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The Gamma function

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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1. *Basics involving* $\Gamma(s)$, $B(\alpha,\beta)$, *etc.*

We take the Gamma function $\Gamma(s)$ to be defined by Euler's integral

$$\Gamma(s) \;=\; \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}(s)>0)$$

Integration by parts proves the functional equation

$$\Gamma(s+1) \;=\; s\cdot\Gamma(s)$$

For $0<s\in\mathbb{Z}$, this relation and induction show the connection to *factorials*,

$$\Gamma(n) \;=\; (n-1)! \qquad\qquad (\text{for } n=1,2,\ldots)$$

From the functional equation, we get a meromorphic continuation of $\Gamma(s)$ to the entire complex plane, except for poles at non-positive integers $-n$. The poles are *simple*, with residue $(-1)^n/n!$ at $-n$.

The identity

$$\int_0^\infty t^s\, e^{-ty}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{y^s} \qquad\qquad (\text{for } y>0 \text{ and } \mathrm{Re}(s)>0)$$

for $y>0$ first follows for $\mathrm{Re}(s)>0$ by replacing t by t/y in the integral. Then

$$\int_0^\infty t^s\, e^{-tz}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{z^s} \qquad\qquad (\text{for } \mathrm{Re}(z)>0 \text{ and } \mathrm{Re}(s)>0)$$

by complex analysis, since both sides are holomorphic in s and agree on the positive reals. This identity allows non-obvious evaluation of a Fourier transform. Namely, let

$$f(x) \;=\; \begin{cases} x^\alpha\cdot e^{-x} & (\text{for } x>0) \\ 0 & (\text{for } x<0)\end{cases}$$

For $\mathrm{Re}(\alpha)>-1$ this function is locally integrable at 0, and in any case is of rapid decay at infinity. We can compute its Fourier transform:

$$\int_{\mathbb{R}} e^{-2\pi i\xi x}\, f(x)\, dx \;=\; \int_0^\infty e^{-2\pi i\xi x}\, x^{\alpha+1}\, e^{-x}\, \frac{dx}{x} \;=\; \int_0^\infty x^{\alpha+1}\, e^{-x(1+2\pi i\xi)}\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha+1)}{(1+2\pi i\xi)^{\alpha+1}}$$

Further, Fourier inversion gives the non-obvious

$$\int_{\mathbb{R}} e^{2\pi i\xi x}\, \frac{1}{(1+2\pi i\xi)^{\alpha+1}}\, d\xi \;=\; \frac{1}{\Gamma(\alpha+1)}\cdot \begin{cases} x^\alpha\cdot e^{-x} & (\text{for } x>0) \\ 0 & (\text{for } x<0)\end{cases}$$

For $\alpha \in \mathbb{Z}$, the same conclusion can be reached by evaluation by residues. Next, we recall the argument that expresses Euler's beta integral in terms of gamma functions, as

$$B(a,b) \;=\; \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Indeed, replacing x by $\frac{t}{t+1} = 1 - \frac{1}{t+1}$ in the integral gives

$$\int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \int_0^\infty \Big(\frac{t}{t+1}\Big)^{a-1}\,\Big(1-\frac{t}{t+1}\Big)^{b-1}\,\frac{dt}{(t+1)^2} \;=\; \int_0^\infty t^a\,\Big(\frac{1}{t+1}\Big)^{a+b}\,\frac{dt}{t}$$

Use the gamma identity in the form

$$\Big(\frac{1}{t+1}\Big)^s \;=\; \frac{1}{\Gamma(s)}\int_0^\infty e^{-u(t+1)}\,u^s\,\frac{du}{u}$$

to rewrite the beta integral further as

$$\frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^{a+b}\,t^a\,e^{-u(t+1)}\,\frac{du}{u}\frac{dt}{t} \;=\; \frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^b\,t^a\,e^{-u}\,e^{-t}\,\frac{dt}{t}\frac{du}{u} \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

as claimed. ///

A similar integral, with one more factor, is Euler's integral for **hypergeometric functions**,

$$F(\alpha,\beta,\gamma;z) \;=\; \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 x^{\beta-1}\,(1-x)^{\gamma-\beta-1}\,(1-xz)^{-\alpha}\,dx$$

where B is the beta function

$$B(a,b) \;=\; \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

This F is the ${}_2F_1$ hypergeometric function, whose *series* definition is

$$F(\alpha,\beta,\gamma;z) \;=\; 1 + \frac{a\,b}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \;=\; \sum_{n=0}^\infty \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

The notation $(a)_n$ is the *Pockhammer* symbol.

