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1. *Fourier analysis on finite abelian groups*

Let G be a finite abelian group, and $L^2(G)$ the space of complex-valued functions [1] on G, with inner product

$$\langle f, \varphi \rangle = \sum_{x \in G} f(x)\,\overline{\varphi(x)}$$

Let G act on $L^2(G)$ by the regular representation [2]

$$(g \cdot f)(x) = f(xg) \qquad \text{(for } x, g \in G \text{ and } f \in L^2(G)\text{)}$$

Let

$$\widehat{G} = \{\text{group homomorphisms } \sigma : G \longrightarrow \mathbb{C}^\times\}$$

be the collection of *characters*[3] on G. Since G is finite, it is torsion, so $|\sigma(x)| = 1$ for all $\sigma \in \widehat{G}$ and $x \in G$. By a standard abuse of notation, σ *also* denotes a one-dimensional complex vector space on which G acts by the character σ, namely

$$g \cdot v = \sigma(g) \cdot v \qquad \text{(for } v \text{ in the vector space, } g \in G\text{)}$$

[1.0.1] Theorem: The regular representation of G decomposes $L^2(G)$ as a direct sum of (scalar multiples of) characters of G, each occuring exactly once. That is,

$$L^2(G) = \bigoplus_{\sigma \in \widehat{G}} \mathbb{C} \cdot \sigma \qquad \text{(as } G \text{ representations)}$$

Proof: On one hand, each $\sigma \in \widehat{G}$ is itself a complex-valued function on G, so [4] is in $L^2(G)$. Thus, all scalar multiples of the *function* σ give a copy of the *representation* σ inside $L^2(G)$. Further, for $\sigma \neq \tau$ in $\widehat{G}$,

[1] Finite groups are both *discrete* and *compact*. One canonical measure on *discrete* groups is *counting* measure. Even without talking about invariant measures on groups, counting measure μ is translation-invariant in the sense that $\mu(X) = \mu(X \cdot g)$ for finite subsets X of the group and for g in the group. But there is another canonical invariant measure on *compact* groups, namely, that which gives the whole group measure 1. Thus, there are at least two canonical normalizations of invariant measure on a finite group.

[2] For *abelian* G, the left and right actions of G on itself by multiplication are the same, so there is scant point in distinguishing them. For non-abelian groups these two actions are distinct, and in that case we would need to keep track of the action of $G \times G$ on G and on functions on G.

[3] In the context of abelian groups, all irreducible complex representations are one-dimensional, and it is not unreasonable to identify any such thing with a group homomorphism of G to $\mathbb{C}^\times$. With such an identification, the *trace* is essentially the same thing as the representation itself. For non-abelian groups and higher-dimensional irreducibles, the characters in the sense of *traces* are not group homomorphisms themselves.

[4] One should observe that the function σ really does behave as required under right translation, namely, $\sigma(xg) = \sigma(x)\sigma(g) = \sigma(g)\sigma(x)$.

$\sigma\tau^{-1}$ is not the trivial character in $\widehat{G}$, so by the cancellation lemma [5]

$$\langle \sigma, \tau \rangle \;=\; \sum_{x \in G} \sigma(x)\,\overline{\tau(x)} \;=\; \sum_{x \in G} \big(\sigma\tau^{-1}\big)(x) \;=\; 0$$

That is, the one-dimensional *subspaces* σ and τ inside $L^2(G)$ are orthogonal, since the *functions* σ and τ are orthogonal. Thus, each σ in $\widehat{G}$ occurs in $L^2(G)$ at least once.

On the other hand, G acts [6] on $L^2(G)$ by commuting *unitary* operators:

$$\langle g \cdot f,\; g \cdot \varphi \rangle \;=\; \sum_{x \in G} f(xg)\,\overline{\varphi}(xg) \;=\; \sum_{x \in G} f(x)\,\overline{\varphi}(x) \;=\; \langle f, \varphi \rangle$$

by replacing x by xg^{-1}. Thus, by the finite-dimensional spectral theorem [7] $L^2(G)$ has an orthogonal basis consisting of simultaneous eigenvectors [8] for G. That is, a simultaneous eigenvector f has the behavior

$$f(xg) \;=\; \sigma(g) \cdot f(x) \qquad\qquad \text{(for some } \sigma \in \widehat{G}\text{, for all } x, g \in G\text{)}$$

In fact, this gives

$$f(x) \;=\; f(1 \cdot x) \;=\; (x \cdot f)(1) \;=\; \sigma(x) \cdot f(1) \;=\; f(1) \cdot \sigma(x)$$

That is, such f is a scalar multiple of the function σ. That is, it is already in the subspace of $L^2(G)$ spanned by the function σ. Thus, each *representation* $\sigma \in \widehat{G}$ occurs *just once* in $L^2(G)$, in the sense that the space of functions in $L^2(G)$ that are eigenvectors for G with eigenvalues given by σ is one-dimensional. [9] ///

[5] The basic *cancellation lemma* that shows that the sum of a non-trivial character χ over a finite (not necessarily non-abelian) group G is 0, by changing variables, as follows. For any $h \in G$, we change variables to obtain $\sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(gh) = \chi(h) \sum_{g \in G} \chi(g)$. Since $\chi(h) \neq 1$ for some $h \in G$, the sum must be 0.

[6] More broadly, the action of a *torsion* group on a finite-dimensional vector space over an algebraically closed field of characteristic zero is *simultaneously* diagonalizable.

[7] The spectral theorem for single *normal* operators, that is, operators T commuting with their adjoints, gives a spectral theorem for individual unitary operators T, since these are normal. Then note that the λ eigenspace V_λ for T is *stabilized* by any operator S that commutes with T, since $T(Sv) = S(Tv) = S(\lambda v) = \lambda \cdot Sv$. Restrictions of unitary operators S to stable subspaces are still unitary, so V_λ has an orthogonal basis of eigenvectors for S. These are necessarily still λ-eigenvectors for T. A downward induction gives the result.

[8] For a simultaneous eigenvector $v \neq 0$ for a group G, the set of eigenvalues λ_g of v for $g \in G$ is not an unstructured set of numbers. Rather, the map $g \to \lambda_g$ is a character of G, since, by associativity, $\lambda_g \lambda_h v = g(hv) = (gh) \cdot v = \lambda_{gh} v$.

[9] Indeed, we just showed that the space of functions in $L^2(G)$ that are simultaneous eigenfunctions for G with eigenvalues given by σ consists exactly of scalar multiples of the function σ.