05f. Counting zeros of $\zeta(s)$

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We count zeros of $\zeta(s)$ in the critical strip $0 \leq \text{Re}(s) \leq 1$ using the argument principle and the Laplace-Stirling asymptotic

$$\Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1) \quad \text{(in } \text{Re}(s) \geq \delta > 0 \text{ as } |s| \to \infty)$$

The convergent Euler product shows that there are no zeros in $\text{Re}(s) > 1$. The functional equation $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s)$ shows that the only zeros of $\zeta(s)$ in $\text{Re}(s) < 0$ are where $\Gamma(\frac{s}{2})$ has poles, namely, negative even integers. These are the trivial zeros of $\zeta(s)$. The counting function of interest is

$$N(T) = \text{number of zeros of } \zeta(s) \text{ in } 0 \leq \text{Im}(s) \leq T \text{ and } 0 \leq \text{Re}(s) \leq 1$$

For any fixed $1 < 2 \in \mathbb{R}$, with $2 \notin \mathbb{Z}$, by the argument principle\footnote{From [Backlund 1914, 1916, 1918]. See also [Titchmarsh/Heath-Brown 1951/1989] pages 212-213. [Backlund 1916] was a thesis done under E. Lindelöf’s supervision.} \[0.1\]

\[ N(T) = \frac{1}{2 \pi i} \int_{R_T} \frac{\zeta'(s)}{\zeta(s)} \, ds + O(1) \]

where $R_T$ is the rectangle connecting $2 \pm iT$ and $1 - 2 \pm iT$, traversed counter-clockwise, deformed slightly to skirt any zeros of $\zeta(s)$. The division by 2 takes into account the double-counting of zeros off the real interval $[0,1]$, and the $O(1)$ term accounts for the pole at $s = 1$. \[0.2\] The following is an estimate of this integral, giving the leading terms in an asymptotic in $T$.

\[ \text{[0.1] Theorem: } N(T) = \frac{1}{2 \pi} \cdot T \log T - \frac{\log 2 \pi e}{2 \pi} \cdot T + O(\log T) = \frac{1}{2 \pi} T \log \frac{T}{2 \pi e} + O(\log T) \]

\[ \text{[0.2] Remark: } \text{The vertical asymptotics of } \Gamma(s) \text{ completely determine the leading terms of the asymptotic expansion, by a direct computation which determines the constants.} \]

\[ \text{Proof: } \text{Using the functional equation } \xi(1-s) = \xi(s), \text{ and the symmetry } \xi(\overline{s}) = \overline{\xi(s)}, \text{ we integrate only upward from 2 to } 2 + iT, \text{ and then left from } 2 + iT \text{ to } \frac{1}{2} + iT. \text{ The argument-principle integral computes } 1/2\pi \text{ times the net change in the imaginary part of } \log \xi(s) \text{ over the given paths, requiring continuity of the logarithm. We compute separately the net changes of the imaginary parts of the summands in} \]

$$\log(\xi(s)) = -\frac{s}{2} \log \pi + \log(s^2) + \log(\xi(s))$$

The net change of imaginary part of the logarithm of $\pi^{-s/2}$ is

$$\text{Im} \left( \log \pi^{-\left(\frac{1}{2} + iT\right)/2} - \log \pi^{-2/2} \right) = \text{Im} \left( -\frac{1}{2} + iT \cdot \log \pi \right) = -\frac{T}{2} \log \pi$$


\[ \text{[2] As usual, } \xi(s) \text{ is the completed zeta function } \xi(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s). \text{ The usual notation is } S(T) = \frac{1}{\pi} \text{ arg } \xi(s), \text{ required to be 0 at } s = 2, \text{ and continuous along the vertical line from } 2 \text{ to } 2 + iT \text{ and then to } \frac{1}{2} + iT. \text{ When there is a zero along } (0,1) + iT, \text{ compute } S(T) \text{ slightly above } T. \]

\[ \text{[3] The } O(1) \text{ term would also accommodate zeros finitely-many other other poles and zeros that conceivably might exist for other zeta-functions and } L\text{-functions.} \]
From \[ \log \Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1) \]
we have
\[ \log \Gamma\left(\frac{s}{2}\right) = (\frac{s}{2} - \frac{1}{2}) \log \frac{s}{2} - \frac{s}{2} + O(1) \]
Thus, the net change from 2 to \( \frac{1}{2} + iT \) is
\[
\text{Im} \left( \log\left(\frac{1}{2} + iT\right) - \log\left(\frac{2}{2}\right) \right) = \text{Im} \left( \left(\frac{1}{2} + iT\right) - \left(\frac{1}{2} + iT\right) \right) + O(1)
\]
\[
= \text{Im} \left( \left(-\frac{1}{4} + iT\right)(\frac{\pi i}{2} + \log \frac{T}{2} + O(\frac{1}{T})) \right) - T + O(1) = \frac{T}{2} \log \frac{T}{2} - T + O(1)
\]
Since \( s = 2 + i\mathbb{R} \) is within the region of absolute convergence of the Euler product, \( \log \zeta(2 + it) \) is bounded on that line, so the net change in the imaginary part of the argument of \( \zeta(s) \) from 2 to \( 2 + iT \) is \( O(1) \).

