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Fourier series, Weyl equidistribution

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- Dirichlet's pigeon-hole principle, approximation theorem
- Kronecker's approximation theorem
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1. Dirichlet's pigeon-hole principle, approximation theorem

The *pigeon-hole* principle[1] formulated by Dirichlet by 1834, observes that when $N+1$ things are partitioned into N disjoint subsets, there is at least one subset containing at least 2 things. The archetypical application is the following:

[1.0.1] Theorem: *(Dirichlet)* For every real α and every integer $N \ge 1$, there are integers p, q with $1 \le q \le N$ such that $|q\alpha - p| \le \frac{1}{N}$.

Proof: For each m in the range $1 \le m \le N+1$, choose $n = n_m$ so that $m\alpha - n_m \in [0,1)$ is the *fractional part*[2] of $m\alpha$. The $N+1$ choices of m produce $N+1$ numbers $m\alpha - n$ in $[0,1)$. Dividing the interval into N subintervals of length $\frac{1}{N}$, by the pigeon-hole principle some subinterval contains both $m\alpha - n$ and $m'\alpha - n'$ for some $1 \le m' < m \le N+1$. That is,

$$\frac{1}{N} \;\ge\; |(m\alpha - n) - (m'\alpha - n')| \;=\; |(m-m')\alpha - (n-n')|$$

so $1 \le q = m - m' \le N$ and $p = n - n'$ meet the requirement of the theorem. ///

[1.0.2] Remark: It is trivial to see that, given a real number r and given a positive integer n, there is integer m such that $|r - \frac{m}{n}| < \frac{1}{n}$. The theorem comes close assuring a smaller error of $\frac{1}{n^2}$, but then we no longer have freedom to choose the denominator.

2. Kronecker's approximation theorem

The following is an example of Kronecker's 1884 generalization of Dirichlet's approximation results, in which Kronecker illustrates a different causal mechanism:

[2.0.1] Theorem: The collection of integer multiples $n\alpha$ of *irrational* real α is *dense* in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Proof: The assertion is equivalent to $\mathbb{Z}\alpha + \mathbb{Z}$ being dense in $\mathbb{R}$. The topological closure Γ of $\mathbb{Z}\alpha + \mathbb{Z}$ in $\mathbb{R}$ is still a subgroup of $\mathbb{R}$. The topologically-closed subgroups of $\mathbb{R}$ are *classifiable*: they are exactly $\{0\}$, free $\mathbb{Z}$-modules $\mathbb{Z}\beta$ on a single generator $\beta \ne 0$, and the whole $\mathbb{R}$. Granting this classification for a moment, if Γ is *not* the whole, then $\Gamma = \mathbb{Z} \cdot \beta$ for some β, and certainly $\mathbb{Z}\alpha + \mathbb{Z} \subset \mathbb{Z}\beta$. Thus, $\mathbb{Z} \subset \mathbb{Z}\beta$, so $1 = n\beta$ for some integer n, and $\beta = \frac{1}{n}$. Similarly, $1 \cdot \alpha = m\beta = \frac{m}{n}$ for some n, so α is rational.

[1] Dirichlet used *Schubfachprinzip*, which is *drawer principle*.

[2] The *fractional part* of a real number α, sometimes denoted $\{\alpha\}$ or $\langle\alpha\rangle$ or α mod 1, in an elementary context is $\langle\alpha\rangle = \alpha - \lfloor\alpha\rfloor$, where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal α. More to the point, the fractional part is really the *image* of α in the quotient $\mathbb{R}/\mathbb{Z}$.

Now we classify topologically-closed subgroups $\Gamma \neq \{0\}$ of $\mathbb{R}$. Since $\Gamma \neq \{0\}$ and is closed under additive inverses, Γ contains *positive* elements.

In the case that there is a *least* positive element γ_o, we claim that $\Gamma = \mathbb{Z}\cdot\gamma_o$. Indeed, given another $0 < \gamma \in \Gamma$, by the archimedean property of the real numbers there is an integer n such that $n\gamma_o \leq \gamma < (n+1)\gamma_o$. In fact, $n\gamma_o = \gamma$, or else $0 < \gamma - n\gamma_o < \gamma_o$, contradicting the minimality of γ_o.

In the case that there is *no* least positive $\gamma_o \in \Gamma$, let $\gamma_1 > \gamma_2 > \ldots > 0$ be an infinite sequence of positive elements of Γ. The infimum γ_o of this sequence is in Γ, since Γ is closed. Replacing γ_j by $\gamma_j - \gamma_o$, we can assume that $\gamma_j \to 0$. Given real β, there is integer n such that $n\gamma_j \leq \beta < (n+1)\gamma_j$ by archimedean-ness, so $\mathbb{Z}\gamma_j$ contains elements within $|\gamma_j|$ of any real number. Since $\gamma_j \to 0$, given $\varepsilon > 0$ there is $|\gamma_j| < \varepsilon$, so every real number is within $\varepsilon > 0$ of $\mathbb{Z}\gamma_j$. Thus, the topologically closed subgroup Γ must be $\mathbb{R}$. ///

