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Pointwise convergence of Fourier series

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A special, self-contained argument gives a good-enough result for immediate purposes. [1]

Consider ($\mathbb{Z}$-)periodic functions on $\mathbb{R}$, that is, complex-valued functions f on $\mathbb{R}$ such that $f(x+n) = f(x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$. For periodic f sufficiently nice so that integrals

$$\widehat{f}(n) \;=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \qquad\qquad (n^{th} \text{ Fourier coefficient of } f)$$

make sense, the **Fourier expansion** of f is

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x} \qquad\qquad (\text{Fourier expansion of } f)$$

We want simple sufficient conditions on f and on points x_o so that

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o} \qquad\qquad (\text{as convergent double sum of complex numbers})$$

Consider periodic *piecewise-C^o* [2] functions which are left-continuous and right-continuous [3] at any discontinuities.

[0.1] Theorem: For periodic f piecewise-C^o functions left-continuous and right-continuous at its discontinuities, for points x_o at which f is C^0 and *left-differentiable* [4] and *right-differentiable*, the Fourier series of f evaluated at x_o *converges* to $f(x)$:

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o}$$

That is, for such functions, at such points, the Fourier series *represents* the function *pointwise*.

[0.2] Remark: The most notable missing conclusion in the theorem is *uniform* pointwise convergence. For more serious applications, pointwise convergence not known to be uniform is often useless.

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[1] The virtue of the argument here is mainly its immediacy and lack of prerequisites. However, this approach is inadequate in most other situations. For example, for many purposes, we want not mere pointwise convergence, but *uniform* pointwise convergence.

[2] A function is *piecewise-C^o* when it is C^o *except* for a discrete set of points, at which it may fail to be continuous.

[3] As usual, a function has a *left-continuous* at x_o if the limit of $f(x)$ as x approaches x_o from the *left* exists. Similarly, f is *right-continuous* if the limit approaching from the *right* exists. Note that there is no purpose in asking whether these limits are the value $f(x_o)$, since if they had that common value, then the function would be continuous at x_o, and the notion of one-sided continuity would be irrelevant.

[4] As usual, a function f is *left-differentiable* at x_o if the limit of $[f(x) - f(x_o)]/[x - x_o]$ exists as x approaches x_o *from the left.* Right-differentiability at x_o is similar. Admittedly, this is a clumsy notion, but is relevant to treatment of functions that are not entirely smooth, but not too badly behaved.

Proof: First, treat the special case $x_o = 0$ and $f(0) = 0$. Representability of $f(0)$ by the Fourier series is the assertion that

$$0 \;=\; f(0) \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)\, e^{2\pi i n\cdot 0} \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)$$

Substituting the defining integral for the Fourier coefficients:

$$\sum_{-M\le n<N} \widehat{f}(n) \;=\; \sum_{-M\le n<N} \int_0^1 f(u)\, e^{-2\pi i n u}\, du$$

$$=\; \int_0^1 \sum_{-M\le n<N} f(u)\, e^{-2\pi i n u}\, du \;=\; \int_0^1 f(u)\cdot \frac{e^{2\pi i M u} - e^{-2\pi i N u}}{1-e^{-2\pi i u}}\, du$$

To prove the representability of $f(0)$ by the Fourier series, we will show that

$$\lim_{\ell\to\pm\infty} \int_0^1 \frac{f(u)\cdot e^{-2\pi i \ell u}}{1-e^{-2\pi i u}}\, du \;=\; 0$$

We claim that the function

$$g(x) \;=\; \frac{f(x)}{1-e^{-2\pi i x}}$$

is piecewise-C^o, and left-continuous and right-continuous at discontinuities. The only issue is at integers, and by the periodicity it suffices to prove continuity at 0. To prove continuity at 0, we can forget about periodicity for a moment, and write

$$\frac{f(x)}{1-e^{-2\pi i x}} \;=\; \frac{f(x)}{x}\cdot\frac{x}{1-e^{-2\pi i x}}$$

The two-sided limit

$$\lim_{x\to 0} \frac{x}{1-e^{-2\pi i x}} \;=\; \frac{d}{dx}\Big|_{x=0} \frac{x}{1-e^{-2\pi i x}} \;=\;$$

exists, by differentiability. Similarly, we have left and right limits

$$\lim_{x\to 0^-} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

and

$$\lim_{x\to 0^+} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

by the one-sided differentiability of f. Combining these two one-sided limits, both limits

$$\lim_{x\to 0^-} \frac{f(x)}{1-e^{-2\pi i x}} \qquad\qquad \lim_{x\to 0^+} \frac{f(x)}{1-e^{-2\pi i x}}$$

exist, proving the one-sided continuity of g at 0.

We want to prove an easy instance of a *Riemann-Lebesgue lemma*, namely, that that the Fourier coefficients of a periodic, piecewise-C^o function g, with left and right limits at discontinuities, go to 0.

