing y by $y/\pi n^2$. This is the desired integral representation.

[2.2.1] Remark: The shifts in the exponent of π and argument of Γ are inescapable and correct, since the functional equation will turn out to be $s \to 1-s$ *with* these shifts in the gamma factor.

[2.3] Functional equation of $\widetilde{\theta}_\chi$

We anticipate that $\widetilde{\theta}_\chi$ has a functional equation similar to that of $\theta(iy) = \sum_n e^{-\pi n^2 y}$, proven by Poisson summation.

As with *even* χ, the minor obstacle is that $\widetilde{\theta}_\chi$ is *not* simply a sum of a nice function over a lattice, so Poisson summation is not immediately applicable. However, since $\chi(n)$ depends only upon n mod N, we can break the sum expressing $\widetilde{\theta}_\chi$ into a finite sum of sums-over-lattices, and then apply Poisson summation.

Specifically,

$$\widetilde{\theta}_\chi(iy) \;=\; \sum_n \chi(n)\, n\sqrt{y}\, e^{-\pi n^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} (\ell N + b)\sqrt{y}\, e^{-\pi(\ell N+b)^2 y}$$

Thus, Poisson summation is to be applied to

$$f((xN+b)\sqrt{y}) \;=\; (xN+b)\sqrt{y}\, e^{-\pi(xN+b)^2 y} \qquad (\text{with } f(x) = xe^{-\pi x^2})$$

This differs from $f(x)$ merely by dilations and translation, and the Fourier transform behaves cogently with respect to such, so something reasonable will come out. By changing variables:

$$\begin{aligned}
\Big(f((xN+b)\sqrt{y})\Big)^{\widehat{}}(\xi) \;&=\; \int_{\mathbb{R}} f((xN+b)\sqrt{y})\, e^{-2\pi \xi x}\, dx \\
&=\; \frac{1}{N}\int_{\mathbb{R}} f((x+b)\sqrt{y})\, e^{-2\pi\xi x/N}\, dx \quad (\text{replacing } x \text{ by } x/N) \\
&=\; \frac{e^{2\pi i \xi b/N}}{N}\int_{\mathbb{R}} f(x\sqrt{y})\, e^{-2\pi\xi x/N}\, dx \quad (\text{replacing } x \text{ by } x-b) \\
&=\; \frac{e^{2\pi i \xi b/N}}{N\sqrt{y}}\int_{\mathbb{R}} f(x)\, e^{-2\pi\xi x/N\sqrt{y}}\, dx \quad (\text{replacing } x \text{ by } /\sqrt{y}) \\
&=\; \frac{e^{2\pi i \xi b/N}}{N\sqrt{y}}\, \widehat{f}(\xi/N\sqrt{y}) \quad (\text{taking Fourier transform})
\end{aligned}$$

The function $f(x) = xe^{-\pi x^2}$ is an *eigenfunction*[2] of Fourier transform:

$$\big(xe^{-\pi x^2}\big)^{\widehat{}}(\xi) \;=\; -i\cdot \xi e^{-\pi\xi^2}$$

Thus, by Poisson summation,

$$\begin{aligned}
\widetilde{\theta}_\chi(iy) \;&=\; \sum_{b \bmod N}\chi(b)\sum_{\ell\in\mathbb{Z}} f((\ell N+b)\sqrt{y}) \;=\; -i\sum_{b \bmod N}\chi(b)\sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi i \ell b/N}}{N\sqrt{y}}\, f\Big(\frac{\ell}{N\sqrt{y}}\Big) \\
&=\; \frac{-i}{N\sqrt{y}}\sum_{\ell\in\mathbb{Z}} f\Big(\frac{\ell}{N\sqrt{y}}\Big)\sum_{b \bmod N}\chi(b)e^{2\pi i\ell b/N}
\end{aligned}$$

As already proven in the treatment of *even* χ, for *primitive* χ the inner sum *vanishes* for $\gcd(\ell, N) > 1$.

[2] This Fourier transform is readily computed: view the integral defining the Fourier transform as a contour integral along the real axis. Moving the contour upward by $i\xi$ immediately computes the Fourier transform, much as was done with the Gaussian itself.

Returning to the functional equation for $\widetilde{\theta}_\chi$, now we *can* make the change of variables: replace b by $b\ell^{-1}$ mod N:

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i}{N\sqrt{y}} \sum_{\ell\in\mathbb{Z}} \chi^{-1}(\ell)\, f\big(\frac{\ell}{N\sqrt{y}}\big) \sum_{b \bmod N} \chi(b) e^{2\pi i b/N}$$

The inner sum has become a Gauss sum $\langle\chi,\psi\rangle$ *not* depending on ℓ,

$$\langle\chi,\psi\rangle \;=\; \sum_{b \bmod N} \chi(b)\,\overline{\psi}(b) \qquad\qquad (\text{with } \psi(b) = e^{-2\pi i b/N})$$

so we have

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}} \cdot \sum_{\ell\in\mathbb{Z}} \chi^{-1}(\ell)\, f\big(\frac{\ell}{N\sqrt{y}}\big) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}} \cdot \widetilde{\theta}_{\chi^{-1}}\big(\frac{i}{N^2 y}\big) \qquad (\text{odd } \chi \text{ primitive mod } N)$$

giving the functional equation

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}} \;\cdot\; \widetilde{\theta}_{\chi^{-1}}\big(\frac{i}{N^2 y}\big) \qquad (\text{odd } \chi \text{ primitive mod } N)$$

[2.3.1] Remark: The functional equation relates $\widetilde{\theta}_\chi$ to $\widetilde{\theta}_{\chi^{-1}}$, rather than to $\widetilde{\theta}_\chi$ itself.