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2. $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin\pi s$

Take $0 < \mathrm{Re}(s) < 1$ for convergence of both integrals, and compute

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty\int_0^\infty u^s\,e^{-u}\cdot v^{1-s}\,e^{-v}\,\frac{du}{u}\,\frac{dv}{v} \;=\; \int_0^\infty\int_0^\infty u\,e^{-u(1+v)}\,v^{1-s}\,\frac{du}{u}\,\frac{dv}{v}$$

by replacing v by uv. Replacing u by $u/(1+v)$ (another instance of the basic *gamma identity*) and noting that $\Gamma(1) = 1$ gives

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv$$

Replace the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of v^{-s} given by $(e^{2\pi i}v)^{-s} = e^{-2\pi i s}v^{-s}$, integrate from $+\infty$ to $\varepsilon > 0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of v^{-s}. For $\mathrm{Re}(-s) > -1$ the integral around the little circle goes to 0 as $\varepsilon \to 0$. Thus,

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \lim_{\varepsilon\to 0} \frac{1}{1-e^{-2\pi i s}} \int_{H_\varepsilon} \frac{v^{-s}}{1+v}\,dv$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\mathrm{Re}(-s) < 0$. Thus, this integral is equal to the limit as $R \to +\infty$ and $\varepsilon \to 0$ of the integral

from R to ε
from ε clockwise back to ε
from ε to R
from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \frac{2\pi i}{1-e^{-2\pi i s}}\cdot(e^{\pi i})^{-s} \;=\; \frac{2\pi i}{e^{\pi i s}-e^{-\pi i s}} \;=\; \frac{\pi}{\sin \pi s}$$

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3. *Duplication:* $\Gamma(s)\cdot\Gamma(s+\frac{1}{2}) \;=\; 2^{1-2s}\cdot\sqrt{\pi}\cdot\Gamma(2s)$

In many regards, this well-known identity is the most conceptually obscure in this whole discussion, at least insofar as the integral representation of the gamma function does not suggest the truth of the relation.

From the Eulerian integral definition,

$$\Gamma(s)\cdot\Gamma(s+\tfrac{1}{2}) \;=\; \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}\cdot\int_0^\infty e^{-u}\,u^{s+\frac{1}{2}}\,\frac{du}{u} \;=\; \int_0^\infty\int_0^\infty e^{-t-u}\,(tu)^s\,u^{\frac{1}{2}}\,\frac{dt}{t}\,\frac{du}{u}$$

Replace t by t^2 and u by u^2, giving

$$4\int_0^\infty\int_0^\infty e^{-t^2-u^2}\,(tu)^{2s-1}\,u\,dt\,du$$

To symmetrize this expression, add it to itself, with the roles of t and u interchanged, and divide by 2, obtaining

$$2\int_0^\infty\int_0^\infty e^{-t^2-u^2}\,(tu)^{2s-1}\,(t+u)\,dt\,du$$

We will make the change of variables $x = t^2+u^2$ and $y = 2tu$. Thus,

$$t+u \;=\; \sqrt{x+y} \qquad\qquad t-u \;=\; \sqrt{x-y}$$

and

$$t \;=\; \tfrac{1}{2}\cdot\Big(\sqrt{x+y}+\sqrt{x-y}\Big) \qquad u \;=\; \tfrac{1}{2}\cdot\Big(\sqrt{x+y}-\sqrt{x-y}\Big)$$

which suggests that we should be sure to have $x - y$, and, to avoid ambiguity, have $t-u>0$ as well. Thus, take twice the integral over $0<u<t$, namely,

$$4\int_0^\infty\int_0^t e^{-t^2-u^2}\,(tu)^{2s-1}\,(t+u)\,du\,dt$$

The change of measure is

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{4\sqrt{x+y}} + \frac{1}{4\sqrt{x-y}} & \frac{1}{4\sqrt{x+y}} - \frac{1}{4\sqrt{x-y}} \\ \frac{1}{4\sqrt{x+y}} - \frac{1}{4\sqrt{x-y}} & \frac{1}{4\sqrt{x+y}} + \frac{1}{4\sqrt{x-y}} \end{vmatrix} = \frac{1}{4} \cdot \frac{1}{\sqrt{x^2-y^2}}$$

Thus, the integral becomes

$$\int_0^\infty \int_0^x e^{-x} \left(\frac{y}{2}\right)^{2s-1} \frac{1}{\sqrt{x-y}}\, dy\, dx \;=\; 2^{1-2s} \int_0^\infty e^{-x}\, x^{2s-\frac{1}{2}}\, dx \cdot \int_0^1 y^{2s-1}\,(1-y)^{-\frac{1}{2}}\, dy$$

$$= \; 2^{1-2s}\, \Gamma(2s+\tfrac{1}{2}) \cdot \int_0^1 y^{2s-1}\,(1-y)^{-\frac{1}{2}}\, dy$$

by replacing y by xy. The integral in y is a beta integral

$$\int_0^1 y^{2s-1}\,(1-y)^{-\frac{1}{2}}\, dy \;=\; \int_0^1 y^{2s-1}\,(1-y)^{\frac{1}{2}-1}\, dy \;=\; B(2s, \tfrac{1}{2}) \;=\; \frac{\Gamma(2s)\,\Gamma(\frac{1}{2})}{\Gamma(2s+\frac{1}{2})}$$

Thus, the whole is

$$2^{1-2s}\, \Gamma(2s+\tfrac{1}{2}) \cdot \frac{\Gamma(2s)\,\Gamma(\frac{1}{2})}{\Gamma(2s+\frac{1}{2})} \;=\; 2^{1-2s}\, \Gamma(2s) \cdot \sqrt{\pi}$$

In summary, we have Legendre's duplication formula,

$$\Gamma(s) \cdot \Gamma(s+\tfrac{1}{2}) \;=\; \sqrt{\pi}\, 2^{1-2s}\, \Gamma(2s)$$
