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# A rationality principle

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The following rationality principle was used by H. Klingen, *Über die Werte der Dedekindschen Zetafunktionen*, Math. Ann. **145** (1961/2), 265-272, to study special values of  $L$ -function on totally real number fields.

**[0.1] Theorem:** The constant term  $c_o$  in the Fourier expansion  $f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi inz}$  of a holomorphic elliptic modular form  $f(z)$  lies in the field  $\mathbb{Q}(c_1, c_2, \dots)$  generated over  $\mathbb{Q}$  by the higher Fourier coefficients.

*Proof:* In fact, we prove that the map  $\sum_n c_n e^{2\pi inz} \rightarrow c_o$  is a finite  $\mathbb{Q}$ -linear combination of maps  $\sum_n c_n e^{2\pi inz} \rightarrow c_n$  with  $n > 0$ , depending on the weight  $2k$ .

**[0.2] Lemma:** Let  $K \subset L$  be fields,  $V$  a finite-dimensional  $L$ -vectorspace, and  $\Lambda \subset \text{Hom}_L(V, L)$ . Suppose that  $\Lambda$  spans  $\text{Hom}_L(V, L)$  over  $L$ , and that  $V_o = \{v \in V : \lambda v \in K, \forall \lambda \in \Lambda\}$  spans  $V$  over  $L$ . For any  $L$ -basis  $B \subset \Lambda$  of  $\text{Hom}_L(V, L)$ , every  $\lambda \in \Lambda$  is a  $K$ -linear (not merely  $L$ -linear) combination of elements of  $B$ .

*Proof:* Let  $\{v_\beta : \beta \in B\}$  be the dual  $L$ -basis to  $B$ , namely,  $\beta(v_\beta) = 1$  and  $\beta'(v_\beta) = 0$  for  $\beta' \neq \beta$ . Since  $\beta(V_o) \subset K$ ,

$$V_o \subset K\text{-span of } \{v_\beta : \beta \in B\}$$

If  $V_o$  failed to contain some  $v_\beta$ , then the  $K$ -dimension of  $V_o$  would be less than the  $L$ -dimension of  $V$ , so  $V_o$  could not span  $V$  over  $L$ . Thus,  $V_o$  is exactly the  $K$ -span of  $\{v_\beta : \beta \in B\}$ . Write  $\lambda \in \Lambda$  as a  $L$ -linear combination of elements of  $B$ :  $\lambda = \sum_{\beta \in B} c_\beta \beta$ , with  $c_\beta \in L$ . Since  $v_{\beta'} \in V_o$ ,

$$c_{\beta'} = \sum_{\beta \in B} c_\beta \beta(v_{\beta'}) = \left( \sum_{\beta \in B} c_\beta \beta \right)(v_{\beta'}) = \lambda(v_{\beta'}) \in K$$

That is,  $\lambda$  is a  $K$ -linear combination of the functionals in  $B$ , as claimed. ///

**[0.3] Claim:** The space of holomorphic modular forms of level one, of a fixed weight  $2k$ , have a  $\mathbb{C}$ -basis consisting of modular forms with *rational* Fourier coefficients.

*Proof:* First, we prove that monomials  $E_4^a E_6^b$  with  $4a + 6b = 2k$  (and non-negative integers  $a, b$ ) span the space of weight  $2k$  modular forms of level one. Every  $2k \geq 4$  has at least one expression  $2k = 4a + 6b$ , and there are no non-zero modular forms of weight 2, and only constants of weight 0. For weights 4, 6, 8, 10, the divisor relation shows that  $E_4, E_6, E_4^2, E_4 E_6$  span these spaces.

Suppose  $2k \geq 12$ . Note that  $\Delta = E_4^3 - E_6^2$  is not identically 0, and is a nowhere-vanishing (except at  $i\infty$ ) cuspform. Given  $f(z) = \sum_n c_n e^{2\pi inz}$ , and any  $a, b$  with  $2k = 4a + 6b$ ,  $f - c_o \cdot E_4^a E_6^b$  is a cuspform, and

$$\frac{f - c_o \cdot E_4^a E_6^b}{\Delta}$$

is of weight  $2k - 12$ , so induction gives the result. The Eisenstein series  $E_4$  and  $E_6$  have rational Fourier coefficients, so  $E_4^a E_6^b$  has rational Fourier coefficients, for all non-negative integers  $a, b$ . Thus, each space of weight  $2k$  modular forms is spanned by modular forms with rational Fourier coefficients. ///

Now we prove the theorem. In the lemma, let  $V$  be the space of modular forms of weight  $2k > 0$ , and  $\Lambda$  the collection of all functionals  $\lambda_n : \sum_n c_n e^{2\pi inz} \rightarrow c_n$ . Certainly  $\Lambda$  spans the dual space  $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ , and the claim shows that the space  $V_o$  of modular forms of weight  $2k$  with rational coefficients spans the whole space over  $\mathbb{C}$ . For weight  $2k > 0$ , the simultaneous kernel of all  $\lambda_n$  with  $n > 0$  consists of constants, and the only constant modular form of positive weight is 0. Thus, we can choose a basis  $B \subset \Lambda$  for  $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  from among functionals  $\lambda_n$  with  $n > 0$ . Thus,  $\lambda_o$  is a  $\mathbb{Q}$ -linear combination of the  $\lambda_n$ 's with  $n > 0$ . ///