(*September 14, 2019*)

Riemann and $\zeta(s)$

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Even more interesting than a Prime Number Theorem is the *precise* relationship between primes and zeros of zeta found by [Riemann 1859]. A similar idea applies to *any* zeta or L-function with *analytic continuation*, *functional equation*, and *Euler product*.

It took 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does *not* yield a Prime Number Theorem, despite giving a precise relationship between primes and zeros of zeta.

The *idea* is that the equality of the Euler product and Riemann-Hadamard product for zeta allows extraction of an *exact formula* for a suitably-weighted counting of primes, a sum over zeros of zeta, via a contour integration of the logarithmic derivatives. As observed by [Guinand 1947] and [Weil 1952], [Weil 1972], the classical formulas are equalities of values of a certain *distribution*, in the sense of *generalized functions*.

An essential supporting point is *meromorphic continuation* of $\zeta(s)$ via *integral representation(s)* of $\zeta(s)$ in terms of *theta function(s)*. The most symmetrical choice of Schwartz-function [1] data for the theta function gives the *functional equation* of of $\zeta(s)$. This theta function is an example of *automorphic form*. [2] Further, these integral representations give *vertical growth estimates*, critical for invocation of Hadamard's theorem on product expansions of entire functions. This is an archetype: in great generality, automorphic forms are the principal device for proof of non-trivial, essential facts about zeta functions and L-functions.

A key mechanism in analytic continuation and functional equation of $\zeta(s)$ is a functional equation relating two theta series, and showing that the most-symmetrical theta series *is an automorphic form*. This follows from the *Poisson summation formula* for Schwartz functions, itself a corollary of the representability of smooth functions by their *Fourier series*. (See the Supplement.)

The *Gamma function* and its *asymptotics* are needed to use the functional equation of $\zeta(s)$ to obtain the growth bound on $\zeta(s)$ necessary to apply the Hadamard product result. Something simpler than Stirling's formula suffices, admitting a simpler proof. (See the Supplements on asymptotics, and on the functional equation of Gamma.)

Another of Riemann's contributions was another meromorphic continuation of $\zeta(s)$ via another integral representation using a Hankel/keyhole contour, which does not give the functional equation, but *does* give the special value result that $\zeta(-n) \in \mathbb{Q}$ for negative integers $-n$.

[1] Recall that a *Schwartz function* on $\mathbb{R}$ is an infinitely differentiable function all of whose derivatives, including itself, are of *rapid decay* at infinity. That is, $(1+x^2)^\ell \cdot |f^{(k)}(x)|$ is bounded for all k, ℓ.

[2] For practical purposes, *modular form* and *automorphic form* are synonyms, although some sources try to attribute delicately precise meanings.

1. *Riemann's explicit formula*

The dramatic [Riemann 1859] on the relation between primes and zeros of the zeta function depended on many ideas undeveloped in Riemann's time. Thus, the following sketch, very roughly following Riemann, is far from a proof, but uncovers supporting ideas *needed* to produce a proof, and shows the reward for doing this follow-up. [3]

Riemann knew from Euler that $\zeta(s)$ has an *Euler product* expansion in a half-plane

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}} \qquad\qquad (\text{for } \operatorname{Re} s > 1)$$

As we discuss just below, [Riemann 1859] proved that $\zeta(s)$ has a *meromorphic continuation* so that $(s-1)\zeta(s)$ is *entire*, with $0 = \zeta(0) = \zeta(-2) = \zeta(-4) = \ldots$. [4] The negative even integer are the *trivial* zeros of $\zeta(s)$. Riemann's first inspiration was to imagine that $\zeta(s)$ has a *product expansion* in terms of its zeros [5]

$$(s-1)\,\zeta(s) \;=\; e^{a+bs} \prod_\rho \Big(1 - \frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^\infty \Big(1 + \frac{s}{2n}\Big)\, e^{-s/2n} \qquad (\rho \text{ non-trivial zero of } \zeta, \text{ for all } s\in\mathbb{C})$$

[Hadamard 1893] proved this. *Then*, more tangible information can be extracted from the equality of the two products

$$(s-1) \prod_p \frac{1}{1-\frac{1}{p^s}} \quad=\quad e^{a+bs} \prod_\rho \Big(1 - \frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^\infty \Big(1 + \frac{s}{2n}\Big)\, e^{-s/2n} \qquad (\operatorname{Re} s > 1)$$

Taking logarithmic derivatives of both sides, using $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$ on the left-hand side:

$$\frac{1}{s-1} - \sum_{m\geq 1,\; p} \frac{\log p}{p^{ms}} \;=\; b \;+\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;+\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

A slight rearrangement:

$$\sum_{m\geq 1,\; p} \frac{\log p}{p^{ms}} \;=\; \frac{1}{s-1} - b \;-\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big) \qquad (\text{for } \operatorname{Re} s > 1)$$

[1.1] Remark: The two sides of the equality of logarithmic derivatives are very different. The logarithmic derivative of the Euler product converges well, but only in right half-planes. The logarithmic derivative of

[3] Riemann's paper was remarkable: he intuited an interesting objective, and sketched the necessary supporting ideas without waiting for corroboration. Thus, he was able to see the interest of the *conclusion*, giving a powerful motivation to investigation of the supporting technical ideas.

[4] It is absolutely not obvious that $\zeta(s)$ vanishes at negative even integers. This will follow from the *functional equation*. Even so, Euler had already done computations that could be interpreted as proving this!

[5] This product expansion idea did not occur in a complete vacuum. First, of course, for non-zero complex $\alpha_1, \ldots, \alpha_n$, $(1 - \frac{z}{\alpha_1}) \ldots (1 - \frac{z}{\alpha_n})$ is the unique polynomial of the form $1 + \ldots$ with the indicated zeros. Euler's evaluation of $\sum_n \frac{1}{n^2}$ by imagining (and later proving) $\sin \pi z = \pi z \prod_n (1 - \frac{z^2}{n^2})$ was well known. Also, Euler's product expansion of the inverse of the Gamma-function $\Gamma(s) = \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t}$ as $\frac{1}{\Gamma(s)} = z e^{\gamma z} \prod_n (1 + \frac{z}{n})\, e^{-z/n}$ was well known to Riemann. But the zeta function $\zeta(s)$ is much less elementary.

the Riemann-Hadamard product does *not* converge strongly, but is *not* restricted to a half-plane, and its poles are exhibited explicitly by the expression.

