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Reproducing kernels, Bergman kernels, Poisson kernels

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The identity map 1_V on a finite-dimensional inner-product vector space V is expressible in terms of any orthonormal basis $\{e_i\}$ as

$$v = 1_V(v) = \sum_i \langle v, e_i \rangle \cdot e_i \quad (\text{for } v \in V)$$

That is, the matrix of the identity map with respect to any basis $\{e_i\}$ is simply the identity matrix:

$$1_V = \sum_i \langle -, e_i \rangle \otimes e_i \in V^* \otimes V$$

One infinite-dimensional analogue involves a Hilbert space V of functions on a measure space Z . The identity *matrix* is replaced by the *reproducing kernel* $K(z, w)$: this should be a function of some kind on $Z \times Z$ giving the identity map on V :

$$f(z) = \int_Z K(z, w) \cdot f(w) dw \quad (\text{for } f \in V)$$

With orthonormal basis $\{f_n\}$ for V , apparently

$$K(z, w) = \sum_n f_n(z) \cdot \overline{f_n(w)}$$

It is completely unclear in what sense this converges, if any. For example, square integrable functions $L^2(\mathbb{T})$ on the circle \mathbb{T} have orthonormal basis $e^{i\theta} \rightarrow e^{in\theta}/\sqrt{2\pi}$, but the corresponding expression

$$K(\theta, \theta') = \sum_i e^{in\theta} \cdot e^{-in\theta'}$$

does not converge *pointwise* for any $\theta, \theta' \in \mathbb{R}$. It does converge *distributionally*, in fact in a Sobolev space, but that is another story, about *Schwartz kernel functions* for a great variety of operators. Here, we are interested in situations where a reproducing kernel does converge pointwise in a more elementary sense.

1. Bergman kernel on the unit disk

Let D be the unit disk in \mathbb{C} , and V the closure in $L^2(D)$ of the vector space of polynomial functions. This is a Hilbert subspace of $L^2(D)$. The inner product of two holomorphic monomials z^m and z^n is

$$\begin{aligned} \langle z^m, z^n \rangle &= \int_D z^m \cdot \overline{z^n} dx dy = \int_0^1 \int_0^{2\pi} (re^{i\theta})^m \cdot (re^{-i\theta})^n d\theta r dr = \int_0^1 \int_0^{2\pi} r^{m+n+1} \cdot e^{i\theta(m-n)} d\theta dr \\ &= \begin{cases} 0 & (\text{for } m \neq n) \\ \frac{\pi}{n+1} & (\text{for } m = n) \end{cases} \end{aligned}$$

Thus, the reproducing kernel here, the *Bergman kernel*, should be

$$K(z, w) = \frac{1}{\pi} \sum_{n \geq 0} (n+1) \cdot z^n \cdot \bar{w}^n \quad (\text{for } |z| < 1 \text{ and } |w| < 1)$$

We can sum the series:

$$\sum_{n \geq 0} (n+1) \cdot z^n \cdot \bar{w}^n = \frac{1}{\bar{w}} \frac{\partial}{\partial z} \left(\sum_{n \geq 0} (z\bar{w})^n \right) = \frac{1}{\bar{w}} \frac{\partial}{\partial z} \frac{1}{1 - z\bar{w}} = \frac{1}{\bar{w}} \frac{\bar{w}}{(1 - z\bar{w})^2} = \frac{1}{(1 - z\bar{w})^2}$$

and the convergence is uniform on compacts. Thus, the Bergman kernel for the disk is

$$K(z, w) = \frac{1}{\pi} \frac{1}{(1 - z\bar{w})^2} \quad (\text{for } |z| < 1 \text{ and } |w| < 1)$$

2. Poisson kernel on the unit disk

With the usual Euclidean Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, a function f on the disk D is *harmonic* when $\Delta f = 0$. Granting standard properties of Gelfand-Pettis integrals, the integral

$$f_n(z) = \int_0^{2\pi} e^{-in\theta} f(e^{i\theta} z) d\theta$$

is still harmonic, because the differential operator passes inside the integral in the obviously plausible fashion:

$$\Delta f_n(z) = \Delta \int_0^{2\pi} e^{-in\theta} f(e^{i\theta} z) d\theta = \int_0^{2\pi} e^{-in\theta} \Delta_z f(e^{i\theta} z) d\theta = \int_0^{2\pi} e^{-in\theta} \cdot 0 d\theta = 0$$

