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The Estermann phenomenon

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The *Estermann phenomenon* is that not every natural Dirichlet series has a meromorphic continuation. One need not look far:

Claim: *(Estermann)* Let $d(n)$ be the number of positive divisors of n. The Dirichlet series

$$\sum_n \frac{d(n)^3}{n^s} \;=\; \zeta(s)^4 \prod_p \Big(1 + 4p^{-s} + p^{-2s}\Big)$$

has a *natural boundary* along $\mathrm{Re}(s) = 0$, in contrast to meromorphically continuable

$$\sum_n \frac{d(n)}{n^s} \;=\; \zeta(s)^2 \qquad\qquad \text{and} \qquad\qquad \sum_n \frac{d(n)^2}{n^s} \;=\; \frac{\zeta(s)^4}{\zeta(2s)}$$

Proof: The cases with meromorphic continuations are treated along the way to examination of the example lacking meromorphic continuation. By the multiplicativity $d(mn) = d(m)d(n)$ for coprime m, n,

$$\sum_n \frac{d(n)}{n^s} \;=\; \prod_p \Big(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots\Big)$$

Recall

$$1 + 2x + 3x^2 + \ldots \;=\; \frac{d}{dx}\Big(1 + x + x^2 + x^3 + \ldots\Big) \;=\; \frac{d}{dx}\frac{1}{1-x} \;=\; \frac{1}{(1-x)^2}$$

Thus,

$$\sum_n \frac{d(n)}{n^s} \;=\; \prod_p \frac{1}{(1-p^{-s})^2} \;=\; \zeta(s)^2$$

Continuing,

$$\sum_n \frac{d(n)^2}{n^s} \;=\; \prod_p \Big(1 + \frac{2^2}{p^s} + \frac{3^2}{p^{2s}} + \ldots\Big)$$

and

$$1 + 2^2x + 3^2x^2 + \ldots \;=\; \frac{d}{dx}\Big(x\frac{d}{dx}\Big(1 + x + x^2 + x^3 + \ldots\Big)\Big)$$

$$= \; \frac{d}{dx}\frac{x}{(1-x)^2} \;=\; \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} \;=\; \frac{1+x}{(1-x)^3} \;=\; \frac{1-x^2}{(1-x)^4}$$

For

$$\sum_n \frac{d(n)^3}{n^s} \;=\; \prod_p \Big(1 + \frac{2^3}{p^s} + \frac{3^3}{p^{2s}} + \ldots\Big)$$

similarly

$$1 + 2^3x + 3^3x^2 + \ldots \;=\; \frac{d}{dx}\Big(x \cdot \frac{1+x}{(1-x)^3}\Big) \;=\; \frac{1+x}{(1-x)^3} + x \cdot \frac{1}{(1-x)^3} + x \cdot \frac{3(1+x)}{(1-x)^4}$$

$$= \; \frac{(1-x^2) + x(1-x) + 3x(1+x)}{(1-x)^4} \;=\; \frac{1 - x^2 + x - x^2 + 3x + 3x^2}{(1-x)^4} \;=\; \frac{1 + 4x + x^2}{(1-x)^4}$$

The numerator is *not* a cyclotomic polynomial, so is *not* a finite product-and-ratio of polynomials $1 - x^\ell$, so there is no obvious analogous expression in terms of $\zeta(s)$, $\zeta(2s)$, $\zeta(3s)$, etc.

The polynomial $1+4x+x^2$ can be written as an arbitrarily large product-and-ratio of binomials $1-x^\ell$, with a leftover polynomial factor of the form $1+cx^{\ell+1}+\ldots$. Thus, $\sum_n d(n)^3/n^s$ can be written as an arbitrarily large product-and-ratio of factors $\zeta(\ell s)$ together with a leftover Euler product convergent in $\mathrm{Re}(s) > \frac{1}{\ell+1}$.

To illustrate this, the first step would be to get rid of the $4x$ term by multiplying by $(1-x)^4$:

$$(1-x)^4 \cdot (1+4x+x^2) \;=\; (1-4x+6x^2-4x^3+x^4)(1+4x+x^2) \;=\; 1-9x^2+16x^3-9x^4+x^6$$

Thus,

$$\prod_p (1+4p^{-s}+p^{-2s}) \;=\; \zeta(s)^4 \cdot \prod_p (1-9p^{-2s}+16p^{-3s}-9p^{-4s}+p^{-4s})$$

Next, to get rid of the $-9x^2$ term, multiply by $(1+x^2)^9 = (1-x^4)^9/(1-x^2)^9$, giving

$$\prod_p (1+4p^{-s}+p^{-2s}) \;=\; \zeta(s)^4 \cdot \frac{\zeta(4s)^9}{\zeta(2s)^9} \cdot \prod_p (1+16p^{-3s}+\ldots)$$

Since $1+4x+x^2$ is not a cyclotomic polynomial, this process does not terminate. Inductively, there is an *infinite* increasing sequence of integers ℓ_j and *non-zero* integers e_j such that

$$1+4x+x^2 \;=\; (1-x)^{e_1}(1-x^2)^{e_2}(1-x^3)^{e_3}\ldots(1-x^{\ell_j})^{e_{\ell_j}} \cdot \big(1+x^{\ell_j+1}P_j(x)\big)$$

with (non-zero) polynomials $P_j(x)$. Certainly

$$D_j(s) \;=\; \prod_p \big(1+p^{-s(\ell_j+1)}P_j(p^{-s})\big)$$

is absolutely convergent and non-vanishing for $\mathrm{Re}(s) > \frac{1}{\ell_j+1}$. Thus, for every j, there is an expression

$$\prod_p (1+4p^{-s}+p^{-2s}) \;=\; D_j(s) \cdot \prod_{1\le i\le j} \zeta(\ell_i \cdot s)^{e_i} \qquad\qquad (\text{for } \mathrm{Re}(s) > \tfrac{1}{\ell_j+1})$$

On one hand, this gives a meromorphic continuation to $\mathrm{Re}(s) > \frac{1}{\ell_j+1}$. On the other hand, since the exponents e_i are non-zero, the infinitely-many zeros of $\zeta(s)$ in the critical strip make the zeros of $\zeta(\ell \cdot s)$ bunch up just to the right of $\mathrm{Re}(s) = 0$ as $\ell \to \infty$. ///

[0.1] Remarks: Continuing in this vein, [Kurokawa 1985a,b] showed that $\sum \frac{a_n^k}{n^s}$ has a natural boundary for $k \ge 3$, where $f(z) = \sum a_n\, e^{2\pi i n z}$ is a modular form,

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