Again diverging slightly from Riemann's original treatment, we intend to apply the Perron identity [6] (see Appendix)

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{Y^s}{s}\,ds \;=\; \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y>1) \\ 0 & (\text{for } 0<Y<1)\end{cases} \qquad (\text{for } \sigma>0)$$

to the log-derivative identity multiplied by X^s/s. Assuming we can apply the Perron identity *term-wise* to X^s/s times the left-hand side, this would give

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\sum_{m,p}\frac{\log p}{p^{ms}}\,ds \;=\; \sum_{m,p}\log p\cdot\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s\cdot p^{-ms}}{s}\,ds \;=\; \sum_{p^m<X}\log p$$

Assuming we can use *residues* term-wise to evaluate X^s/s times the right-hand side, with $\sigma>1$, this would give

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\cdot\Big(\frac{1}{s-1}-b\;-\;\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big)\;-\;\sum_n\Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)\Big)\,ds$$

$$=\;(X-1)\;-\;b\;-\;\sum_\rho\Big(\frac{X^\rho}{\rho}+\frac{1}{-\rho}+\frac{1}{\rho}\Big)\;-\;\sum_n\Big(\frac{X^{-2n}}{-2n}+\frac{1}{2n}-\frac{1}{2n}\Big)\;=\;X\;-\;(b+1)\;-\;\sum_\rho\frac{X^\rho}{\rho}\;+\;\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

Thus, we would have [vonMangoldt 1893]'s reformulation of **Riemann's Explicit Formula:**

$$\sum_{p^m<X}\log p \;=\; X\;-\;(b+1)\;-\;\sum_\rho\frac{X^\rho}{\rho}\;+\;\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

Slightly more precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence, we should say

$$\sum_{p^m<X}\log p \;=\; X\;-\;(b+1)\;-\;\lim_{T\to\infty}\sum_{|\mathrm{Im}\,(\rho)|<T}\frac{X^\rho}{\rho}\;+\;\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

[1.2] Remark: As in Riemann's original, the above sketch has many potential gaps, despite clear intentions. The existence and convergence of the Hadamard product needs both *generalities* about Weierstraß-Hadamard product expressions for entire functions of prescribed growth, grounded in basic complex analysis, and *specifics* about the growth of the *analytic continuation* of $\zeta(s)$. The analytic continuation of $\zeta(s)$ is discussed in the next section, and growth properties later. The growth properties depend on Stirling-Laplace asymptotics of the Gamma function $\Gamma(s)$, and the *Phragmén-Lindelöf* theorem. Background on these topics in complex analysis is recalled in supplements.

[1.3] Non-trivial zeros ρ of $\zeta(s)$

The convergent Euler product shows that $\zeta(s)\ne0$ in the half-plane $\mathrm{Re}\,(s)>1$. Subsequent considerations [7] show that the only possible non-trivial zeros are in the *critical*

[6] Perron's identity is completely standard by now, but was not part of Riemann's approach. Invocation of the Perron identity allows a somewhat simpler approach than Riemann's original, due to von Mangoldt and others.

[7] The analytic continuation and functional equation (below), and relatively elementary properties of the Gamma function $\Gamma(s)=\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t}$

strip $0 \le \mathrm{Re}(s) \le 1$. In 1896, Hadamard and de la Vallée-Poussin independently proved that there are no zeros on the edges $\mathrm{Re}(s) = 0, 1$ of the critical strip, which they used to prove the *Prime Number Theorem.*

The functional equation shows that if ρ is a non-trivial zero, then $1 - \rho$ is a non-trivial zero. The property $\zeta(\bar{s}) = \overline{\zeta(s)}$ shows that if ρ is a non-trivial zero, then $\bar{\rho}$ is a non-trivial zero.

[1.4] The Riemann Hypothesis

After the main term X in the right-hand side of the explicit formula, the next-largest terms would be the X^ρ/ρ summands, with $0 \le \mathrm{Re}(\rho) \le 1$ due to the Euler product and functional equation. The *Riemann Hypothesis* is that all the non-trivial zeros ρ have $\mathrm{Re}(\rho) = \frac{1}{2}$. With a bound like $T \log T$ on the number of zeros below height T, the Riemann hypothesis is equivalent to an error term of order $X^{\frac{1}{2}+\varepsilon}$ in the Prime Number Theorem, for all $\varepsilon > 0$.

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2. *Analytic continuation and functional equation of $\zeta(s)$ via θ*

The following ideas gained publicity and importance from Riemann, but were apparently known to some degree before Riemann.

The key is that the completed zeta function has an *integral representation* in terms of an *automorphic form*, the simplest *theta function*. Both the *analytic continuation* and the *functional equation* of zeta follow from this integral representation using a parallel functional equation of the theta function, the latter demonstrated by *Poisson summation*, which comes from *Fourier series*.

Modernizing this idea, the analytic continuation can be separated from the functional equation, as in the following.

[2.1] Elementary-but-insufficiently-enlightening argument for analytic continuation

Simple calculus can extend the domain of $\zeta(s)$ as far to the left as we want. The idea is to pay attention to *quantitative* aspects of the integral test. First, by comparison to $\int_1^\infty \frac{dx}{x^s}$, the sum $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$ converges for $\mathrm{Re}(s) > 1$.

To push this further, it is standard to proceed as follows.

$$\zeta(s) - \frac{1}{s-1} \;=\; \zeta(s) - \int_1^\infty \frac{dx}{x^s} \;=\; \sum_n \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\Big) \;=\; \sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

Even for complex s, we have a Taylor-Maclaurin expansion with *error term*[8]

$$(n+1)^{1-s} \;=\; \big(n \cdot (1 + \frac{1}{n})\big)^{1-s} \;=\; n^{1-s} \cdot \big(1 + \frac{1-s}{n} + O(\tfrac{1}{n^2})\big) \;=\; \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$

The constant in the big-O term is *uniform* in n for fixed s. Thus,

$$\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big] \;=\; \frac{1}{n^s} - \frac{1}{n^s} + \frac{1}{s-1} O\big(\frac{1}{n^{s+1}}\big) \;=\; O\big(\frac{1}{n^{s+1}}\big)$$

That is, for fixed[9] $\mathrm{Re}(s) > 0$, we have *absolute convergence* of

$$\sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big) \qquad (\text{for } \mathrm{Re}(s) > 0)$$

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[8] Landau's *big-O* notation $f(x) = g(x) + O(h(x))$ means that, as $x \to \infty$ (or some other limit), $|f(x) - g(x)|$ is bounded by a constant multiple of $|h(x)|$.

[9] In fact, the big-O constant is also *uniform* for s in compacts inside $\mathrm{Re}(s) > 0$. Thus, the series converges *locally uniformly on compacts*, so does give a *holomorphic* function.

in the larger region $\mathrm{Re}(s) > 0$.

[2.2] Remark: Iterating the idea of approximating sums by integrals gives a comparable extension to $\mathrm{Re}(s) > -\ell$ for all ℓ, as Euler already effectively found, systematically by *Euler-Maclaurin summation*. However, such continuations give no clues about functional equations, and certainly not about Riemann's explicit formula.