By design, this integral picks out the part of f equivariant by $e^{i\theta} \rightarrow e^{in\theta}$: by a change of variables, replacing θ' by $\theta' - \theta$,

$$f_n(e^{i\theta} z) = \int_0^{2\pi} e^{-in\theta'} f(e^{i\theta'} \cdot e^{i\theta} z) d\theta' = e^{in\theta} \cdot \int_0^{2\pi} e^{-in\theta'} f(e^{i\theta'} z) d\theta' = e^{in\theta} \cdot f_n(z)$$

Thus, $f_n(e^{i\theta} r) = e^{in\theta} \varphi_n(r)$ for a function φ_n of radius alone. Writing $f(e^{i\theta} r) = F(r, \theta)$, the Laplacian in radial coordinates is $F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta}$. The differential equation satisfied by φ_n is obtained by separation of variables:

$$0 = \Delta(e^{in\theta} \varphi_n(r)) = \left(\varphi_n'' + \frac{1}{r} \varphi_n' + \frac{1}{r^2} (in)^2 \varphi_n \right) \cdot e^{in\theta}$$

so

$$\varphi_n'' + \frac{1}{r} \varphi_n' - \frac{n^2}{r^2} \varphi_n = 0$$

This is an Euler-type equation, meaning of the form $u'' + \frac{b}{r} u' + \frac{c}{r^2} u = 0$. Solutions are of the form $u(r) = r^\lambda$ for λ a solution of the indicial equation $\lambda(\lambda - 1) + b\lambda + c = 0$, and also $r^\lambda \log r$ for a double root λ . For $n \neq 0$, the solutions are $r \rightarrow r^{\pm|n|}$, but $r^{-|n|}$ blows up at $r \rightarrow 0^+$, so this solution cannot appear in a component $f_n(re^{i\theta}) = e^{in\theta} \varphi_n(r)$ of f continuous on the disk. Likewise, the second solution $\log r$ for $n = 0$ cannot occur. Thus,

$$f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta} = \sum_{n \geq 0} c_n z^n + \sum_{n < 0} c_n \bar{z}^{|n|}$$

The boundary value of $\sum_{n \geq 0} c_n z^n + \sum_{n < 0} c_n \bar{z}^{|n|}$ appears to be the Fourier series $\sum_n c_n e^{in\theta}$. The not-classically-convergent reproducing kernel on the circle

$$K_1(e^{i\theta}, e^{i\theta'}) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\theta} e^{-in\theta'} \quad (\text{convergent in a Sobolev space, not pointwise})$$

becomes classically convergent upon replacing $e^{in\theta}$ by $r^{|n|}e^{in\theta}$ with $0 \leq r < 1$, and we obtain the Poisson kernel:

$$\begin{aligned} K(re^{i\theta}, e^{i\theta'}) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-in\theta'} = \frac{1}{2\pi} \left(\frac{1}{1 - re^{i\theta} \cdot e^{-i\theta'}} + \frac{re^{-i\theta} \cdot e^{i\theta'}}{1 - re^{-i\theta} \cdot e^{i\theta'}} \right) \\ &= \frac{1}{2\pi} \frac{(1 - re^{-i\theta} \cdot e^{i\theta'}) + re^{-i\theta} \cdot e^{i\theta'}(1 - re^{i\theta} \cdot e^{-i\theta'})}{(1 - re^{i\theta} \cdot e^{-i\theta'})(1 - re^{-i\theta} \cdot e^{i\theta'})} = \frac{1}{2\pi} \frac{1 - r^2}{(1 - re^{i\theta} \cdot e^{-i\theta'})(1 - re^{-i\theta} \cdot e^{i\theta'})} \\ &= \frac{1}{2\pi} \frac{1 - |z|^2}{(1 - z \cdot e^{-i\theta'})(1 - \bar{z} \cdot e^{i\theta'})} = \frac{1}{2\pi} \frac{1 - |z|^2}{|1 - z \cdot e^{-i\theta'}|^2} \end{aligned}$$

That is, certainly for smooth functions f on the circle, and even for *distributions* f on the circle,

$$z \longrightarrow \int_0^{2\pi} K(z, e^{i\theta'}) f(e^{i\theta'}) d\theta'$$

is a harmonic function on the open disk with boundary values (in a range of possible senses) f .