[2.3.2] Remark: The *flip* in the argument y is $y \to 1/N^2 y$, not simply $y \to 1/y$. The fixed-point of this map is $1/N$, influencing the break-up of the integral in proof of the analytic continuation and functional equation, next.

[2.4] Analytic continuation and functional equation

Starting from the integral representation,

$$\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\, L(s,\chi) \;=\; \int_0^\infty y^{\frac{s}{2}}\, \frac{\widetilde{\theta}_\chi(iy)}{2}\, \frac{dy}{y} \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\, \frac{\widetilde{\theta}_\chi(iy)}{2}\, \frac{dy}{y} + \int_0^{1/N} y^{\frac{s}{2}}\, \frac{\widetilde{\theta}_\chi(iy)}{2}\, \frac{dy}{y}$$

As for $\zeta(s)$, the integral from $1/N$ to ∞ is nicely convergent for all $s \in \mathbb{C}$, and gives an entire function. The goal is to use the functional equation to convert the integral from 0 to $1/N$ to the other sort, and then symmetrize the resulting expression as much as possible. That is, replace y by $1/N^2 y$ in the problemmatical integral from 0 to $1/N$, obtaining

$$\tfrac{1}{2}\int_{1/N}^\infty (1/N^2 y)^{\frac{s}{2}}\, \widetilde{\theta}_\chi\Big(\frac{i}{N^2 y}\Big)\, \frac{dy}{y} \;=\; \tfrac{1}{2}\, N^{-s} \int_{1/N}^\infty y^{-\frac{s}{2}}\, \widetilde{\theta}_\chi\Big(\frac{i}{N^2 y}\Big)\, \frac{dy}{y}$$

Replacing χ by χ^{-1} in the functional equation above gives

$$\widetilde{\theta}_{\chi^{-1}}(iy) \;=\; \frac{-i\langle\chi^{-1},\psi\rangle}{N\sqrt{y}} \cdot \widetilde{\theta}_\chi\big(\frac{i}{N^2 y}\big)$$

Rearranging,

$$\widetilde{\theta}_\chi\big(\frac{i}{N^2 y}\big) \;=\; \frac{-iN\sqrt{y}}{\langle\chi^{-1},\psi\rangle}\widetilde{\theta}_{\chi^{-1}}(iy)$$

Substituting this gives

$$\frac{-iN^{1-s}}{\langle\chi^{-1},\psi\rangle} \int_{1/N}^\infty y^{\frac{1-s}{2}}\, \frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\, \frac{dy}{y}$$

That is,

$$\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\, L(s,\chi) \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\, \frac{\widetilde{\theta}_\chi(iy)}{2}\, \frac{dy}{y} \;+\; \frac{-iN^{1-s}}{\langle\chi^{-1},\psi\rangle} \int_{1/N}^\infty y^{\frac{1-s}{2}}\, \frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\, \frac{dy}{y}$$

This gives the *analytic continuation*, but is not symmetrized.

Recall that for χ *primitive*,

$$|\langle\chi,\psi\rangle| \;=\; \sqrt{N}$$

We do not pretend to know the *argument* of the Gauss sum, only its *size*: let

$$\varepsilon(\chi) \;=\; \frac{-i\sqrt{N}}{\langle\chi^{-1},\psi\rangle}$$

Then $|\varepsilon(\chi)|=1$ and

$$\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big(y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)N^{\frac{1}{2}-s}y^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Ignoring $\varepsilon(\chi)$ for a moment, symmetrize the powers of the conductor N, by multiplying through by $N^{\frac{s}{2}}$, giving

$$N^{\frac{s}{2}}\,\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Regarding $\varepsilon(\chi)$: since χ is *odd*,

$$\overline{\langle\chi^{-1},\psi\rangle} \;=\; \langle\chi,\overline{\psi}\rangle \;=\; \sum_b \chi(b)\overline{\psi}(-b) \;=\; \chi(-1)\sum_b \chi(b)\overline{\psi}(b) \;=\; \chi(-1)\langle\chi,\psi\rangle \;=\; -\langle\chi,\psi\rangle \qquad \text{(for odd } \chi)$$

Thus,

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; \frac{-i\sqrt{N}\cdot -i\sqrt{N}}{\langle\chi^{-1},\psi\rangle\cdot\langle\chi,\psi\rangle} \;=\; \frac{-N}{-\overline{\langle\chi,\psi\rangle}\cdot\langle\chi,\psi\rangle} \;=\; 1$$

giving the symmetry

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad \text{(for } \chi \text{ odd)}$$

Thus, we have

$$N^{\frac{s}{2}}\,\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\int_{1/N}^\infty \Big(\varepsilon(\chi^{-1}(Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; (Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{(1-s)+1}{2}}\,\Gamma(\tfrac{(1-s)+1}{2})\,L(1-s,\chi^{-1}) \qquad (\chi \text{ odd})$$

giving the functional equation

$$N^{\frac{s}{2}}\,\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{2-s}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1}) \qquad \text{(odd } \chi \text{ primitive mod } N)$$

where

$$\varepsilon(\chi) \;=\; \frac{-i\sqrt{N}}{\langle\chi^{-1},\psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1 \qquad \text{(for } \chi \text{ odd, primitive modulo } N)$$