[2.3] Slight modernization of Riemann's argument We update Riemann's idea to avoid needless artifacts. Both the original and this update are archetypes. [10] Let $f(x)$ be *any* very well-behaved function on $\mathbb{R}$, that is, infinitely differentiable, and it and all its derivatives are rapidly decreasing at infinity. These are *Schwartz functions*, after [Schwartz 1950/51]. Further, take f *even*, that is $f(-x) = f(x)$. The even Schwartz function f is a *dummy*, insofar as only its general properties are used. In effect, Riemann's choice was the Gaussian $f(x) = e^{-\pi x^2}$, based on connections to Jacobi's theta functions, as we see along the way. A *theta function*[11] associated to the even Schwartz function f is

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(y\cdot n) \qquad\qquad (\text{for } y > 0)$$

and associated *Gamma function*[12]

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, f(t)\, \frac{dt}{t}$$

First, we have the *integral representation*, from which will follow the meromorphic continuation and functional equation:

[2.4] Proposition: $$\int_0^\infty y^s\, \frac{\theta_f(y) - f(0)}{2}\, \frac{dy}{y} \;=\; \Gamma_f(s)\cdot\zeta(s) \qquad\qquad (\text{for } \mathrm{Re}(s) > 1)$$

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[10] Riemann's original line of argument was brought to completion by [Hecke 1918/20]. [Müntz 1924] demonstrated the mildly surprising fact that the special features of the Gaussian were essentially irrelevant to Riemann's argument, presaging the Iwasawa-Tate fairly complete clarification of this, in terms of the *local functional equations*. Substantial modernization occurred in [Matchett 1946], [Iwasawa 1950/52], [Iwasawa 1952], and [Tate 1950/1967]. In particular, these sources observed that certain details involving *theta functions* were less essential than previously believed. Nevertheless, the *automorphic* nature of theta functions was *also* important in its own right.

[11] Again, Riemann used $f(u) = e^{-\pi u^2}$, and, consistent with an existing convention at the time, in effect defined

$$\theta(iy) \;=\; \sum_{n\in\mathbb{Z}} f(\sqrt{y}\cdot n) \qquad\qquad (\text{with Gaussian } f(u) = e^{-\pi u^2})$$

That is, the argument of θ is iy rather than y, and $\sqrt{y}$ enters on the right side, rather than y. Further, the Gaussian extends to an *entire* function, and this theta function extends to a holomorphic function, the simplest *Jacobi theta function*, on the upper half-plane $\mathfrak{H}$:

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad\qquad (\text{with } z \in \mathfrak{H})$$

[12] With Gaussian $f(x) = e^{-\pi x^2}$, this construction gives an exponential multiple of the standard Gamma function at $\frac{s}{2}$:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, e^{-\pi x^2}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\int_0^\infty t^{\frac{s}{2}}\, e^{-\pi x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}} \int_0^\infty t^{\frac{s}{2}}\, e^{-x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})$$

Proof: The $n = 0$ (constant) term $f(0)$ of $\theta_f(y)$ is the only summand not rapidly decreasing. The even-ness of f makes the $\pm n$ terms have equal contributions to $\theta_f(y)$. Thus, interchanging sum and integral, and replacing y by y/n,

$$\int_0^\infty y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y} \;=\; \sum_{n\geq 1} \int_0^\infty y^s \; f(yn) \; \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \int_0^\infty y^s \; f(y) \; \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \; \Gamma_f(s)$$

as claimed. ///

[2.5] Remark: The measure $\frac{dy}{y}$ is the natural multiplication-invariant measure on the positive reals.

[2.6] Theorem: The completed zeta function $\Gamma_f(s) \cdot \zeta(s)$ has a meromorphic continuation to $s \in \mathbb{C}$, and $s(s-1) \cdot \Gamma_f(s) \cdot \zeta(s)$ is *entire.*

[2.7] Remark: Repeated integration by parts shows that $\Gamma_f(s)$ itself has a meromorphic continuation:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s \, f(t) \; \frac{dt}{t} = \int_0^\infty \frac{t^{s+1}}{s} \; f'(t) \; \frac{dt}{t} = \int_0^\infty \frac{t^{s+2}}{s(s+1)} \; f''(t) \; \frac{dt}{t} = \int_0^\infty \frac{t^{s+3}}{s(s+1)(s+2)} \; f'''(t) \; \frac{dt}{t} = \ldots$$

Since all the derivatives of f are of rapid decay, these expressions give an extension of $\Gamma_f(s)$ to $s \in \mathbb{C}$ except for at worst $s = 0, -1, -2, -3, \ldots$.

Proof: Break the integral of the integral representation into two parts:

$$\Gamma_f(s) \cdot \zeta(s) \;=\; \int_1^\infty y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y} \;+\; \int_0^1 y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y}$$

It is not hard to check that $\frac{\theta_f(y) - \theta_f(0)}{2}$ is rapidly decreasing at $+\infty$, so the integral on $[1, \infty)$ is absolutely convergent (and uniformly for s in compacts) for all $s \in \mathbb{C}$.

The behavior of $\theta_f(y)$ as $y \to 0^+$ is harder to analyze, and is best done by the following device.

The trick is to convert the integral on $[0, 1]$ to an integral over $[1, \infty)$, up to two elementary terms. The new integral over $[1, \infty)$ will involve the theta function $\theta_{\widehat{f}}$ attached to the *Fourier transform*

$$\widehat{f}(x) \;=\; \int_{\mathbb{R}} e^{-2\pi i x \xi} \; f(\xi) \; d\xi$$

of f. We grant for the moment that Fourier transform maps the Schwartz space to itself, as is directly verifiable in concrete examples such as the Gaussian $f(x) = e^{-\pi x^2}$. Simply by changing variables in the integral, we recall a homogeneity property of the Fourier transform:

$$\widehat{f}(x/y) \;=\; \int_{\mathbb{R}} e^{-2\pi i \frac{x}{y} \xi} \; f(\xi) \; d\xi \;=\; |y| \int_{\mathbb{R}} e^{-2\pi i x \xi} \; f(y\xi) \; d\xi \;=\; |y| \cdot (f \circ y)\widehat{\;}(x)$$

by replacing ξ by ξy in the integral, where $(f \circ y)(\xi) = f(y\xi)$. We grant ourselves the standard *Poisson summation formula*

$$\sum_{n \in \mathbb{Z}} F(n) \;=\; \sum_{n \in \mathbb{Z}} \widehat{F}(n) \qquad \text{(for Schwartz functions } F\text{)}$$

(See the Appendix for proof.) Letting $F(x) = f(yx)$ and using the homogeneity property of Fourier transform, this is

$$\sum_{n \in \mathbb{Z}} f(y \cdot n) \;=\; \sum_{n \in \mathbb{Z}} \frac{1}{y} \, \widehat{f}\Big(\frac{1}{y} \cdot n\Big) \qquad \text{(for } y > 0\text{)}$$