3. Bergman kernel on the unit ball in \mathbb{C}^n

Let B be the unit ball in \mathbb{C}^n . Because B is stable under coordinate-wise rotations

$$(z_1, \dots, z_n) \longrightarrow (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$$

the hermitian inner products of monomials (in multi-index notation)

$$\int_B z^\alpha \cdot \bar{z}^\beta = \int_B (z_1^{\alpha_1} \dots z_n^{\alpha_n}) \cdot \overline{(z_1^{\beta_1} \dots z_n^{\beta_n})}$$

must vanish unless $\alpha = \beta$: changing variables,

$$\int_B z^\alpha \cdot \bar{z}^\beta = e^{i(\alpha_1 - \beta_1)\theta_1} \dots e^{i(\alpha_n - \beta_n)\theta_n} \cdot \int_B z^\alpha \cdot \bar{z}^\beta$$

That is, unless $\alpha_j = \beta_j$ for every j , some choice of angles θ_j will make the factor on the right-hand side 0, so the integral must be 0. With $\alpha = \beta$,

$$\begin{aligned} \int_B z^\alpha \cdot \bar{z}^\alpha &= \int_B |z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n} = \int_0^1 r^{2\alpha_1 + \dots + 2\alpha_n + (2n-1)} dr \int_{S^{2n-1}} \left| \frac{z_1}{r} \right|^{2\alpha_1} \dots \left| \frac{z_n}{r} \right|^{2\alpha_n} \\ &= \frac{1}{2\alpha_1 + \dots + 2\alpha_n + 2n} \int_{S^{2n-1}} \left| \frac{z_1}{r} \right|^{2\alpha_1} \dots \left| \frac{z_n}{r} \right|^{2\alpha_n} \end{aligned}$$

To integrate a homogeneous degree-zero function f over a sphere, integrate $f(x/|x|) \cdot \varphi(|x|)$ over \mathbb{C}^n for any non-negative φ on $(0, \infty)$, and divide by the integral of $\varphi(|x|)$ over \mathbb{C}^n . Uniqueness of the invariant integral up to scalars assures that this is independent of the auxiliary function φ . Given α , take $\varphi(r) = r^{2\alpha_1 + \dots + 2\alpha_n} e^{-\pi r^2}$. Letting $|z|^2 = |z_1|^2 + \dots + |z_n|^2$, the normalizing constant is

$$\begin{aligned} \int_{\mathbb{C}^n} |z|^{2\alpha_1 + \dots + 2\alpha_n} e^{-\pi|z|^2} &= |S^{2n-1}| \int_0^\infty r^{2\alpha_1 + \dots + 2\alpha_n + 2n} e^{-\pi r^2} \frac{dr}{r} \\ &= \frac{1}{2} |S^{2n-1}| \int_0^\infty r^{\alpha_1 + \dots + \alpha_n + n} e^{-\pi r} \frac{dr}{r} = \pi^{-(\alpha_1 + \dots + \alpha_n + n)} \cdot \frac{1}{2} |S^{2n-1}| \cdot \Gamma(\alpha_1 + \dots + \alpha_n + n) \end{aligned}$$

and

$$\int_{\mathbb{C}^n} \left| \frac{z_1}{r} \right|^{2\alpha_1} \dots \left| \frac{z_n}{r} \right|^{2\alpha_n} \cdot r^{2\alpha_1 + \dots + 2\alpha_n} e^{-\pi r^2} = \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n} e^{-\pi r^2} = \prod_j \int_{\mathbb{C}} |z_j|^{2\alpha_j} e^{-\pi |z_j|^2}$$

The j^{th} integral over \mathbb{C} is

$$\int_{\mathbb{C}} |z_j|^{2m} e^{-\pi |z_j|^2} = 2\pi \int_0^\infty r^{2\alpha_j+2} e^{-\pi r^2} \frac{dr}{r} = \pi \int_0^\infty r^{\alpha_j+1} e^{-\pi r} \frac{dr}{r} = \pi^{\alpha_j} \cdot \Gamma(\alpha_j + 1)$$

Thus,

$$\begin{aligned} \int_{S^{2n-1}} \left| \frac{z_1}{r} \right|^{2\alpha_1} \dots \left| \frac{z_n}{r} \right|^{2\alpha_n} &= \frac{\pi^{\alpha_1} \Gamma(\alpha_1 + 1) \dots \pi^{\alpha_n} \Gamma(\alpha_n + 1)}{\pi^{-(\alpha_1 + \dots + \alpha_n + n)} \cdot \frac{1}{2} |S^{2n-1}| \cdot \Gamma(\alpha_1 + \dots + \alpha_n + n)} \\ &= \frac{\pi^n \Gamma(\alpha_1 + 1) \dots \Gamma(\alpha_n + 1)}{\frac{1}{2} |S^{2n-1}| \cdot \Gamma(\alpha_1 + \dots + \alpha_n + n)} \end{aligned}$$

and

$$\int_B z^\alpha \cdot \bar{z}^\alpha = \frac{1}{2\alpha_1 + \dots + 2\alpha_n + 2n} \int_{S^{2n-1}} \left| \frac{z_1}{r} \right|^{2\alpha_1} \dots \left| \frac{z_n}{r} \right|^{2\alpha_n} = \frac{\pi^n \Gamma(\alpha_1 + 1) \dots \Gamma(\alpha_n + 1)}{|S^{2n-1}| \cdot \Gamma(\alpha_1 + \dots + \alpha_n + n + 1)}$$