The essential property of g is that on $[0,1]$ it is approximable by *step functions*[5] in the sense[6] that, given $\varepsilon > 0$ there is a *step function* $s(x)$ such that

$$\int_0^1 |s(x) - g(x)|\,dx \;<\; \varepsilon$$

With such s,

$$|\widehat{s}(n) - \widehat{g}(n)| \;\le\; \int_0^1 |s(u) - g(u)|\,du \;<\; \varepsilon \qquad\qquad (\text{for all } \varepsilon > 0)$$

Thus, it suffices to prove that Fourier coefficients of *step functions* go to 0, and, thus, that Fourier coefficients of *characteristic functions of intervals* go to 0. The latter is an easy computation:

$$\int_a^b e^{-2\pi i\ell x}\,dx \;=\; [\frac{e^{-2\pi i\ell x}}{-2\pi i\ell}]_a^b \;=\; \frac{e^{-2\pi i\ell b} - e^{-2\pi i\ell a}}{-2\pi i\ell} \;\longrightarrow\; 0 \qquad\qquad (\text{as } \ell \to \pm\infty)$$

This proves a RIemann-Lebesgue lemma for any function L^1-approximable by step functions. Thus, the Fourier coefficients of g go to 0, proving that the Fourier series of f converges to $f(0)$ when f is C^1 at 0.

For arbitrary $x_o \in [0,1]$, replacing f by $f - f(x_o)$ reduces to the case that $f(x_o) = 0$. Note that the continuity of f at x_o is necessary for this reduction. Replacing $f(x)$ by $\varphi(x) = f(x + x_o)$ reduces to the case x_o, noting that the effect on the Fourier expansion is to multiply the Fourier coefficients by constants:

$$\widehat{\varphi}(n) \;=\; \int_0^1 f(x + x_o)\,e^{-2\pi inx}\,dx \;=\; \int_{x_o}^{1+x_o} f(x)\,e^{-2\pi in(x-x_o)}\,dx \;=\; e^{2\pi inx_o}\int_{x_o}^{1+x_o} f(x)\,e^{-2\pi inx}\,dx$$

For any $\mathbb{Z}$-periodic function h, using the periodicity, such a shifted integral can be converted back to an integral over $[0,1]$:

$$\int_{x_o}^{1+x_o} h(x)\,dx \;=\; \int_{x_o}^1 h(x)\,dx \;+\; \int_1^{1+x_o} h(x)\,dx \;=\; \int_{x_o}^1 h(x)\,dx \;+\; \int_0^{x_o} h(x+1)\,dx$$

$$= \int_{x_o}^1 h(x)\,dx \;+\; \int_0^{x_o} h(x)\,dx \;=\; \int_0^1 h(x)\,dx$$

Thus,

$$\widehat{\varphi}(n) \;=\; e^{2\pi inx_o}\int_{x_o}^{1+x_o} f(x)\,e^{-2\pi inx}\,dx \;=\; e^{2\pi inx_o}\int_0^1 f(x)\,e^{-2\pi inx}\,dx \;=\; e^{2\pi inx_o}\,\widehat{f}(n)$$

Thus, the result at $x_o = 0$ for $\varphi(x) = f(x + x_o)$ gives the general case:

$$f(x_o) \;=\; \varphi(0) \;=\; \sum_n \widehat{\varphi}(n) \;=\; \sum_n \widehat{f}(n)\,e^{2\pi inx_o}$$

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[5] As usual, a *step function* φ is a function that assumes only finitely-many values, and is of the form

$$\varphi(x) \;=\; \begin{cases} y_1 & (\text{for } x_0 \le x < x_1) \\ y_2 & (\text{for } x_1 \le x < x_2) \\ \ldots & \\ y_{k-1} & (\text{for } x_{k-2} \le x < x_{k-1}) \\ y_k & (\text{for } x_{k-1} \le x < x_k) \end{cases}$$

for some collection of intervals $[x_0, x_1), [x_1, x_2), \ldots, [x_{k-1})$ and corresponding values $y_1, \ldots, y_k$.

[6] In standard language, this assertion of approximability is that continuous functions on $[0,1]$ can be approximated by step functions *in L^1-norm.* The L^1 norm $|f|_{L^1}$ of a function on $[0,1]$ is simply the integral of the absolute value: $\int_0^1 |f(x)|\,dx$.

Thus, we have proven that piecewise-C^1 functions with left and right limits at discontinuities are pointwise represented by their Fourier series at points where they're differentiable. ///

[0.3] Remark: In fact, the argument above shows that for a function f and point x_o such that

$$\frac{f(x) - f(x_o)}{e^{2\pi i x} - e^{2\pi i x_o}}$$

is in $L^1[0,1]$, the Fourier series at x_o converges to $f(x_o)$. This holds, for example, when f satisfies a *Lipschitz condition*

$$|f(x) - f(x_o)| \;\le\; |x - x_o|^\alpha \qquad\qquad (\text{as } x \to x_o, \text{ with some } \alpha > 0)$$

and is in $L^1[0,1]$.
