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Gauss sums and harmonic analysis

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[This document is
http://www.math.umn.edu/~garrett/m/mfms/notes_2019-20/06f_Gauss_sums.pdf]

In the context of harmonic analysis on finite abelian groups, *Gauss sums* reflect the interaction of *addition* and *multiplication* on the finite ring \mathbb{Z}/N . We need a few basic facts.

Let ψ be an *additive character* $\psi : \mathbb{Z}/N \rightarrow \mathbb{C}^\times$ on the additive group of \mathbb{Z}/N , and require that ψ *not* be well-defined modulo N' for any proper divisor N' of N .

Let χ be a *multiplicative character* $\chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times$, extended by 0 to non-invertible elements of \mathbb{Z}/N . The corresponding Gauss sum is essentially an inner product in space of \mathbb{C} -valued functions on \mathbb{Z}/N :

$$g(\chi, \psi) = \sum_{x \bmod N} \chi(x) \cdot \psi(x)$$

The character χ is *primitive* mod N if it is *not* well-defined on \mathbb{Z}/N' for any proper divisor N' of N . Then say that χ has *conductor* N . This distinction has immediate consequences:

[0.1] **Claim:** $g(\chi, \psi) = 0$ for χ *not* primitive mod N .

Proof: Non-primitivity means that there is N' a proper divisor of N such that χ is well-defined modulo N' . This means that $\chi(1 + kN') = \chi(1) = 1$ for all $k \in \mathbb{Z}$. Then

$$g(\chi, \psi) = \sum_{x \bmod N} \chi(x) \cdot \psi(x) = \sum_{x \bmod N} \chi(x(1 + kN')^{-1}) \cdot \psi(x)$$

where the inverse means in \mathbb{Z}/N . Replacing x by $x(1 + kN')$ gives

$$g(\chi, \psi) = \sum_{x \bmod N} \chi(x) \cdot \psi(x(1 + kN')) = \sum_{x \bmod N} \chi(x) \cdot \psi(x) \cdot \psi(x \cdot kN')$$

By the *cancellation lemma*, summing $\psi(xkN')$ over $k \bmod N/N'$ produces either N/N' or 0 depending whether $k \rightarrow \psi(xkN')$ is the trivial character or not. The character $k \rightarrow \psi(xkN')$ is trivial exactly when $N|xN'$. Since N' is a *proper* divisor of N , this can happen only when x has a non-trivial common factor with N . But then $\chi(x) = 0$. Thus,

$$\begin{aligned} \frac{N}{N'} \cdot g(\chi, \psi) &= \sum_{k \bmod N/N'} g(\chi, \psi) = \sum_{k \bmod N/N'} \left(\sum_{x \bmod N} \chi(x) \cdot \psi(x) \cdot \psi(x \cdot kN') \right) \\ &= \sum_{x \bmod N} \chi(x) \cdot \psi(x) \left(\sum_{k \bmod N/N'} \psi(x \cdot kN') \right) = \sum_{x \bmod N} 0 = 0 \end{aligned}$$

since in every summand either $\chi(x) = 0$ or the inner sum over k is 0. ///

[0.2] **Claim:** For primitive $\chi \bmod N$,

$$|g(\chi, \psi)|^2 = N$$

Proof: Start the computation in the obvious fashion, writing $\psi(a) = e^{2\pi ia/N}$. Let Σ' denote sum over $(\mathbb{Z}/N)^\times$, and Σ'' denote sum over $\mathbb{Z}/N - (\mathbb{Z}/N)^\times$.

$$\left| \sum_{a \bmod N} \chi(a) \psi(a) \right|^2 = \sum'_{a,b} \chi(a) \psi(a) \bar{\chi}(b) \psi(-b)$$

Replacing a by ab , this becomes

$$\sum'_{a,b} \chi(a) \psi((a-1) \cdot b)$$

We claim that, because χ has conductor N (and not smaller!)

$$\sum'_a \chi(a) \psi((a-1) \cdot b) = 0 \quad (\text{for } \gcd(b, N) > 1)$$

To see this, let p be a prime dividing $\gcd(b, N)$. That N is the conductor of χ is to say that χ is *primitive* mod N , meaning that χ does not factor through *any* quotient $\mathbb{Z}/(N/p)$. That is, there is some $\eta = 1 \bmod N/p$ such that $\chi(\eta) \neq 1$.

Since $p|b$, and $\eta = 1 \bmod N/p$,

$$(a\eta - 1) \cdot b = (a-1)b + a(\eta-1)b = (a-1)b \bmod N$$

Thus, replacing a by ηa ,

$$\sum'_a \chi(a) \psi((a-1) \cdot b) = \sum'_a \chi(a\eta) \psi((a\eta-1) \cdot b) = \chi(\eta) \sum'_a \chi(a) \psi((a-1) \cdot b)$$

Thus, the sum over a is 0. Thus, we can drop the coprimality constraint:

$$\sum'_{a,b} \chi(a) \psi((a-1) \cdot b) = \sum_{a,b} \chi(a) \psi((a-1) \cdot b)$$

For $a \neq 1$, the inner sum over b is 0, because the sum of a non-trivial character over a finite group is 0. For $a = 1$ the sum over b gives N . ///
