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Special values of $L(s,\chi)$

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The contour-integration device applied earlier to Riemann's zeta is equally applicable to Dirichlet L-functions. It immediately proves that values of $L(s,\chi)$ at non-positive integers lie in the field generated over $\mathbb{Q}$ by the values of χ, necessarily roots of unity. It also shows the connection to the Laurent coefficients of more general analogues of the function $1/(e^t-1)$ which appeared in the discussion of $\zeta(s)$.

[1.1] An integral representation of $\Gamma(s)\cdot L(s,\chi)$

[1.2] Claim: For a Dirichlet character χ modulo $N>1$,

$$\int_0^\infty t^s \,\frac{\sum_{j=1}^{N-1}\chi(j)\,e^{-tj}}{1-e^{-tN}}\,\frac{dt}{t} \;=\; \Gamma(s)\cdot L(s,\chi) \qquad\qquad (\text{for } \mathrm{Re}(s)>1)$$

Proof: Expanding geometric series and using the Gamma identity $\int_0^\infty t^s\,e^{-ty}\,dt/t = y^{-s}\,\Gamma(s)$ for $\mathrm{Re}(s)>1$ and $y>0$,

$$\int_0^\infty t^s \,\frac{\sum_{j=1}^{N-1}\chi(j)\,e^{-tj}}{1-e^{-tN}}\,\frac{dt}{t} \;=\; \int_0^\infty t^s\sum_{j=1}^N \chi(j)e^{-tj}\cdot\Big(1+e^{-tN}+e^{-2tN}+e^{-3tN}+\ldots\Big)\,\frac{dt}{t}$$

$$=\;\sum_{j=1}^N\chi(j)\int_0^\infty t^s\Big(e^{-tj}+e^{-t(N+j)}+e^{-t(2N+j)}+e^{-t(3N+j)}+\ldots\Big)\,\frac{dt}{t}$$

$$=\;\sum_{j=1}^N\chi(j)\cdot\Gamma(s)\cdot\Big(\frac{1}{j^s}+\frac{1}{(N+j)^s}+\frac{1}{(2N+j)^s}+\frac{1}{(3N+j)^s}+\ldots\Big)\;=\;\Gamma(s)\cdot\sum_{n=1}^\infty\frac{\chi(n)}{n^s}\;=\;\Gamma(s)\cdot L(s,\chi)$$

as asserted. ///

[1.3] Keyhole/Hankel contour The usual *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon>0$, counterclockwise around a circle of radius ε at 0, back to ε on the real line, and outbound back to $+\infty$ along the real line.

The typical application is to evaluation of integrals of the form $\int_0^\infty t^s\,f(t)\;dt$ with f holomorphic on a neighborhood of $[0,+\infty)$, with $-1<\mathrm{Re}(s)$, and with suitable convergence. In such an example, analytically continuing counterclockwise around 0 has no impact on $f(t)$, but, significantly, t^s changes by a factor $e^{2\pi is}$, since

$$t^s \;=\; (|t|\cdot e^{i\theta})^s \;=\; |t|^s\cdot e^{i\theta s} \qquad\qquad (\text{and } \theta \text{ goes from 0 to } 2\pi)$$

With whatever choice of out-bound value of t^s, the inbound version of t^s is $t^s\cdot e^{2\pi is}$. The absolute value of the integrand goes to 0 as $|t|\to 0$, so the integral over the small circle goes to 0 as $\varepsilon\to 0$, as do the integrals to and from $0,\varepsilon$ along the real line.