The ratio of Gamma values is almost the inverse of a multinomial coefficient: letting $|\alpha| = \alpha_1 + \dots + \alpha_n$,

$$\frac{\Gamma(|\alpha| + n + 1)}{\Gamma(\alpha_1 + \dots + \alpha_n + n + 1)} = \frac{(|\alpha| + n)!}{\alpha_1! \dots \alpha_n!} = \binom{|\alpha|}{\alpha_1 \dots \alpha_n} \cdot (|\alpha| + 1) \dots (|\alpha| + n)$$

and

$$(|\alpha| + 1) \dots (|\alpha| + n) = \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} t^{|\alpha|+n}$$

Thus, up to a constant, the reproducing kernel $K(z, w)$ is

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} t^n \sum_{\alpha} \binom{\alpha_1 + \dots + \alpha_n + n}{\alpha_1 \dots \alpha_n} z^\alpha \bar{w}^\alpha t^{|\alpha|} &= \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} t^n \sum_{N \geq 0} t^N \sum_{|\alpha|=N} \binom{N}{\alpha_1 \dots \alpha_n} z^\alpha \bar{w}^\alpha \\ &= \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} t^n \sum_{N \geq 0} t^N (z_1 \bar{w}_1 + \dots + z_n \bar{w}_n)^N = (z \cdot \bar{w})^{-n} \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} \sum_{\ell \geq 0} (t z \cdot \bar{w})^\ell \\ &= (z \cdot \bar{w})^{-n} \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=1} \frac{1}{1 - t z \cdot \bar{w}} = (z \cdot \bar{w})^{-n} \cdot \frac{(n-1)! \cdot (z \cdot \bar{w})^n}{(1 - z \cdot \bar{w})^{n+1}} = \frac{(n-1)!}{(1 - z \cdot \bar{w})^{n+1}} \end{aligned}$$

Rather than organizing all the constants above, granting that $K(z, w) = c/(1 - z \cdot \bar{w})^{n+1}$ for some constant c , the constant can be determined by integrating against the function 1 and evaluating at 0:

$$1 = \int_B K(0, w) \cdot 1 = \int_B c \cdot 1 = \text{vol}(B)$$

so $c = 1/\text{vol}(B)$, and the Bergman kernel for the complex n -ball B is

$$K(z, w) = \frac{1}{\text{vol}(B)} \cdot \frac{1}{(1 - z \cdot \bar{w})^{n+1}}$$

4. Reproducing kernel for holomorphic cuspforms

Recall that a holomorphic *cuspform* f on the upper half-plane \mathfrak{H} for $\Gamma = SL_2(\mathbb{Z})$ of weight $2k$ is a holomorphic function with the *automorphy* property

$$f(\gamma z) = (cz + d)^{2k} \cdot f(z) \quad (\text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma)$$

and a decay constraint at $i\infty$ describable by requiring that its Fourier expansion has no non-positive-index terms:

$$f(x + iy) = \sum_{n \geq 1} c_n e^{2\pi i n(x + iy)}$$

(Hecke showed that it follows that the coefficients are of *polynomial growth*). The Petersson inner product on holomorphic cuspforms of weight $2k$ for Γ is

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathfrak{H}} f(z) \cdot \overline{g(z)} y^{2k} \frac{dx dy}{y^2}$$

It is completely unreasonable to expect to explicitly identify an orthonormal basis for all these spaces of cuspforms. Instead, we anticipate that the reproducing kernel is obtained by *winding up* a suitable function $\varphi(z, w)$ on $\mathfrak{H} \times \mathfrak{H}$, holomorphic in z and anti-holomorphic in w :

$$K(z, w) = \sum_{\gamma} \varphi(\gamma z, w) \cdot \frac{1}{(cz + d)^{2k}} \quad (\text{summed over } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma)$$

We want $\varphi(z, w)$ so that the winding-up creates $K(z, w)$ a holomorphic cuspform in z and anti-holomorphic cuspform in w . Thus, for $\gamma \in \Gamma$, we would want

$$K(z, \gamma w) = (c\bar{w} + d)^{2k} \cdot K(z, w) \quad (\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma)$$