Thus,

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(yn) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{y}\widehat{f}(n) \;=\; \frac{1}{y}\cdot\theta_{\widehat{f}}\Big(\frac{1}{y}\Big)$$

This gives a way to flip the interval $[0,1]$ to $[1,\infty)$, by replacing y by $1/y$, accommodating the anomalous terms for $n=0$ separately:

$$\int_0^1 y^s\,\frac{\theta_f(y)-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-\frac{1}{y}\widehat{f}(0)}{2}+\frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{-s}\,\frac{y\theta_{\widehat{f}}(y)-y\widehat{f}(0)}{2}+\int_0^1 y^s\,\frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\int_0^1 y^{s-1}\,\frac{dy}{y} \;-\; \frac{f(0)}{2}\int_0^1 y^s\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

The integral on $[1,\infty)$ is entire in s, since $\theta_{\widehat{f}}(y)-\widehat{f}(0)$ is rapidly decreasing at ∞. The two elementary terms have obvious meromorphic continuations. Thus,

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}+y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

Again, the integral is *entire*, and the elementary terms give the only poles, which are at $s=0,1$. ///

[2.8] Remark: The expression

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}+y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

gives a bit more information than the bare statement of the theorem, namely, it tells the residues of the poles at $s=0,1$, and shows a certain potential symmetry, as in the following.

For f with $\widehat{f}=f$ Riemann's original symmetrical result is recovered:

[2.9] Theorem: *(Riemann)* The *completed* zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s\in\mathbb{C}$, except for simple poles at $s=0,1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

Proof: Various means show that $f(x)=e^{-\pi x^2}$ is its own Fourier transform. Thus, the expression in the proof of the previous theorem becomes symmetrical in $s\leftrightarrow 1-s$, and the artifact of the coefficient of $\frac{1}{2}$ on both sides can be discarded. ///

[2.10] Remark: The leading factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ should *not* be construed as objectionable in any way, but, rather, as something that really does *belong* with $\zeta(s)$. The $\pi^{-s/2}\Gamma(\frac{s}{2})$ is called the **gamma factor** for $\zeta(s)$.

In the context of the *Euler product* the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime*[13] ∞.

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3. *The Hankel/keyhole contour and* $\zeta(-n) \in \mathbb{Q}$

This contour-integration device here is one of Riemann's proofs of analytic continuation of $\zeta(s)$. It immediately proves that values of $\zeta(s)$ at non-positive integers are *rational*, and shows the connection to the Laurent coefficients of $1/(e^t - 1)$ at $t = 0$.

[3.1] An integral representation of $\Gamma(s) \cdot \zeta(s)$ Although the integral representation of $\zeta(s)$ using a theta function is perhaps better in the long run, there is a more elementary one:

[3.2] Claim: For $\text{Re}(s) > 1$,

$$\Gamma(s) \cdot \zeta(s) \;=\; \int_0^\infty \frac{t^{s-1}\,dt}{e^t - 1}$$

Proof: Expand a geometric series, exchange sum and integral, and change variables:

$$\int_0^\infty \frac{t^{s-1}\,dt}{e^t - 1} \;=\; \int_0^\infty \frac{t^{s-1}\,e^{-t}\,dt}{1 - e^{-t}} \;=\; \int_0^\infty t^{s-1} \sum_{n\geq 1} e^{-nt}\,dt \;=\; \sum_{n\geq 1} \int_0^\infty t^s\,e^{-nt}\,\frac{dt}{t}$$

$$=\; \sum_{n\geq 1} \frac{1}{n^s} \int_0^\infty t^s\,e^{-t}\,\frac{dt}{t} \;=\; \Gamma(s) \cdot \sum_{n\geq 1} \frac{1}{n^s} \;=\; \Gamma(s) \cdot \zeta(s)$$

as claimed. ///

[3.3] Keyhole/Hankel contour The *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon > 0$, counterclockwise around a circle of radius ε at 0, back to ε on the real line, and outbound back to $+\infty$ along the real line.

The usual elementary application is to evaluation of integrals similar to $\int_0^\infty \frac{t^s\,dt}{t^2+1}$, with $0 < \text{Re}(s) < 1$. In such an example, analytically continuing counterclockwise around 0 has no impact on the denominator, but, significantly, the numerator changes by a factor $e^{2\pi i s}$, since

$$t^s \;=\; (|t| \cdot e^{i\theta})^s \;=\; |t|^s \cdot e^{i\theta s} \qquad\qquad \text{(and } \theta \text{ goes from 0 to } 2\pi\text{)}$$

We want the out-bound value of t^s to be real-valued for real s, so the inbound version of t^s must be $t^s \cdot e^{2\pi i s}$. The absolute value of the integrand goes to 0 as $|t| \to 0$, so the integral over the small circle goes to 0 as $\varepsilon \to 0$, as do the integrals to and from $0, \varepsilon$ along the real line.

Thus, letting H_ε be the Hankel contour with circle of radius $\varepsilon > 0$,

$$\lim_{\varepsilon\to 0} \int_{H_\varepsilon} \frac{t^s\,dt}{t^2+1} \;=\; \lim_{\varepsilon\to 0} \Big(\int_{+\infty}^{\varepsilon} \frac{(t \cdot e^{2\pi i})^s\,dt}{t^2+1} + (\text{integral over little circle}) + \int_{\varepsilon}^{+\infty} \frac{t^s\,dt}{t^2+1} \Big)$$

---

[13] An insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other p-adic completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime ∞ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes p and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* ∞. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there's little point in fretting about this.

$$= \; (1 - e^{2\pi i s}) \int_0^\infty \frac{t^s \, dt}{t^2+1}$$

In this elementary example, the trick is to further modify H_ε by *not* going all the way to $+\infty$ outbound, but stopping at $+R$ for large positive R, traversing clockwise a large circle of radius R back to the positive real axis, and then inbound to ε. The integrals from R to and from $+\infty$ go to 0 as $R \to +\infty$, as does the integral over the large circle, since

$$\Big|\text{integral over big circle}\Big| \le \; \text{length} \cdot \text{max value} \le \; 2\pi R \cdot \frac{R^{\mathrm{Re}\,(s)}}{R^2 - 1}$$

For each R, ε, this gives a path integral (counter-clockwise) over a *closed* path. By *residues*, this picks up $2\pi i$ times the sum of the residues inside the path. Thus, we discover that the integrals do not depend on the parameters $0 < \varepsilon < 1 < R$. Keeping track of the relevant versions of t^s,

$$(1 - e^{2\pi i s}) \int_0^\infty \frac{t^s \, dt}{t^2+1} \; = \; 2\pi i \cdot \Big((\text{residue at } t = i) + (\text{residue at } t = -i) \Big)$$

$$= \; 2\pi i \cdot \Big(\frac{e^{\frac{1}{2}\pi i s}}{i+i} + \frac{e^{\frac{3}{2}\pi i s}}{-i-i} \Big) \; = \; \pi \cdot (e^{\frac{1}{2}\pi i s} - e^{\frac{3}{2}\pi i s})$$

