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First proof of $L(1,\chi) \neq 0$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

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The subtle element in Dirichlet's theorem about primes in arithmetic progressions is *non-vanishing of L-functions* $L(s,\chi)$ at $s = 1$.

Here, we give a rather ugly and unexplanatory proof. However, the argument has few prerequisites. It uses Landau's Lemma, which we prove here.

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1. *First proof of non-vanishing on* $\mathrm{Re}(s) = 1$

We prove that $L(1,\chi) \neq 0$ for all χ mod N, *granting* that these L-functions have meromorphic extensions to some neighborhood of $s = 1$. We also need to know that for the trivial character χ_o mod N the L-function $L(s,\chi_o)$ has a *simple* pole at $s = 1$.

[1.1] Theorem: For a Dirichlet character χ mod N other than the trivial character χ_o mod N,

$$L(1,\chi) \neq 0$$

Proof: To prove that the L-functions $L(s,\chi)$ do not vanish at $s = 1$, and in fact do not vanish on the whole line [1] $\mathrm{Re}(s) = 1$, direct arguments involve tricks similar to what we do here.

First, for χ whose square is *not* the trivial character χ_o modulo N, the standard trick is to consider

$$\lambda(s) \;=\; L(s,\chi_o)^3 \cdot L(s,\chi)^4 \cdot L(s,\chi^2)$$

Then, letting $\sigma = \mathrm{Re}(s)$, from the Euler product expressions for the L-functions noted earlier, in the region of convergence,

$$|\lambda(s)| \;=\; \left|\exp\left(\sum_{m,p} \frac{3 + 4\chi(p^m) + \chi^2(p^m)}{mp^{ms}}\right)\right| \;=\; \exp\left|\sum_{m,p} \frac{3 + 4\cos\theta_{m,p} + \cos 2\theta_{m,p}}{mp^{m\sigma}}\right|$$

where for each m and p we let

$$\theta_{m,p} \;=\; (\text{the argument of } \chi(p^m)) \in \mathbb{R}$$

The trick, presumably found after considerable experimentation, is that for *any* real θ

$$3 + 4\cos\theta + \cos 2\theta \;=\; 3 + 4\cos\theta + 2\cos^2\theta - 1 \;=\; 2 + 4\cos\theta + 2\cos^2\theta \;=\; 2(1+\cos\theta)^2 \geq 0$$

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[1] Non-vanishing of $\zeta(s)$ on the whole line $\mathrm{Re}(s) = 1$ yields the Prime Number Theorem: let $\pi(x)$ be the number of primes less than x. Then $\pi(x) \sim x/\ln x$, meaning that the limit of the ratio of the two sides as $x \to \infty$ is 1. This was first proven in 1896, separately, by Hadamard and de la Vallée Poussin. The same sort of argument also gives an analogous *asymptotic* statement about primes in each congruence class modulo N, namely that $\pi_{a,N}(x) \sim x/[\varphi(N)\cdot\ln x]$, where $\gcd(a,N) = 1$ and φ is Euler's totient function.

Therefore, miraculously, all the terms inside the large sum being exponentiated are non-negative, and

$$|\lambda(s)| \geq e^0 \;=\; 1$$

In particular, if $L(1,\chi) = 0$ were to be 0, then, since $L(s,\chi_o)$ has a simple pole at $s = 1$ and since $L(s,\chi^2)$ does *not* have a pole (since $\chi^2 \neq \chi_o$), the multiplicity ≥ 4 of the 0 in the product of L-functions would overwhelm the three-fold pole, and $\lambda(1) = 0$. This would contradict the inequality just obtained.