A similar argument would prove Kronecker's multi-dimensional version:

[2.0.2] Theorem: *(Kronecker)* An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of real numbers is linearly independent over $\mathbb{Q}$ if and only if the topological closure of the collection $\{N \cdot \alpha : N = 1, 2, 3, \ldots\}$ of multiples of α is *dense* in the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. ///

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3. *Weyl equidistribution criterion*

The idea of a sequence of real numbers $\alpha_1, \alpha_2, \ldots$ being *equidistributed* modulo $\mathbb{Z}$, that is, in $\mathbb{R}/\mathbb{Z}$, is a *quantitative* strengthening of a merely *qualitative* density assertion. More generally, the notion of equidistribution of a sequence of finite sets $S_1, S_2, \ldots$ can be quantified, specializing to the case of a sequence $\alpha_1, \alpha_2, \ldots$ of points by taking $S_n = \{\alpha_1, \ldots, \alpha_n\}$.

[3.0.1] Remark: In fact, to allow *repeats* of points, we are really looking at sequences of *multi-sets*, meaning that each element has a *multiplicity* that can be greater than 1, as opposed to genuine *sets*, which collapse multiplicities. For example, $\{a, b\} = \{a, b, b\} = \{a, b, b, b\} = \ldots$.

One possible equidistribution requirement is that the number of α_n mod $\mathbb{Z}$ with $1 \leq n \leq N$ in any subinterval $[a, b]$ of $[0, 1]$ is asymptotically $N \cdot |b - a|$ as $N \to \infty$. It is simpler to weaken the requirement a little, in effect replacing the collection of characteristic functions of subintervals $[a, b]$ with arbitrary continuous functions on $\mathbb{R}/\mathbb{Z}$, and, further to consider the collection of all *smooth* functions on $\mathbb{R}/\mathbb{Z}$.

Say that a sequence of points $\alpha_1, \alpha_2, \ldots$ is *equidistributed mod* $\mathbb{Z}$ when

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} f(\alpha_n) \;=\; \int_0^1 f(x)\, dx \qquad \text{(for every } f \in C_c^\infty(\mathbb{R}/\mathbb{Z})\text{)}$$

Say that a sequence of (multi-) sets $S_1, S_2, \ldots$ is *equidistributed mod* $\mathbb{Z}$ when

$$\lim_{N \to +\infty} \frac{1}{\#S_N} \sum_{\alpha \in S_N} f(\alpha) \;=\; \int_0^1 f(x)\, dx \qquad \text{(for every } f \in C_c^\infty(\mathbb{R}/\mathbb{Z})\text{)}$$

We want to allow possible repetition in the elements in the S_N, so these are not properly *set*, since sets do not allow repetition: $\{a, b\} = \{a, b, b\} = \{a, b, b, b, \} = \ldots$. Thus, these S_N's are *multi-sets*, or *lists*. Still, we will use set-element notation, and write $\alpha \in S_N$ to refer to all elements of the list S_N.

Let $\psi_n(x) = e^{2\pi i n x}$.

[3.0.2] Theorem: *(Weyl)* A sequence of (multi-) sets $\{S_N\}$ is equidistributed mod $\mathbb{Z}$ if

$$\lim_N \frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha) \;=\; 0 \qquad \text{(for every } n \neq 0\text{)}$$

In particular, for irrational real α, the sequence $\alpha, 2\alpha, 3\alpha, \ldots$ is equidistributed modulo $\mathbb{Z}$.

[3.0.3] Remark: The other direction of implication in the theorem is trivial, since the exponentials are smooth functions. That is, Weyl's theorem says that it suffices to check equidistribution relative to a restricted list of smooth functions, the exponentials ψ_n.

[3.0.4] Remark: The sums

$$\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha) \;=\; \frac{1}{\#S_N} \sum_{\alpha \in S_N} e^{2\pi i n\alpha}$$

are reasonably construed as the *Fourier coefficients* of the *measure* $f \to \sum_{\alpha \in S_N} f(\alpha)$.

Proof: *Smooth* $\mathbb{Z}$-periodic f has Fourier expansion $\sum_n \widehat{f}(n)\, \psi_n$ converging absolutely and uniformly to f.

$$\frac{1}{\#S_N} \sum_{\alpha \in S_N} f(\alpha) \;=\; \frac{1}{\#S_N} \sum_{\alpha \in S_N} \Big(\sum_n \widehat{f}(n)\psi_n(\alpha)\Big) \;=\; \frac{1}{\#S_N} \sum_n \widehat{f}(n)\Big(\sum_{\alpha \in S_N} \psi_n(\alpha)\Big)$$

$$= \;\widehat{f}(0) + \sum_{n \neq 0} \widehat{f}(n) \cdot \Big(\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha)\Big)$$