That is,

$$\int_0^\infty \frac{t^s \, dt}{t^2+1} \; = \; \pi \cdot \frac{e^{\frac{1}{2}\pi i s} - e^{\frac{3}{2}\pi i s}}{1 - e^{2\pi i s}} \; = \; \pi \cdot \frac{e^{-\frac{1}{2}\pi i s} - e^{\frac{1}{2}\pi i s}}{e^{-\pi i s} - e^{\pi i s}} \; = \; \frac{\pi}{e^{\frac{1}{2}\pi i s} + e^{-\frac{1}{2}\pi i s}} \; = \; \frac{\pi/2}{\cos \frac{\pi s}{2}}$$

This is a charming and useful device, but a different secondary trick is applied to $\zeta(s)$:

[3.4] Evaluation of $\zeta(-n)$ The first part of the Hankel contour discussion gives

$$\Gamma(s) \cdot \zeta(s) \; = \; \int_0^\infty \frac{t^{s-1} \, dt}{e^t - 1} \; = \; \frac{1}{1 - e^{2\pi i (s-1)}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1} \, dt}{e^t - 1} \; = \; \frac{1}{1 - e^{2\pi i s}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1} \, dt}{e^t - 1}$$

Rewrite this as

$$\zeta(s) \; = \; \frac{1}{\Gamma(s) \cdot (1 - e^{2\pi i s})} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1} \, dt}{e^t - 1}$$

At $s = -n \in \{0, -1, -2, -3, -4, \ldots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1 - e^{2\pi i s}$ cancel, giving a finite, computable value. Second, the function t^{-n-1} is *single-valued*, so the inbound and outbound integrals of the Hankel contour simply *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $\frac{t^{-n-1}}{e^t - 1}$ at 0.

The periodicity of $1 - e^{2\pi i s}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi i s} \; = \; 1 - \Big(1 + \frac{2\pi i s}{1!} + \frac{(2\pi i s)^2}{2!} + \ldots \Big) \; = \; -2\pi i s + \text{higher}$$

Grant for the moment that the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\zeta(-n) \; = \; \frac{1}{\frac{(-1)^n}{n!} \cdot (-2\pi i)} \cdot 2\pi i \cdot \mathrm{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1} \; = \; (-1)^{n+1} \cdot n! \cdot \mathrm{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1}$$

$$= \; (-1)^{n+1} \cdot n! \cdot \big(-1^{th} \text{ Laurent coefficient of } \tfrac{t^{-n-1}}{e^t - 1} \text{ at } t = 0 \big)$$

$$= \; (-1)^{n+1} \cdot n! \cdot \big(n^{th} \text{ Laurent series coefficient of } \tfrac{1}{e^t - 1} \text{ at } t = 0 \big)$$

The Laurent coefficients of $\frac{1}{e^t-1}$ are more-or-less Bernoulli numbers. These are not completely elementary objects, but are certainly *rational*. Thus, $\zeta(-n) \in \mathbb{Q}$.

[3.5] Vanishing $\zeta(-2)=\zeta(-4)=\ldots=0$ A slightly finer analysis of the generating function $\frac{1}{e^t-1}$ yields the vanishing of $\zeta(s)$ at negative even integers, as follows.

First, $\frac{1}{e^t-1}$ is very close to being *odd* as a function of t:

$$\frac{1}{e^t-1} + \frac{1}{e^{-t}-1} \;=\; \frac{1}{e^t-1} + \frac{e^t}{1-e^t} \;=\; \frac{1}{e^t-1} - \frac{e^t}{e^t-1} \;=\; \frac{1-e^t}{e^t-1} \;=\; -1$$

Thus,

$$\Big(\frac{1}{e^t-1} + \tfrac{1}{2}\Big) + \Big(\frac{1}{e^{-t}-1} + \tfrac{1}{2}\Big) \;=\; 0$$

and $\frac{1}{e^t-1} + \frac{1}{2}$ is *odd*, so all its non-vanishing Laurent coefficients are odd-degree. Thus, for even $-2n < 0$,

$$\zeta(-2n) \;=\; (-1)^{2n+1}\,(2n)!\,(-2n^{th} \text{ Laurent coefficient of } \frac{1}{e^t-1}) \;=\; 0$$

[3.6] Laurent expansion of $\frac{1}{e^t-1}$ We compute a few terms of the Laurent expansion near $t=0$:

$$\frac{1}{e^t-1} \;=\; \frac{1}{(1+t+t^2/2+t^3/6+\ldots)-1} \;=\; \frac{1}{t+t^2/2+t^3/6+\ldots} \;=\; \frac{1}{t}\cdot\frac{1}{1+t/2+t^2/6+\ldots}$$

$$=\; \frac{1}{t}\cdot\Big(1-(t/2+t^2/6+\ldots)+(t/2+t^2/6+\ldots)^2-(t/2+t^2/6+\ldots)^3+\ldots\Big)$$

by expanding the geometric series for $\frac{1}{1+(t/2+\ldots)}$. Ignoring t^4 and higher-order terms,

$$\frac{t}{e^t-1} \;=\; 1-(\frac{t}{2}+\frac{t^2}{6}+\frac{t^3}{24}) + \Big((\frac{t}{2})^2+2\cdot\frac{t}{2}\cdot\frac{t^2}{6}\Big) - (\frac{t}{2})^3+\ldots$$

$$=\; 1-\frac{1}{2}t+(-\frac{1}{6}+\frac{1}{4})t^2+(-\frac{1}{24}+\frac{1}{6}-\frac{1}{8})t^3+\ldots \;=\; 1-\frac{1}{2}t+\frac{1}{12}t^2+0\cdot t^3+\ldots$$

That is,

$$\frac{1}{e^t-1} \;=\; \frac{1}{t}-\frac{1}{2}+\frac{1}{12}t+0\cdot t^2+\ldots$$

[3.7] Residues of $\Gamma(s)$ Finally, we determine the residues of $\Gamma(s)$. Certainly

$$\Gamma(1) \;=\; \int_0^\infty t^1\, e^{-t}\,\frac{dt}{t} \;=\; \int_0^\infty e^{-t}\,dt \;=\; 1$$

From the functional equation $s\Gamma(s)=\Gamma(s+1)$, near $s=0$

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{1+\text{higher}}{s} \;=\; \frac{1}{s} + (\text{holomorphic at } s=0)$$

Thus, the residue at 0 is 1. Iterating the functional equation,

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{\Gamma(s+2)}{(s+1)s} \;=\; \frac{\Gamma(s+3)}{(s+2)(s+1)s} \;=\; \ldots \;=\; \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+2)(s+1)s}$$

Thus, the leading Laurent term at $s = -n$ is

$$\frac{1}{s+n} \cdot \frac{\Gamma(s+n+1)}{(s+n-1)\ldots(s+2)(s+1)s}\Big|_{s=-n} = \frac{1}{s+n} \cdot \frac{\Gamma(-n+n+1)}{(-n+n-1)\ldots(-n+2)(-n+1)(-n)}$$

$$= \frac{1}{s+n} \cdot \frac{1}{(-1)(-2)(-3)\ldots(-n+2)(-n+1)(-n)} = \frac{1}{s+n} \cdot \frac{(-1)^n}{n!}$$

That is, the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$ as claimed.

