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# *Simplest cases of Deligne's conjectures on special values of $L$-functions*

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As earlier, we have functional equation $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$ for $\zeta(s)$ itself, and

$$N^{\frac{s}{2}}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{1-s}{2}}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad (\textit{even}\ \chi \text{ primitive mod } N)$$

where

$$\varepsilon(\chi) \;=\; \frac{\sqrt{N}}{\langle \chi^{-1},\psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1, \qquad \varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad (\textit{even}\ \chi \text{ primitive mod } N)$$

and

$$N^{\frac{s}{2}}\,\pi^{-\frac{s+1}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{1-s}{2}}\,\pi^{-\frac{2-s}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1}) \qquad (\textit{odd}\ \chi \text{ primitive mod } N)$$

where

$$\varepsilon(\chi) \;=\; \frac{-i\sqrt{N}}{\langle \chi^{-1},\psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1, \qquad \varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad (\textit{odd}\ \chi \text{ primitive modulo } N)$$

At the same time, in all cases the values at *non-positive* integers $s$ are *algebraic*, in fact, lying in a cyclotomic field $\mathbb{Q}(\chi)$, and $L(-n,\chi)^\sigma = L(-n,\chi^\sigma)$ for Galois automorphisms $\sigma$. The two results can easily be combined to *try* to obtain information about values at *positive* integers $n>1$, via

$$\begin{cases} \zeta(n) &=& \dfrac{\pi^{-\frac{1-n}{2}}\Gamma(\frac{1-n}{2})}{\pi^{-\frac{n}{2}}\Gamma(\frac{n}{2})}\zeta(1-n) \\ L(n,\chi) &=& \varepsilon(\chi)\cdot\dfrac{N^{\frac{1-n}{2}}\,\pi^{-\frac{1-n}{2}}\,\Gamma(\frac{1-n}{2})}{N^{\frac{n}{2}}\,\pi^{-\frac{n}{2}}\,\Gamma(\frac{n}{2})}\,L(1-n,\chi^{-1}) \quad (\textit{even}\ \chi \text{ primitive mod } N) \\ L(n,\chi) &=& \varepsilon(\chi)\cdot\dfrac{N^{\frac{1-n}{2}}\,\pi^{-\frac{2-n}{2}}\,\Gamma(\frac{2-n}{2})}{N^{\frac{n}{2}}\,\pi^{-\frac{n+1}{2}}\,\Gamma(\frac{n+1}{2})}\,L(1-n,\chi^{-1}) \quad (\textit{odd}\ \chi \text{ primitive mod } N) \end{cases}$$

We know that $\Gamma(s)$ is rational at $0<s\in\mathbb{Z}$ and is a rational multiple of $\sqrt{\pi}$ at half-integers, these relations would nicely specify $\zeta(n)$ and $L(n,\chi)$ *except* when the Gamma function has a *pole* at the value appearing in the numerator. That is, avoiding poles of $\Gamma$,

$$\zeta(2n) \;=\; \frac{\pi^{-\frac{1-2n}{2}}\Gamma(\frac{1}{2}-n)}{\pi^{-\frac{2n}{2}}\Gamma(n)}\zeta(1-2n) \;=\; \pi^{2n-\frac{1}{2}}\cdot\Gamma(\tfrac{1}{2})\cdot(\text{rational}) \;=\; \pi^{2n}\cdot(\text{rational})$$

Similarly,

$$L(2n,\chi) \;=\; \varepsilon(\chi)\cdot\sqrt{N}\cdot\pi^{2n}\cdot L(1-2n,\chi^{-1})\cdot(\text{rational}) \qquad (\textit{even}\ \chi \text{ primitive mod } N)$$

and we know how the algebraic numbers $\varepsilon(\chi)$ and $L(1-2n,\chi^{-1})$ transform under Galois. For odd characters,

$$L(2n+1,\chi) \;=\; \varepsilon(\chi)\cdot\sqrt{N}\cdot\pi^{2n+1}\cdot\, L(-2n,\chi^{-1})\cdot(\text{rational}) \qquad (\textit{odd}\ \chi \text{ primitive mod } N)$$

and we know how the algebraic numbers $\varepsilon(\chi)$ and $L(-2n,\chi^{-1})$ transform under Galois. This has been understood since the middle of the 19th century, by these and several other methods.

*However*, knowing $\zeta(1-n)$ and $L(1-n,\chi^{-1})$ will *not* determine the values of $\zeta(n)$ and $L(n,\chi)$ in the cases where $\Gamma$ has a pole. Rather, the values $\zeta'(1-n)$ and $L'(1-n,\chi^{-1})$ of the *derivatives* (and the corresponding residues of $\Gamma$) determine $\zeta(n)$ and $L(n,\chi)$. However, we have no comparable information about these values!

That is, we understand $\zeta(2n)$ and $L(2n,\chi)$ for positive even $2n$ for *even* $\chi$, and $L(2n+1,\chi)$ for *odd* $\chi$, but *not* the other half of the values.

[Deligne 1977/79] formulated a conjecture based on these examples and a handful of others available at that time. Roughly, he defined a *critical value* of an $L$-function to be an integer $n$ so that the Gamma-factor appropriate to the $L$-function has no pole at $n$, nor at $1-n$, when the functional equation is normalized. He conjectured that special values of $L$-functions at their critical values would be expressible in terms of algebraic numbers and certain canonical (presumed) transcendentals akin to $\pi$. Later we will see more examples consistent with Deligne's conjecture.

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Very little progress has been made in understanding things like $\zeta(3)$. [Apéry 1979] was the first progress, proving that $\zeta(3)$ is irrational, though not proving anything about $\zeta(3)/\pi^3$ at all! Some further simplifications and progress in this direction was made in [Rivoal 2000] and [Zudilin 2001], [Zudilin 2002].

From yet another side, it has been known for probably 100 years, to Minkowski, Siegel, and others, that natural normalizations of the total volume of *quotients* such as $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ involve values of zeta functions, including the problemmatical ones. For example, up to reasonable normalizations, the volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ is $\zeta(2)\cdot\zeta(3)\cdot\zeta(4)\cdot\zeta(5)\ldots\zeta(n)$. T. Tamagawa observed that the *adelic* form of groups such as $SL_n$ have *canonical* measures, so adelic versions $SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A})$ of quotients have canonically defined volumes. [Weil 1961] elaborates this viewpoint.

This does not express $\zeta(3)$ in simpler terms, but it does connect it to geometry, as in [Borel 1977], [Bloch 1984], [Beilinson 1985] and subsequent papers by these authors and collaborators.

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