The Estermann phenomenon

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The Estermann phenomenon is that not every natural Dirichlet series has a meromorphic continuation. One need not look far:

Claim: (Estermann) Let \( d(n) \) be the number of positive divisors of \( n \). The Dirichlet series
\[
\sum_n \frac{d(n)^3}{n^s} = \zeta(s)^4 \prod_p \left( 1 + 4p^{-s} + p^{-2s} \right)
\]
has a natural boundary along \( \text{Re}(s) = 0 \), in contrast to meromorphically continuable
\[
\sum_n \frac{d(n)}{n^s} = \zeta(s)^2 \quad \text{and} \quad \sum_n \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}
\]

Proof: The cases with meromorphic continuations are treated along the way to examination of the example lacking meromorphic continuation. By the multiplicativity \( d(mn) = d(m)d(n) \) for coprime \( m, n \),
\[
\sum_n \frac{d(n)}{n^s} = \prod_p \left( 1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots \right)
\]
Recall
\[
1 + 2x + 3x^2 + \ldots = \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \ldots \right) = \frac{d}{dx} \frac{1}{1 - x} = \frac{1}{(1 - x)^2}
\]
Thus,
\[
\sum_n \frac{d(n)}{n^s} = \prod_p \frac{1}{(1 - p^{-s})^2} = \zeta(s)^2
\]
Continuing,
\[
\sum_n \frac{d(n)^2}{n^s} = \prod_p \left( 1 + \frac{2^2}{p^s} + \frac{3^2}{p^{2s}} + \ldots \right)
\]
and
\[
1 + 2^2x + 3^2x^2 + \ldots = \frac{d}{dx} \left( x \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \ldots \right) \right) = \frac{d}{dx} \frac{x}{(1 - x)^2} = \frac{1}{(1 - x)^2} + \frac{2x}{(1 - x)^3} = \frac{1 + x}{(1 - x)^3} = \frac{1 - x^2}{(1 - x)^4}
\]
For
\[
\sum_n \frac{d(n)^3}{n^s} = \prod_p \left( 1 + \frac{3^3}{p^s} + \frac{3^3}{p^{2s}} + \ldots \right)
\]
similarly
\[
1 + 2^4x + 3^4x^2 + \ldots = \frac{d}{dx} \left( x \frac{1 + x}{(1 - x)^3} \right) = \frac{1 + x}{(1 - x)^3} + x \cdot \frac{1}{(1 - x)^3} + x \cdot \frac{3(1 + x)}{(1 - x)^4} = \frac{(1 - x^2) + x(1 - x) + 3x(1 + x)}{(1 - x)^4} = \frac{1 - x^2 + x - x^2 + 3x + 3x^2}{(1 - x)^4} = \frac{1 + 4x + x^2}{(1 - x)^4}
\]
The numerator is not a cyclotomic polynomial, so is not a finite product-and-ratio of polynomials $1 - x^s$, so there is no obvious analogous expression in terms of $\zeta(s)$, $\zeta(2s)$, $\zeta(3s)$, etc.

The polynomial $1 + 4x + x^2$ can be written as an arbitrarily large product-and-ratio of binomials $1 - x^\ell$, with a leftover polynomial factor of the form $1 + cx^{\ell+1} + \ldots$. Thus, $\sum_n d(n)n^s$ can be written as an arbitrarily large product-and-ratio of factors $\zeta(\ell s)$ together with a leftover Euler product convergent in $\Re(s) > \frac{1}{\ell s + 1}$.

To illustrate this, the first step would be to get rid of the $4x$ term by multiplying by $(1 - x)^4$:

$$(1 - x)^4 \cdot (1 + 4x + x^2) = (1 - 4x + 6x^2 - 4x^3 + x^4)(1 + 4x + x^2) = 1 - 9x^2 + 16x^3 - 9x^4 + x^6$$

Thus,

$$\prod_p (1 + 4p^{-s} + p^{-2s}) = \zeta(s)^4 \cdot \prod_p (1 - 9p^{-2s} + 16p^{-3s} - 9p^{-4s} + p^{-4s})$$

Next, to get rid of the $-9x^2$ term, multiply by $(1 + x^2)^9 = (1 - x^4)^9/(1 - x^2)^9$, giving

$$\prod_p (1 + 4p^{-s} + p^{-2s}) = \zeta(s)^4 \cdot \frac{\zeta(4s)^9}{\zeta(2s)^9} \cdot \prod_p (1 + 16p^{-3s} + \ldots)$$

Since $1 + 4x + x^2$ is not a cyclotomic polynomial, this process does not terminate. Inductively, there is an infinite increasing sequence of integers $\ell_j$ and non-zero integers $e_j$ such that

$$1 + 4x + x^2 = (1 - x)^{e_1}(1 - x^3)^{e_2}(1 - x^{\ell_3})^{e_3} \cdots (1 - x^{\ell_j})^{e_j} \cdot (1 + x^{\ell_{j+1}}P_j(x))$$

with (non-zero) polynomials $P_j(x)$. Certainly

$$D_j(s) = \prod_p (1 + p^{-s(\ell_j + 1)}P_j(p^{-s}))$$

is absolutely convergent and non-vanishing for $\Re(s) > \frac{1}{\ell_j + 1}$. Thus, for every $j$, there is an expression

$$\prod_p (1 + 4p^{-s} + p^{-2s}) = D_j(s) \cdot \prod_{1 \leq i \leq j} \zeta(\ell_i \cdot s)^{e_i} \quad \text{(for $\Re(s) > \frac{1}{\ell_j + 1}$)}$$

On one hand, this gives a meromorphic continuation to $\Re(s) > \frac{1}{\ell_j + 1}$. On the other hand, since the exponents $e_j$ are non-zero, the infinitely-many zeros of $\zeta(s)$ in the critical strip make the zeros of $\zeta(\ell \cdot s)$ bunch up just to the right of $\Re(s) = 0$ as $\ell \to \infty$.

[0.1] Remarks: Continuing in this vein, [Kurokawa 1985a,b] showed that $\sum a_n e^{2\pi ins}$ is a modular form.

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**Bibliography**