Thus, letting H_ε be the Hankel contour with circle of radius $\varepsilon>0$, for $-1<\mathrm{Re}(s)$,

$$\lim_{\varepsilon\to 0}\int_{H_\varepsilon} t^s\,\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}\,\frac{dt}{t} \;=\; (1-e^{2\pi is})\int_0^\infty t^s\,\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}\,\frac{dt}{t} \;=\; (1-e^{2\pi is})\cdot\Gamma(s)\cdot L(s,\chi)$$

[1.4] Evaluation of $\zeta(-n)$ Rewrite this as

$$L(s,\chi) \;=\; \frac{1}{\Gamma(s)\cdot(1-e^{2\pi is})}\cdot\lim_{\varepsilon\to 0}\int_{H_\varepsilon} t^{s-1}\,\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}\;dt$$

At $s=-n\in\{0,-1,-2,-3,-4\ldots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1-e^{2\pi is}$ cancel, giving a finite, computable value. Second, the function t^{-n-1} is *single-valued*, so the inbound and outbound integrals of the Hankel contour *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $t^{-n-1}\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}$ at $t=0$.

The periodicity of $1-e^{2\pi is}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1-e^{2\pi is} \;=\; 1-\Big(1+\frac{2\pi is}{1!}+\frac{(2\pi is)^2}{2!}+\ldots\Big) \;=\; -2\pi is+\text{higher}$$

The residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$L(s,\chi) \;=\; \frac{1}{\frac{(-1)^n}{n!}\cdot(-2\pi i)}\cdot 2\pi i\cdot \mathrm{Res}_{t=0}\, t^{-n-1}\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}$$

$$=\; (-1)^{n+1}\cdot n!\cdot\Big(n^{th}\text{ Laurent coefficient of }\tfrac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}\text{ at }t=0\Big)$$

Those Laurent coefficients are generalized Bernoulli numbers, and lie in $\mathbb{Q}(\chi)$, the field generated over $\mathbb{Q}$ by values of χ.

Further, these Laurent coefficients are finite linear combinations of values of χ, so the Galois group $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ acts compatibly on these Laurent coefficients: for $\sigma\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$,

$$\Big(n^{th}\text{ Laurent coefficient of }\frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}\Big)^\sigma \;=\; n^{th}\text{ Laurent coefficient of }\frac{\sum_{j=1}^{N-1}\chi^\sigma(j)e^{-tj}}{1-e^{-Nt}}$$

where χ^σ is the character $\chi^\sigma(j)=\chi(j)^\sigma$ obtained by applying σ to the values, which are roots of unity.

Thus, we have proven

[1.5] Theorem: $L(-n,\chi)\in\mathbb{Q}(\chi)$ for $-n=0,-1,-2,-3,\ldots$ and $L(-n,\chi)^\sigma=L(-n,\chi^\sigma)$. ///

[1.6] Vanishing of $L(-n,\chi)$ depending on parity The *parity* of χ is determined by $\chi(-1)$: for $\chi(-1)=1$, χ is *even*, while for $\chi(-1)=-1$, χ is *odd*. For the function

$$f(t) \;=\; \frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}}$$

parity refers to the comparison of $f(-t)$ to $f(t)$: for $f(-t)=f(t)$ the function would be *even*, and for $f(-t)=-f(t)$, the function is *odd*. Direct computation gives

$$f(-t) \;=\; \frac{\sum_{j=1}^{N-1}\chi(j)e^{-tj}}{1-e^{-Nt}} \;=\; \frac{\sum_{j=1}^{N-1}\chi(j)e^{Nt-tj}}{e^{Nt}-1} \;=\; \frac{\sum_{j=1}^{N-1}\chi(N-j)e^{tj}}{e^{Nt}-1} \;=\; \frac{\sum_{j=1}^{N-1}\chi(-j)e^{tj}}{e^{Nt}-1}$$

by replacing j by $N-j$. Since $\chi(-j)=\chi(-1)\chi(j)$, by adjusting the denominator, this is

$$f(-t) \;=\; -\chi(-1)\frac{\sum_{j=1}^{N-1}\chi(j)e^{tj}}{1-e^{Nt}} \;=\; -\chi(-1)\cdot f(t)$$

That is, the parity of $f(t)$ is *opposite* the parity of χ. In particular, the *even* Laurent coefficients at $t=0$ are 0 for χ *even*, and the *odd* Laurent coefficients at $t=0$ are 0 for χ *odd*. We have proven

[1.7] Claim: $L(-n,\chi)=0$ for $-n\le 0$ of the same parity as χ. ///
