(*September 15, 2019*)

Guinand's explicit formula

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We take for granted the basic analytical properties of Riemann's zeta function and the Gamma function. In particular, with $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ the Gamma factor for $\zeta(s)$, and $\xi(s) = \Gamma_{\mathbb{R}}(s)\cdot\zeta(s)$, the functional equation is $\xi(1-s) = \xi(s)$. The completed zeta function ξ has poles only at $s = 0, 1$, and these simple. We use the normalization of Fourier transform [1]

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}} e^{ix\xi}\, g(x)\, dx$$

[0.1] Theorem: *(Guinand)* For $g \in C_c^\infty(\mathbb{R})$, letting $\rho = \frac{1}{2} + i\gamma$ whether or not ρ is on the cricial line,

$$\sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\gamma) \;-\; \widehat{g}(i/2) - \widehat{g}(-i/2)$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \Big(\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\big(\tfrac{1}{2}+it\big) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\big(\tfrac{1}{2}-it\big)\Big)\,\widehat{g}(t)\, dt \;-\; \sum_{p^m} \frac{\log p}{\sqrt{p^m}}\Big(g(m\log p) + g(-m\log p)\Big)\Big)$$

Proof: The idea is in [Guinand 1947], and there is essentially only one route to take. [2] Consider

$$I \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{\xi'(s)}{\xi(s)}\,\widehat{g}(t)\, ds \qquad\qquad (\text{with } s = \tfrac{1}{2}+it \text{ whether or not } \mathrm{Re}\,(s) = \tfrac{1}{2})$$

The integral makes sense because $\widehat{g}(s)$ is *entire*. It converges because $\widehat{g}$ is rapidly decreasing in $\mathrm{Re}\,(s)$, for every fixed $\mathrm{Im}\,(s)$. The logarithmic derivative has simple poles at the zeros and poles of $\xi(s)$ with residues the order of vanishing, namely, the two simple poles at $s = 0, 1$, trivial zeros at $s = -2, -4, -6, \ldots$, and non-trivial zeros ρ in the critical strip. In terms of $t = (s - \frac{1}{2})/i$, these are at $\pm i/2$. Moving the contour [3] to $\mathrm{Re}\,(s) = -\varepsilon$ captures $t = \pm i/2$ and the non-trivial zeros ρ, written similarly as $\rho = \frac{1}{2} + i\gamma$, whether or not we know that $\mathrm{Re}\,(\gamma) = \frac{1}{2}$:

$$I \;=\; -\widehat{g}(i/2) - \widehat{g}(i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\gamma) + \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=-\varepsilon} \frac{\xi'(s)}{\xi(s)}\,\widehat{g}(t)\, ds$$

From the functional equation, $\frac{\xi'}{\xi}(s) \;=\; -\frac{\xi'}{\xi}(1-s)$, and the integral on $\mathrm{Re}\,(s) = -\varepsilon$ is equal to an integral on $\mathrm{Re}\,(s) = 1+\varepsilon$:

$$I \;=\; -\widehat{g}(i/2) - \widehat{g}(-i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\gamma) - \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)}\,\widehat{g}(t)\, ds$$

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[1] Often, there might be a 2π somewhere, or a different sign in the exponent, but those details are irrelevant. The present choice is for convenience in the situation at hand.

[2] Compare [Weil 1952/1972], [Rudnick-Sarnak 1996] page 277 ff., or [Iwaniec-Kowalski 2004] page 508 ff.

[3] To legitimize this contour move, we should identify a sequence $T_j \to +\infty$ such that the integrals along $[-\varepsilon + iT_j, 1+\varepsilon+iT_j]$ and $[-\varepsilon - iT_j, 1+\varepsilon-iT_j]$ go to 0. Since $\widehat{g}(t)$ is rapidly decreasing, this is not delicate. However, it still does need an asymptotic estimate of the number of zeros of $\zeta(s)$ to height T, which comes from the functional equation, Stirling-Laplace asymptotics for $\Gamma(s)$, and Hadamard's theorem on growth versus zeros of entire functions.

Rearranging,

$$\sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\gamma) - \widehat{g}(i/2) - \widehat{g}(-i/2) \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{\xi'(s)}{\xi(s)}\,\widehat{g}(t)\;ds + \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)}\,\widehat{g}(t)\;ds$$

Separating the archimedean factors,

$$\frac{\xi'(s)}{\xi(s)} \;=\; \frac{\zeta'(s)}{\zeta(s)} + \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)}$$

The contour integrals of the archimedean part can be shifted to $\mathrm{Re}\,(s) = \frac{1}{2}$ without picking up any residues, giving

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)}\,\widehat{g}(t)\;ds + \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\Gamma'_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(1-s)}\,\widehat{g}(t)\;ds \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(t)\Big(\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\big(\tfrac{1}{2}+it\big) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\big(\tfrac{1}{2}-it\big)\Big)\;dt$$

The logarithmic derivative of the finite-prime part, in a right half-plane, is [4]

$$\frac{\zeta'(s)}{\zeta(s)} \;=\; -\sum_{p^m\geq 1} \frac{\log p}{p^{ms}}$$

giving

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{\zeta'(s)}{\zeta(s)}\,\widehat{g}(t)\;ds \;=\; -\sum_{p,m} \log(p)\,\frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=1+\varepsilon} \frac{1}{p^{ms}}\cdot\widehat{g}(t)\;ds$$

Moving the contour to $\mathrm{Re}\,(s) = \frac{1}{2}$ gives $p^{ms} = \sqrt{p^m}\cdot e^{itm\log p}$, and with $ds = d(\frac{1}{2}+it) = i\,dt$, the integral becomes

$$-\sum_{p,m} \frac{\log p}{\sqrt{p^m}}\,\frac{1}{2\pi}\int_{\mathbb{R}} e^{-itm\log p}\,\widehat{g}(t)\;dt$$

Here, Fourier inversion is

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(t)\,e^{-itx}\;dt \;=\; g(x)$$

Thus, by Fourier inversion,

$$-\sum_{p,m} \log p\,\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{p^m}}\cdot e^{-itm\log p}\,\widehat{g}(t)\;dt \;=\; -\sum_{p,m} \frac{\log p}{\sqrt{p^m}}\;g(m\log p)$$

The $1-s$ term is treated similarly, giving the assertion of the theorem. ///

[Guinand 1947] A. P. Guinand, *Some Fourier transforms in prime-number theory*, Quart. J. Math., Oxford **18** (1947), 53-64.

[Iwaniec-Kowalski 2004] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Coll. Publ., 2004.

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[4] A different notational choice is to sum over all integers, rather than over powers of primes, and use the *von Mangoldt* function

$$\Lambda(n) \;=\; \begin{cases} \log p & (\text{for } n = p^k,\ \text{prime } p,\ 1 \leq k \in Z) \\[2ex] 0 & (\text{otherwise}) \end{cases}$$

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