The subtle computation concerns the net change in the argument of \( \zeta(s) \) from \( 2 + iT \) to \( \frac{1}{2} + iT \). We recall a version of part of a relevant lemma from [Titchmarsh/Heath-Brown 1951/1989], page 213, which uses Jensen’s Lemma to approximate the number of zeros, hence, the change in argument, in terms of the growth of a meromorphic function. We will apply the following lemma to \( f(s) = \zeta(s) \):

**[0.3] Lemma:** Let \( f \) be a holomorphic function on a vertical strip \(-1 \leq \sigma \leq 2 + 1\), except possibly for a simple pole at \( s = 1 \). Suppose that \( f(\sigma) = f(\bar{s}) \). Assume a lower bound \( \text{Re} f(2 + it) \geq m > 0 \), and a family of upper bounds
\[ |f(\sigma + it)| \leq M(T) \quad \text{for } \frac{1}{4} \leq \sigma \leq 4 \text{ and } 1 \leq t \leq T \]
Then, for \( T \) not the vertical coordinate of a zero of \( f \), there is the upper bound for change in argument from \( 2 + iT \) to \( \frac{1}{2} + iT \)
\[ | \arg f\left(\frac{1}{2} + iT\right) - \arg f(2 + iT) | \leq \frac{\pi}{\log((2 - \frac{1}{4})/(2 - \frac{1}{2}))} \cdot \left( \log M(T) + \log \frac{1}{m} \right) + \pi \]

**[0.4] Remark:** Naturally, some of the details are insignificant, being mere artifacts of the proof. At the same time, we give a more specific version of the result than [Titchmarsh/Heath-Brown 1951/1989].

**Proof:** Let \( q \) be the number of vanishings of \( \sigma \rightarrow \text{Re} f(\sigma + iT) \) between \( 2 + iT \) and \( \frac{1}{2} + iT \). The vanishing points divide the interval into \( q + 1 \) subintervals on each of which either \( \text{Re} f \geq 0 \) or \( \text{Re} f \leq 0 \). In particular, the value of \( f \) stays in either the right or left half-plane, so \( \arg f \) cannot change more than \( \pi \) in each subinterval. Thus,
\[ | \arg f\left(\frac{1}{2} + iT\right) - \arg f(2 + iT) | \leq (q + 1) \cdot \pi \]
Using \( f(\sigma) = \overline{f(\bar{\sigma})} \), the count \( q \) is the number of zeros of \( g(z) = \frac{1}{2}\left(f(z + iT) + f(z - iT)\right) \) on the real interval \( \frac{1}{2} \leq z \leq 2 \). Certainly this count is at most the number of zeros of \( g(z) \) in the disk \( |z - 2| \leq 2 - \frac{1}{2} \).

Let \( \nu(r) \) be the number of zeros of \( g \) in \( |z - 2| \leq r \). Setting up application of Jensen’s lemma,[4] we have an upper bound for \( q \):
\[ \int_{0}^{2^{\frac{1}{2}}} \frac{\nu(r)}{r^2} \, dr \geq \int_{2^{\frac{1}{2}}}^{2} \frac{\nu(r)}{r^2} \, dr \geq \nu(2 - \frac{1}{2}) \cdot \log \left( \frac{2 - \frac{1}{4}}{2 - \frac{1}{2}} \right) \geq q \cdot \log \left( \frac{2 - \frac{1}{4}}{2 - \frac{1}{2}} \right) \]

[4] Jensen’s Lemma usually appears as follows: for holomorphic \( f \) on \( |z| \leq r \), with no zeros on \( |z| = r \), and with \( f(0) \neq 0 \),
\[
\log |f(0)| - \sum_{\rho} \log |\rho| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad \text{(summed over zeros } |\rho| < r \text{ of } f)\]
Jensen’s lemma leads to an upper bound for the integral:

\[
\int_0^{2-\frac{1}{4}} \frac{\nu(r)}{r} \, dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(2 + (2 - \frac{1}{4})e^{i\theta})| \, d\theta - \log |g(2)| \leq \log M(T+2) + \log \frac{1}{m}
\]

giving the lemma. ///

To apply lemma to \( f(s) = \zeta(s) \), we show that since \( \Re \zeta(s) \) has a lower bound \( m > 0 \) on \( 2 + iR \). Indeed,

\[
\Re \zeta(2 + it) \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} > 1 - \int_1^{\infty} \frac{dx}{x^2} = 1 - \frac{1}{2-1} = 0
\]

Because of the strict inequality, there is a strictly positive lower bound \( m \).

On vertical lines, \( \zeta(\sigma + it) \) has a polynomial bound, from the functional equation, and from Phragmén-Lindelöf. Thus, the net change in the argument of \( \zeta(s) \) from \( 2 + iT \) to \( \frac{1}{2} + iT \) is \( O(\log T) \).

Thus, altogether, the argument principle gives

\[
N(T) = \frac{1}{2} \cdot \frac{1}{2\pi} \cdot 4 \cdot \left( \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} - \frac{T}{2} \log \pi \right) + O(\log T)
\]

\[
= \frac{1}{\pi} \cdot \left( \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} (1 + \log \pi) \right) + O(\log T) = \frac{T \log T}{2\pi} - \frac{T}{2\pi} \log 2 - \frac{T}{2\pi} (1 + \log \pi) + O(\log T)
\]

\[
= \frac{1}{2\pi} \cdot T \log T - \frac{\log 2\pi e}{2\pi} \cdot T + O(\log T)
\]

which is the asserted asymptotic. ///

Bibliography:

[Backlund 1914] R.J. Backlund, Sur les zéros de la fonction \( \zeta(s) \) de Riemann, C.R. 158 (1914), 1979-81.