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4. *Appendix: Perron identity*

These contour-integral identities extract information from spectral identities and function-theoretic identities. *One* spectral identity is transformed into *another*, by a Fourier transform. Choices are made to heighten an *asymmetry*, wherein one side is seemingly elementary, and the other is whatever it must be.

[4.1] Heuristic The best-known identity starts from the *idea* that for $\sigma > 0$

$$\int_{\sigma-i\infty}^{\sigma-i\infty} \frac{X^s}{s}\, ds = \begin{cases} 1 & (\text{for } X > 1) \\ \\ 0 & (\text{for } 0 < X < 1) \end{cases} \qquad \text{(convergence?)}$$

The *idea* of the proof of this identity is that, for $X > 1$, the contour of integration slides indefinitely to the left, eventually vanishing, picking up the residue at $s = 0$, while for $0 < X < 1$, the countour slides indefinitely to the right, eventually vanishing, picking up *no* residues.

The *idea* of the application is that this identity can extract *counting* information from a meromorphic continuation of a Dirichlet series: for example, from

$$\sum_n \frac{a_n}{n^s} = f(s) \qquad \text{(left-hand side convergent for } \mathrm{Re}\, s > 1)$$

we would have

$$\sum_{n<X} a_n = \text{sum of residues of } X^s\, f(s)/s$$

That is, the *counting* function $\sum_{n<X} a_n$ is *extracted* from the analytic object $\sum_\lambda a_n/n^s$ by the contour integration. With f a logarithmic derivative, such as $f(s) = \zeta'(s)/\zeta(s)$, the poles of f are mostly the zeros of ζ.

However, the tails of these integrals are fragile.

[4.2] Simple precise assertion The elegant simplicity of the idea about moving lines of integration must be elaborated for correctness: for fixed $\sigma > 0$, for $T > 0$, we claim that

$$\int_{\sigma-iT}^{\sigma-iT} \frac{X^s}{s}\, ds = \begin{cases} 1 + O_\sigma(\frac{X^\sigma}{T\cdot\log X}) & (\text{for } X > 1) \\ \\ O_\sigma(\frac{X^\sigma}{T\cdot|\log X|}) & (\text{for } 0 < X < 1) \end{cases}$$

The proof is a precise form of the idea of sliding vertical contours. That is, for $X > 1$, consider the contour integral around the rectangle with *right* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $-B + iT$, $-B - iT$, with $B \to +\infty$. For $0 < X < 1$ consider the contour integral around the rectangle with *left* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $B + iT$, $B - iT$, with $B \to +\infty$.

For both $X > 1$ and $0 < X < 1$, the $\pm(B \pm iT)$ edge of the rectangle is dominated by

$$\int_{-T}^{T} \frac{e^{-B|\log X|}}{|B \pm it|}\,dt \;\ll\; T \cdot \frac{e^{-B|\log X|}}{B} \;\to\; 0 \qquad\qquad (\text{as } B \to +\infty)$$

in both cases, the top and bottom edges of the rectangle are dominated by

$$X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{|(\sigma \pm u) + iT|}\,du \;\ll\; X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{T}\,du \;\ll\; \frac{X^\sigma}{T \cdot |\log X|}$$

This proves the claim. Replacing X by e^X in the estimate gives the equivalent

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{e^{sX}}{s}\,ds \;=\; \begin{cases} 1 + O_\sigma(\frac{e^{\sigma X}}{T \cdot X}) & (\text{for } X > 0) \\ O_\sigma(\frac{e^{\sigma X}}{T \cdot |X|}) & (\text{for } X < 0) \end{cases}$$

[4.3] Hazards

When the quantity X above is summed, especially if the summation is over a set whose precise specifications are difficult, the denominators of the big-O error terms may blow up. In situations such as

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma + iT} \Big(\sum_j a_j\, e^{-sX_j}\Big) \frac{e^{sX}}{s}\,ds \;=\; \sum_{j\,:\,X_j < X} a_j \;+\; \sum_j a_j \cdot O_\sigma\Big(\frac{e^{\sigma(X - X_j)}}{T \cdot |X - X_j|}\Big)$$

the distribution of the values X_j has an obvious effect on the convergence of the error term.

[4.4] The other side of the equation

A desired and plausible conclusion such as

$$\lim_T \frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} f(s)\,\frac{e^{sX}}{s}\,ds \;=\; \big(\text{sum of } \mathrm{Res}_{s=\rho}\, f(s) \cdot \tfrac{e^{\rho X}}{\rho}\big)$$

summed over poles ρ of f in the left half-plane $\mathrm{Re}\,s < \sigma$, requires that the contour integrals over the other three sides of the rectangle with side $\sigma \pm iT$ go to 0, and that the tails of the vertical integral go to 0. The integral over the large rectangle will be evaluated with X large positive, so the decay condition applies to f to the *left*. The left side of the rectangle will go to 0 for large enough positive X when $f(s)$ has at worst exponential growth to the left, that is, when $f(s) \ll e^{-C \cdot |\mathrm{Re}\,s|}$ for *some* large-enough C and $\mathrm{Re}\,s \to -\infty$. The top and bottom are more fragile, since e^{sX}/s does not have strong decay vertically.

Not unexpectedly, the *poles* of f near $\sigma + iT$ may *bunch up* as T grows, so that a countour integral must be **threaded** between them, and the corresponding integral will be somewhat larger simply because of proximity to these poles. This contribution to vertical growth of f is significant in examples, and motivates alternatives.

