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Jensen's formula

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The *Jensen* formula usually appears as follows: for holomorphic f on $|z| \le r$, no zeros on $|z| = r$, and $f(0) \ne 0$,

$$\log|f(0)| - \sum_\rho \log\left|\frac{\rho}{r}\right| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \qquad \text{(summed over zeros } |\rho| < r \text{ of } f\text{)}$$

This is a corollary of the *mean-value property* for values of a *harmonic* function u [1] in the interior of a disk in terms of its values on the boundary:

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \qquad \text{(for } |z| < 1\text{)}$$

The Poisson formula in turn follows from the *mean-value property*, that $u(0)$ is the average of values of u over the circle.

For non-vanishing holomorphic f, $\log|f|$ is harmonic, so the Poisson formula applies. When f has (finitely-many) zeros ρ in $|z| < 1$, an auxiliary function such as

$$F(z) \;=\; f(z)/\prod_\rho (z-\rho)$$

has *no* zeros there, and the Poisson formula applies to $\log|F|$.

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1. *Mean-value property*

Among other features, in two dimensions harmonic functions form a useful, strictly larger class of functions including holomorphic functions. For example, harmonic functions still enjoy a *mean-value* property, as holomorphic functions do:

[1.1] Theorem: *(Mean-value property)* For harmonic u on a neighborhood of the closed unit disk,

$$u(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$$

Proof: Consider the rotation-averaged function

$$v(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}\cdot z)\,d\theta \qquad \text{(for } |z| \le 1\text{)}$$

[1] A continuously twice-differentiable function on $\mathbb{R}^2$ is *harmonic* when it is annihilated by the *Laplacian* $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In some contexts, a harmonic function is understood to be *real*-valued.

Since the Laplacian Δ is *rotation-invariant*, v is a rotation-invariant *harmonic* function. In polar coordinates, for rotation-invariant functions $v(z) = f(|z|)$, the Laplacian is

$$\Delta v \;=\; \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f(\sqrt{x^2+y^2}) \;=\; \frac{\partial}{\partial x}\big(\frac{x}{|z|} f'(|z|)\big) + \frac{\partial}{\partial y}\big(\frac{y}{|z|} f'(|z|)\big)$$

$$=\; \frac{1}{|z|} f' - \frac{x^2}{|z|^3} f' + \frac{x^2}{|z|^2} f'' + \frac{1}{|z|} f' - \frac{y^2}{|z|^3} f' + \frac{y^2}{|z|^2} f'' \;=\; f'' + \frac{1}{|z|} f'$$

The ordinary differential equation $f'' + f'/r = 0$ on an interval $(0, R)$ is an equation of *Euler type*, meaning expressible in the form $r^2 f'' + Brf' + Cf = 0$ with constants B, C. In general, such equations are solved by letting $f(r) = r^\lambda$, substituting, dividing through by r^λ, and solving the resulting *indicial equation* for λ:

$$\lambda(\lambda - 1) + A\lambda + B \;=\; 0$$

Distinct roots λ_1, λ_2 of the indicial equation produce linearly independent solutions r^{λ_1} and r^{λ_2}. However, as in the case at hand, a repeated root λ produces a second solution $r^\lambda \cdot \log r$. Here, the indicial equation is $\lambda^2 = 0$, so the general solution is $a + b \log r$. When $b \neq 0$, the solution $a + b\log r$ blows up as $r \to 0^+$. Since $f(0) = v(0) = u(0)$ is finite, it must be that $b = 0$. Thus, a *rotation-invariant* harmonic function on the disk is *constant*. Thus, its average over a circle is its central value, proving the mean-value property for harmonic functions. ///

[1.2] Remark: One might worry about commutation of the Laplacian with the integration above. In the first place, it is clear that we *must* have this commutativity. Second, the best and most final argument for such is in terms of *Gelfand-Pettis* (also called *weak*) integrals of function-valued functions, rather than temporary elementary arguments.