After some experimentation, as Petersson already knew, a reasonable choice is

$$\varphi(z, w) = \frac{1}{(z - \bar{w})^{2k}}$$

and then

$$\begin{aligned} K(z, w) &= \sum_{\gamma} \frac{1}{(\gamma z - \bar{w})^{2k}} \frac{1}{(cz + d)^{2k}} = \sum_{\gamma} \frac{1}{\left(\frac{az+b}{cz+d} - \bar{w}\right)^{2k}} \frac{1}{(cz + d)^{2k}} \\ &= \sum_{\gamma} \left(\frac{1}{az + b - cz\bar{w} - d\bar{w}} \right)^{2k} = \sum_{\gamma} \left(\frac{1}{(1 - \bar{w}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}} \right)^{2k} \end{aligned}$$

demonstrating the symmetry of $K(z, w)$ in z and \bar{w} . For a holomorphic cuspform f , with $w = u + iv$, *unwinding* gives

$$\begin{aligned} \int_{\Gamma \backslash \mathfrak{H}} K(z, w) f(w) v^{2k} \frac{du dv}{v^2} &= \int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma} \varphi(\gamma z, w) f(\gamma w) \text{Im}(\gamma w)^{2k} \frac{du dv}{v^2} = \int_{\mathfrak{H}} \varphi(z, w) f(w) v^{2k} \frac{du dv}{v^2} \\ &= \int_{\mathfrak{H}} \frac{1}{(z - \bar{w})^{2k}} \left(\sum_n c_n e^{2\pi i n w} \right) v^{2k} \frac{du dv}{v^2} = \sum_n c_n \int_{\mathfrak{H}} \frac{1}{(z - \bar{w})^{2k}} e^{2\pi i n w} v^{2k} \frac{du dv}{v^2} \end{aligned}$$

The integral is

$$\int_{\mathfrak{H}} \frac{1}{(z - \bar{w})^{2k}} e^{2\pi i n w} v^{2k} \frac{du dv}{v^2} = \int_0^\infty \int_{\mathbb{R}} \frac{1}{((x + iy) - (u - iv))^{2k}} e^{2\pi i n(u+iv)} v^{2k} \frac{du dv}{v^2}$$

Replacing u by $u + x$ makes this into

$$\begin{aligned} & e^{2\pi i n x} \int_0^\infty \int_{\mathbb{R}} \frac{1}{(iy - (u - iv))^{2k}} e^{2\pi i n(u+iv)} v^{2k} \frac{du dv}{v^2} \\ &= e^{2\pi i n x} \int_0^\infty e^{-2\pi n v} \left(\int_{\mathbb{R}} \frac{1}{(u - i(v + y))^{2k}} e^{2\pi i n u} du \right) v^{2k} \frac{dv}{v^2} \end{aligned}$$

The inner integral is evaluated by residues:

$$\int_{\mathbb{R}} \frac{1}{(u - i(v + y))^{2k}} e^{2\pi i n u} du = 2\pi i \cdot \frac{(2\pi i n)^{2k-1}}{(2k-1)!} \cdot e^{2\pi i n \cdot i(v+y)} = \frac{(2\pi i)^{2k} \cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n(v+y)}$$

so the whole integral is

$$\begin{aligned} & \int_0^\infty e^{-2\pi n v} v^{2k-1} \cdot \frac{(2\pi i)^{2k} \cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n \cdot (v+y)} \frac{dv}{v} = \frac{(2\pi i)^{2k}}{\Gamma(2k)} n^{2k-1} e^{-2\pi n y} \int_0^\infty e^{-4\pi n v} v^{2k-1} \frac{dv}{v} \\ &= \frac{(2\pi i)^{2k}}{\Gamma(2k)} n^{2k-1} e^{-2\pi n y} \cdot (4\pi n)^{1-2k} \cdot \Gamma(2k-1) = (2i)^{2k} 4^{1-2k} \pi \cdot e^{-2\pi n y} / (2k-1) \\ &= (-1)^k 2^{2-2k} \pi e^{-2\pi n y} / (2k-1) \end{aligned}$$

Thus,

$$\int_{\mathfrak{H}} \frac{1}{(z - \bar{w})^{2k}} e^{2\pi i n w} v^{2k} \frac{du dv}{v^2} = \frac{(-1)^k 2^{2-2k} \pi}{2k-1} \cdot e^{2\pi i n z}$$

and we have the desired reproducing property

$$\int_{\Gamma \backslash \mathfrak{H}} K(z, w) f(w) v^{2k} \frac{du dv}{v^2} = \frac{(-1)^k 2^{2-2k} \pi}{2k-1} \sum_n c_n e^{2\pi i n z} = \frac{(-1)^k 2^{2-2k} \pi}{2k-1} \cdot f(z)$$