[4.5] Variant identities

When e^{sX}/s is altered to help convergence of the integral against the *counting* aspect is inevitably altered. The proofs of variants follow the same straightforward line as above for the simplest case. Rather than replacing e^{sX}/s with e^{sX}/s^2, a better effect is achieved with $e^{sX}/s(s+1)$. In fact, for $\theta > 0$ and $1 \le \ell \in \mathbb{Z}$

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}\,ds \;=\; \begin{cases} \frac{1}{\ell!\theta^\ell}(1 - e^{-\theta X})^\ell + O_\sigma(\frac{e^{\sigma X}}{T^2 \cdot X}) & (\text{for } X > 0) \\ O_\sigma(\frac{e^{\sigma X}}{T^2 \cdot |X|}) & (\text{for } X < 0) \end{cases}$$

Indeed, the residues at the poles $0, -\theta, -2\theta, \ldots, -\ell\theta$ sum to

$$\frac{e^{0\cdot X}}{(0+\theta)(0+2\theta)\cdots(0+(\ell-1)\theta)(0+\ell\theta)} + \frac{e^{-\theta\cdot X}}{(-\theta+0)(-2\theta+\theta)\cdots(-\theta+(\ell-1)\theta)(-\theta+\ell\theta)}$$
$$+ \frac{e^{-2\theta\cdot X}}{(-2\theta+0)(-2\theta+\theta)\cdots(-2\theta+\ell\theta)} + \ldots + \frac{e^{-\ell\theta\cdot X}}{(-\ell\theta+0)(-\ell\theta+\theta)\cdots(-\ell\theta+(\ell-1)\theta)}$$
$$= \frac{1}{\ell!\,\theta^\ell} - \frac{e^{-\theta X}}{1!\,(\ell-1)!\,\theta^\ell} + \frac{e^{-2\theta X}}{2!\,(\ell-2)!\,\theta^\ell} + \ldots \pm \frac{e^{-\ell\theta X}}{\ell!\,0!\,\theta^\ell} \;=\; \frac{(1-e^{-\theta X})^\ell}{\ell!\,\theta^\ell}$$

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5. *Appendix: Poisson summation*

The simplest form of the Poisson summation formula is

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad\qquad \text{(for suitable functions } f\text{, with Fourier transform } \widehat{f}\text{)}$$

with Fourier transform

$$\text{Fourier transform of } f \;=\; \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$$

[5.1] The idea A good heuristic for the truth of the assertion of Poisson summation is the following. Given f a function on $\mathbb{R}$, form the *periodic* version of f

$$F(x) \;=\; \sum_{n\in\mathbb{Z}} f(x+n)$$

A periodic function should be represented by its *Fourier series*, so

$$F(x) \;=\; \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x} \int_0^1 F(x)\, e^{-2\pi i \ell x}\, dx$$

The Fourier *coefficients* of F expand to be seen as the Fourier *transform* of f:

$$\int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i\ell x}\, dx$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \widehat{f}(\ell)$$

Evaluating at 0, we should have

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; F(0) \;=\; \sum_{\ell\in\mathbb{Z}} \widehat{f}(\ell)$$

[5.2] What would it take to legitimize this? Certainly f must be of sufficient decay so that the integral for its Fourier transform is convergent, and so that summing its translates by $\mathbb{Z}$ is convergent. We'd want f to be continuous, probably differentiable, so that we can talk about pointwise values of F, and to make plausible the hope that the Fourier series of F converges to F pointwise. For f and several derivatives rapidly decreasing, the Fourier transform $\widehat{f}$ will be of sufficient decay so that its sum over $\mathbb{Z}$ does converge.

A simple sufficient hypothesis for convergence is that f be in the *Schwartz space* of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, that is,

$$\text{Schwartz space} \;=\; \{\text{smooth } f \;:\; \sup_x\,(1+x^2)^\ell |f^{(i)}(x)| < \infty \text{ for all } i,\ell\}$$

Representability of a periodic function by its Fourier series is a serious question, with several possible senses. The following section gives a result sufficient for the moment.

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6. *Appendix: pointwise convergence of Fourier series*

A special, self-contained argument gives a good-enough result for immediate purposes. [14]

Consider ($\mathbb{Z}$-)periodic functions on $\mathbb{R}$, that is, complex-valued functions f on $\mathbb{R}$ such that $f(x+n)=f(x)$ for all $x\in\mathbb{R}$, $n\in\mathbb{Z}$. For periodic f sufficiently nice so that integrals

$$\widehat{f}(n) \;=\; \int_0^1 f(x)\,e^{-2\pi i n x}\,dx \qquad\qquad (n^{th} \text{ Fourier coefficient of } f)$$

make sense, the **Fourier expansion** of f is

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{2\pi i n x} \qquad\qquad (\text{Fourier expansion of } f)$$

We want simple sufficient conditions on f and on points x_o so that

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{2\pi i n x_o} \qquad\qquad (\text{as convergent double sum of complex numbers})$$

Consider periodic *piecewise-C^o* [15] functions which are left-continuous and right-continuous [16] at any discontinuities.

[6.1] Theorem: For periodic f piecewise-C^o functions left-continuous and right-continuous at its discontinuities, for points x_o at which f is C^0 and *left-differentiable*[17] and *right-differentiable*, the Fourier series of f evaluated at x_o *converges* to $f(x)$:

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{2\pi i n x_o}$$

---

[14] The virtue of the argument here is mainly its immediacy and lack of prerequisites. However, this approach is inadequate in most other situations. For example, for many purposes, we want not mere pointwise convergence, but *uniform* pointwise convergence.

[15] A function is *piecewise-C^o* when it is C^o *except* for a discrete set of points, at which it may fail to be continuous.

[16] As usual, a function has a *left-continuous* at x_o if the limit of $f(x)$ as x approaches x_o from the *left* exists. Similarly, f is *right-continuous* if the limit approaching from the *right* exists. Note that there is no purpose in asking whether these limits are the value $f(x_o)$, since if they had that common value, then the function would be continuous at x_o, and the notion of one-sided continuity would be irrelevant.

[17] As usual, a function f is *left-differentiable* at x_o if the limit of $[f(x)-f(x_o)]/[x-x_o]$ exists as x approaches x_o *from the left*. Right-differentiability at x_o is similar. Admittedly, this is a clumsy notion, but is relevant to treatment of functions that are not entirely smooth, but not too badly behaved.

That is, for such functions, at such points, the Fourier series *represents* the function *pointwise*.

[6.2] Remark: The most notable missing conclusion in the theorem is *uniform* pointwise convergence. For more serious applications, pointwise convergence not known to be uniform is often useless.