[1.3] Remark: The solutions $a + b\log r$ do indeed exhaust the possible solutions: given $f'' + f'/r = 0$ on $(0, R)$, we see $r \cdot f'$ is *constant* because

$$\frac{\partial}{\partial r}\big(r \cdot f'\big) \;=\; r \cdot f'' + f' \;=\; r \cdot \big(-f'/r\big) + f' \;=\; 0$$

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2. *Connections to holomorphic functions*

With the notation

$$\frac{\partial}{\partial z} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad\qquad \frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

we have

$$\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \overline{z}} \;=\; \frac{\partial}{\partial \overline{z}} \circ \frac{\partial}{\partial z} \;=\; \frac{1}{4} \cdot \Delta$$

A holomorphic function u satisfies the Cauchy-Riemann equation $\partial u/\partial \overline{z} = 0$, so *every holomorphic function is harmonic*. Similarly, every *every conjugate-holomorphic function is harmonic*. Thus, for holomorphic f, the real and imaginary parts

$$\mathrm{Re}\big(f(z)\big) \;=\; \tfrac{1}{2}\Big(f(z) + \overline{f(z)}\Big) \qquad\qquad \mathrm{Im}\big(f(z)\big) \;=\; \frac{1}{2i}\Big(f(z) - \overline{f(z)}\Big)$$

are *harmonic*, and real-valued.

The class of harmonic functions includes useful non-holomorphic real-valued functions. For example, (real-valued) *logarithms of absolute values of non-vanishing holomorphic functions are harmonic:*

$$\log|f(z)| \;=\; \tfrac{1}{2} \cdot \Big(\log f + \log \overline{f}\Big) \;=\; \tfrac{1}{2} \cdot \Big(\text{holomorphic} + \text{anti-holomorphic}\Big)$$

so is annihilated by $\Delta = 4\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar{z}}$.

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3. *Jensen's formula*

[3.1] Theorem: For holomorphic f on an open containing $|z| \le r$, with no zeros on $|z| = r$, and with $f(0) \ne 0$,

$$\log|f(0)| - \sum_\rho \log\left|\frac{\rho}{r}\right| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \qquad\qquad \text{(summed over zeros } |\rho| < r \text{ of } f)$$

Proof: First, for clarity, take $r = 1$. Letting

$$F(z) \;=\; \frac{f(z)}{\prod_\rho (z-\rho)}$$

we have

$$\log|F(z)| \;=\; \log|f(z)| - \sum_\rho \log|\rho - z|$$

By the Poisson formula applied to $\log|F|$,

$$\log|f(0)| - \sum_\rho \log|\rho| \;=\; \log|F(0)| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| - \sum_\rho \log|\rho - e^{i\theta}|\,d\theta$$

A small trick:

$$\int_0^{2\pi} \log|\rho - e^{i\theta}|\,d\theta \;=\; \int_0^{2\pi} \log|1 - e^{-i\theta}\rho|\,d\theta \;=\; \mathrm{Re}\int_0^{2\pi} \log(1 - e^{-i\theta}\rho)\,d\theta \;=\; \mathrm{Re}(0) \;=\; 0$$

because $\log(1 - w\rho)$ is holomorphic on an open containint $|w| \le 1$. For general $r > 0$, take $g(z) = f(z/r)$ and apply the above argument to g. ///

[3.2] Remark: Letting $\nu(t)$ be the number of zeros of size less than t, we can also rewrite

$$-\sum_\rho \log\left|\frac{\rho}{r}\right| \;=\; \sum_\rho (\log r - \log|\rho|) \;=\; \sum_\rho \int_{|\rho|}^r \frac{dt}{t} \;=\; \int_0^r \nu(t)\,\frac{dt}{t}$$

Bibliography:

[Jensen 1899] J.L.V.W. Jensen, *Sur un novel et important théorèm de la théorie des fonctions*, Acta Math **22** (1899), 359-364.