For $\chi^2 = \chi_o$, instead consider

$$\lambda(s) \;=\; L(s,\chi) \cdot L(s,\chi_o) \;=\; \exp\left(\sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}}\right)$$

If $L(1,\chi) = 0$, then this would cancel the simple pole of $L(s,\chi_o)$ at 1, giving a non-zero finite value at $s = 1$. The series inside the exponentiation is a *Dirichlet series with non-negative coefficients*, and for real s

$$\sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}} \geq \sum_{p,\, m \text{ even}} \frac{1+1}{mp^{ms}} \;=\; \sum_{p,\, m} \frac{1+1}{2mp^{2ms}} \;=\; \sum_{p,\, m} \frac{1}{mp^{2ms}} \;=\; \log\zeta(2s)$$

Since $\zeta(2s)$ has a simple pole at $s = \frac{1}{2}$ the series

$$\log\left(L(s,\chi) \cdot L(s,\chi_o)\right) \;=\; \sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}} \geq \log\zeta(2s)$$

necessarily blows up as $s \to \frac{1}{2}^+$. But by *Landau's Lemma* below, a Dirichlet series with non-negative coefficients cannot blow up as $s \to s_o$ along the real line unless the function represented by the series fails to be holomorphic at s_o. Since the function given by $\lambda(s)$ is holomorphic at $s = 1/2$, this gives a contradiction to the supposition that $\lambda(s)$ is holomorphic at $s = 1$ (which had allowed this discussion at $s = 1/2$). That is, $L(1,\chi) \neq 0$. ///

[1.2] Remark: Again, the above argument is quick, but unilluminating. We will give better proofs later.

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2. *Landau's Lemma on Dirichlet series with positive coefficients*

[2.1] Theorem: *(Landau)* Let

$$f(s) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with real coefficients $a_n \geq 0$. Suppose that the series defining $f(s)$ converges for $\mathrm{Re}(s) > \sigma_o$. Suppose further that the function f extends to a function holomorphic in a neighborhood of $s = \sigma_o$. Then, in fact, the series defining $f(s)$ converges for $\mathrm{Re}(s) > \sigma_o - \varepsilon$ for some $\varepsilon > 0$.

Proof: First, by replacing s by $s - \sigma_o$ reduce to the case that $\sigma_o = 0$. Since the function $f(s)$ given by the series is holomorphic on $\mathrm{Re}(s) > 0$ and on a neighborhood of 0, there is $\varepsilon > 0$ such that $f(s)$ is holomorphic on $|s-1| < 1 + 2\varepsilon$, and the power series for the function converges nicely on this open disk. Differentiating the original series termwise (Abel's theorem), evaluate the derivatives of $f(s)$ at $s = 1$ as

$$f^{(i)}(1) \;=\; \sum_n \frac{(-\log n)^i\, a_n}{n} \;=\; (-1)^i \sum_n \frac{(\log n)^i\, a_n}{n}$$

and Cauchy's formulas yield, for $|s-1| < 1 + 2\varepsilon$,

$$f(s) \;=\; \sum_{i \geq 0} \frac{f^{(i)}(1)}{i!}\,(s-1)^i$$

In particular, for $s = -\varepsilon$, we are assured of the convergence to $f(-\varepsilon)$ of

$$f(-\varepsilon) \;=\; \sum_{i\geq 0} \frac{f^{(i)}(1)}{i!}\,(-\varepsilon - 1)^i$$

Since $(-1)^i f^{(i)}(1)$ is a *positive* Dirichlet series, move the powers of -1 a little to obtain

$$f(-\varepsilon) \;=\; \sum_{i\geq 0} \frac{(-1)^i f^{(i)}(1)}{i!}\,(\varepsilon + 1)^i$$

The series

$$(-1)^i f^{(i)}(1) \;=\; \sum_n (\log n)^i \frac{a_n}{n}$$

has positive terms, so the double series, convergent, with positive terms,

$$f(-\varepsilon) \;=\; \sum_{n,i} \frac{a_n\,(\log n)^i}{i!}(1+\varepsilon)^i \frac{1}{n}$$

can be rearranged to

$$f(-\varepsilon) \;=\; \sum_n \frac{a_n}{n}\left(\sum_i \frac{(\log n)^i (1+\varepsilon)^i}{i!}\right) \;=\; \sum_n \frac{a_n}{n} n^{(1+\varepsilon)} \;=\; \sum_n \frac{a_n}{n^{-\varepsilon}}$$

That is, the latter series converges (absolutely). ///