For any cut-off b for the Fourier series, noting $\widehat{f}(0) = \int_0^1 f(x)\, dx$,

$$\frac{1}{\#S_N} \sum_{\alpha \in S_N}^{N} f(\alpha) - \int_0^1 f(x)\, dx \;\le\; \Big|\sum_{n \neq 0} \widehat{f}(n) \cdot \Big(\frac{1}{\#S_N} \cdot \sum_{\alpha \in S_N} \psi_n(\alpha)\Big)\Big| \;\le\; \sum_{n \neq 0} |\widehat{f}(n)| \cdot \Big|\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha)\Big|$$

$$\le\; \sum_{0 < |n| \le b} |\widehat{f}(n)| \cdot \Big|\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha)\Big| \;+\; \sum_{|n| > b} |\widehat{f}(n)| \cdot 1$$

Since the Fourier series of f converges absolutely, given $\varepsilon > 0$ there is large-enough b so that $\sum_{|n|>b} |\widehat{f}(n)| < \varepsilon$. With that b, since $\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha) \to 0$ for each fixed $n \neq 0$, and since there are only finitely-many n with $0 < |n| \le b$, for large-enough N

$$\sum_{0 < |n| \le b} |\widehat{f}(n)| \cdot \Big|\frac{1}{\#S_N} \sum_{\alpha \in S_N} \psi_n(\alpha)\Big| \;<\; \varepsilon$$

Thus,

$$\Big|\frac{1}{\#S_N} \sum_{\alpha \in S_N} f(\alpha) \;-\; \widehat{f}(0)\Big| \;<\; 2\varepsilon$$

That is, $\frac{1}{\#S_N} \sum_{\alpha \in S_N} f(\alpha) \longrightarrow \int_0^1 f(x)\, dx$, and Weyl's criterion suffices for equidistribution.

In the example of integer multiples of an irrational number, by summing a geometric series, with $n \neq 0$,

$$\frac{1}{N} \sum_{\ell=1}^{N} e^{2\pi i n \cdot \ell\alpha} \;=\; \frac{1}{N} \frac{e^{2\pi i n\alpha} - e^{2\pi i n(N+1)\alpha}}{1 - e^{2\pi i n\alpha}}$$

The irrationality of α and $n \neq 0$ assure that the denominator does not vanish. Thus,

$$\frac{1}{N} \sum_{\ell=1}^{N} e^{2\pi i n \cdot \ell\alpha} \;\le\; \frac{1}{N} \cdot \frac{2}{|1 - e^{2\pi i n\alpha}|} \;\to\; 0 \qquad \text{(for each fixed } n \neq 0)$$

proving equidistribution of $\{\ell\alpha\}$. ///

[3.0.5] Remark: The proof only used absolute convergence pointwise of the Fourier series of f to f. Infinite-differentiability of f assures this, but much less is needed. On the other hand, if we allow merely-continuous functions f, then we must invoke Weierstrass approximation, since the Fourier series of typical continuous functions do not converge to them pointwise.

[3.1] Higher-dimension analogues

Fourier series of functions f on $\mathbb{R}^n/\mathbb{Z}^n$ can be written

$$\sum_{\xi\in\mathbb{Z}^n} \widehat{f}(\xi)\, e^{2\pi i\xi\cdot x}$$

where $\xi\cdot x = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n$, and

$$\widehat{f}(\xi) \;=\; \int_0^1 \ldots \int_0^1 e^{-2\pi i \xi\cdot x}\, f(x)\, dx$$

The notion of *equidistribution* modulo $\mathbb{Z}^n$ of a sequence $\{\alpha_\ell\}$ in $\mathbb{R}^n$ is the same:

$$\lim_N \frac{1}{N} \sum_{1\le\ell\le N} f(\alpha_\ell) \;=\; \int_0^1 \ldots \int_0^1 f(x)\, dx \qquad \text{(for all } \mathbb{Z}^n\text{-periodic smooth } f \text{ on } \mathbb{R}^n)$$

Assuming we know that Fourier series of a nice $\mathbb{Z}^n$-periodic function f on $\mathbb{R}^n$ converges absolutely to f pointwise, the same argument proves

[3.1.1] Theorem: *(Weyl)* A sequence $\{\alpha_\ell\}$ in $\mathbb{R}^n$ is equidistributed modulo $\mathbb{Z}^n$ if and only if

$$\lim_N \frac{1}{N} \sum_{1\le\ell\le N} e^{2\pi i \xi\cdot\alpha_\ell} \;=\; 0 \qquad \text{(for all } 0\neq\xi\in\mathbb{Z}^n)$$

For example, for real numbers $\beta_1, \ldots, \beta_n$, the sequence $\alpha_\ell = \ell\cdot(\beta_1,\ldots,\beta_n)$ is equidistributed modulo $\mathbb{Z}^n$ if and only if $1, \beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$. ///

[3.1.2] Remark: Weyl's criterion for equidistribution can be applied to *compact topological groups* K of various sorts, in place of $\mathbb{R}^n/\mathbb{Z}^n$, especially compact *Lie groups*, because of the spectral decomposition of $L^2(K)$ analogous to Fourier series on $\mathbb{R}^n/\mathbb{Z}^n$. In general, Stone-Weierstrass approximation must be sued.

[3.1.3] Remark: Weyl treated a more serious problem, that of equidistribution modulo $\mathbb{Z}$ of sequences $\alpha_\ell = P(\ell)$ with polynomial P.

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