Proof: First, treat the special case $x_o = 0$ and $f(0) = 0$. Representability of $f(0)$ by the Fourier series is the assertion that

$$0 \;=\; f(0) \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)\, e^{2\pi i n\cdot 0} \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)$$

Substituting the defining integral for the Fourier coefficients:

$$\sum_{-M\le n<N} \widehat{f}(n) \;=\; \sum_{-M\le n<N} \int_0^1 f(u)\, e^{-2\pi i n u}\, du$$

$$= \int_0^1 \sum_{-M\le n<N} f(u)\, e^{-2\pi i n u}\, du \;=\; \int_0^1 f(u)\cdot \frac{e^{2\pi i M u} - e^{-2\pi i N u}}{1 - e^{-2\pi i u}}\, du$$

To prove the representability of $f(0)$ by the Fourier series, we will show that

$$\lim_{\ell\to\pm\infty} \int_0^1 \frac{f(u)\cdot e^{-2\pi i \ell u}}{1 - e^{-2\pi i u}}\, du \;=\; 0$$

We claim that the function

$$g(x) \;=\; \frac{f(x)}{1 - e^{-2\pi i x}}$$

is piecewise-C^o, and left-continuous and right-continuous at discontinuities. The only issue is at integers, and by the periodicity it suffices to prove continuity at 0. To prove continuity at 0, we can forget about periodicity for a moment, and write

$$\frac{f(x)}{1 - e^{-2\pi i x}} \;=\; \frac{f(x)}{x}\cdot \frac{x}{1 - e^{-2\pi i x}}$$

The two-sided limit

$$\lim_{x\to 0} \frac{x}{1 - e^{-2\pi i x}} \;=\; \frac{d}{dx}\Big|_{x=0} \frac{x}{1 - e^{-2\pi i x}} \;=$$

exists, by differentiability. Similarly, we have left and right limits

$$\lim_{x\to 0^-} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

and

$$\lim_{x\to 0^+} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

by the one-sided differentiability of f. Combining these two one-sided limits, both limits

$$\lim_{x\to 0^-} \frac{f(x)}{1 - e^{-2\pi i x}} \qquad\qquad \lim_{x\to 0^+} \frac{f(x)}{1 - e^{-2\pi i x}}$$

exist, proving the one-sided continuity of g at 0.

We want to prove an easy instance of a *Riemann-Lebesgue lemma*, namely, that that the Fourier coefficients of a periodic, piecewise-C^o function g, with left and right limits at discontinuities, go to 0.

The essential property of g is that on $[0,1]$ it is approximable by *step functions* [18] in the sense [19] that, given $\varepsilon > 0$ there is a *step function* $s(x)$ such that

$$\int_0^1 |s(x) - g(x)|\, dx \;<\; \varepsilon$$

With such s,

$$|\widehat{s}(n) - \widehat{g}(n)| \;\le\; \int_0^1 |s(u) - g(u)|\, du \;<\; \varepsilon \qquad \text{(for all } \varepsilon > 0)$$

Thus, it suffices to prove that Fourier coefficients of *step functions* go to 0, and, thus, that Fourier coefficients of *characteristic functions of intervals* go to 0. The latter is an easy computation:

$$\int_a^b e^{-2\pi i\ell x}\, dx \;=\; [\frac{e^{-2\pi i\ell x}}{-2\pi i\ell}]_a^b \;=\; \frac{e^{-2\pi i\ell b} - e^{-2\pi i\ell a}}{-2\pi i\ell} \;\longrightarrow\; 0 \qquad \text{(as } \ell \to \pm\infty)$$

This proves a RIemann-Lebesgue lemma for any function L^1-approximable by step functions. Thus, the Fourier coefficients of g go to 0, proving that the Fourier series of f converges to $f(0)$ when f is C^1 at 0.

For arbitrary $x_o \in [0,1]$, replacing f by $f - f(x_o)$ reduces to the case that $f(x_o) = 0$. Note that the continuity of f at x_o is necessary for this reduction. Replacing $f(x)$ by $\varphi(x) = f(x + x_o)$ reduces to the case x_o, noting that the effect on the Fourier expansion is to multiply the Fourier coefficients by constants:

$$\widehat{\varphi}(n) \;=\; \int_0^1 f(x + x_o)\, e^{-2\pi i n x}\, dx \;=\; \int_{x_o}^{1+x_o} f(x)\, e^{-2\pi i n(x - x_o)}\, dx \;=\; e^{2\pi i n x_o} \int_{x_o}^{1+x_o} f(x)\, e^{-2\pi i n x}\, dx$$

For any $\mathbb{Z}$-periodic function h, using the periodicity, such a shifted integral can be converted back to an integral over $[0,1]$:

$$\int_{x_o}^{1+x_o} h(x)\, dx \;=\; \int_{x_o}^1 h(x)\, dx \;+\; \int_1^{1+x_o} h(x)\, dx \;=\; \int_{x_o}^1 h(x)\, dx \;+\; \int_0^{x_o} h(x+1)\, dx$$

$$= \int_{x_o}^1 h(x)\, dx \;+\; \int_0^{x_o} h(x)\, dx \;=\; \int_0^1 h(x)\, dx$$

Thus,

$$\widehat{\varphi}(n) \;=\; e^{2\pi i n x_o} \int_{x_o}^{1+x_o} f(x)\, e^{-2\pi i n x}\, dx \;=\; e^{2\pi i n x_o} \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \;=\; e^{2\pi i n x_o}\, \widehat{f}(n)$$

Thus, the result at $x_o = 0$ for $\varphi(x) = f(x + x_o)$ gives the general case:

$$f(x_o) \;=\; \varphi(0) \;=\; \sum_n \widehat{\varphi}(n) \;=\; \sum_n \widehat{f}(n)\, e^{2\pi i n x_o}$$

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[18] As usual, a *step function* φ is a function that assumes only finitely-many values, and is of the form

$$\varphi(x) \;=\; \begin{cases} y_1 & \text{(for } x_0 \le x < x_1) \\ y_2 & \text{(for } x_1 \le x < x_2) \\ \ldots & \\ y_{k-1} & \text{(for } x_{k-2} \le x < x_{k-1}) \\ y_k & \text{(for } x_{k-1} \le x < x_k) \end{cases}$$

for some collection of intervals $[x_0, x_1), [x_1, x_2), \ldots, [x_{k-1})$ and corresponding values $y_1, \ldots, y_k$.

[19] In standard language, this assertion of approximability is that continuous functions on $[0,1]$ can be approximated by step functions *in L^1-norm*. The L^1 norm $|f|_{L^1}$ of a function on $[0,1]$ is simply the integral of the absolute value: $\int_0^1 |f(x)|\, dx$.

Thus, we have proven that piecewise-C^1 functions with left and right limits at discontinuities are pointwise represented by their Fourier series at points where they're differentiable. ///

[6.3] Remark: In fact, the argument above shows that for a function f and point x_o such that

$$\frac{f(x) - f(x_o)}{e^{2\pi i x} - e^{2\pi i x_o}}$$

is in $L^1[0,1]$, the Fourier series at x_o converges to $f(x_o)$. This holds, for example, when f satisfies a *Lipschitz condition*

$$|f(x) - f(x_o)| \;\le\; |x - x_o|^\alpha \qquad\qquad (\text{as } x \to x_o, \text{ with some } \alpha > 0)$$

and is in $L^1[0,1]$.

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