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Introduction to zeta integrals and L-functions for GL_n

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We give a quick introduction to Fourier-Whittaker expansions of cuspforms on GL_n, and integral representations of associated L-functions, following a part of Jacquet, Piatetski-Shapiro, and Shalika's extensions of Hecke's work for GL_2.

All known ways to analytically continue automorphic L-functions involve *integral representations* using the corresponding automorphic forms. The simplest cases, extending Hecke's treatment of GL_2, need no further analytic devices and very little manipulation beyond Fourier-Whittaker expansions. [1] Poisson summation is a sufficient device for several accessible classes of examples, as in Riemann, [Hecke 1918,20], [Tate 1950], [Iwasawa 1952], and [Godement-Jacquet 1972], and including treatment of the degenerate Eisenstein series needed for the $GL_n \times GL_n$ Rankin-Selberg convolutions. [2]

For f a cuspform on GL_n the most natural L-function obtained by an integral representation is the *Hecke-type* integral representation, also involving a cuspform F on GL_{n-1},

$$\Lambda(s, f \otimes F) \;=\; \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})} |\det h|^{s-\frac{1}{2}} \cdot f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \cdot F(h)\, dh$$

More properly, the integral is a *zeta integral* $Z(s, f \times F)$, since within a given automorphic representation there is an essentially unique choice of automorphic form giving the correct local factors everywhere locally in that zeta integral: see [Jacquet-PS-Shalika 1981], [Jacquet-Shalika 1990], [Cogdell-PS 2003]. At *good* primes this is not an issue, but the general case must subsume the theory of *newforms*, as well as coping with complications at archimedean places. [3]

Another simple natural case, $m = n$, is the *Rankin-Selberg* integral using an auxiliary (degenerate) Eisenstein series

$$E_s(g) \;=\; \sum_{\gamma \in P_k^{n-1,1}\backslash GL_n(k)} \varphi_s(\gamma \cdot g)$$

where $P^{n-1,1}$ has Levi component $GL_{n-1} \times GL_1$, and

$$\varphi_s \;=\; \bigotimes_v \varphi_{s,v}$$

with $\varphi_{s,v}$ in a (degenerate) induced representation from $P_v^{n-1,1}$. The zeta integral attached to two cuspforms f, F on GL_n is

$$\Lambda(s, f \otimes F) \;=\; \int_{GL_n(k)\backslash GL_n(\mathbb{A})} E_s(h) \cdot f(h) \cdot F(h)\, dh$$

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[1] See [Hecke 1937a,b]. Apparently the extension to $GL_{n-1} \times GL_n$ was considered so apparent that it was not explicitly mentioned in [Jacquet-PS-Shalika 1979], which was concerned with $GL_1 \times GL_3$ as prototype for $GL_m \times GL_n$.

[2] We do not discuss examples relying on meromorphic continuation of non-trivial Eisenstein series, as in [Langlands 1967/1976, 1971] and [Shahidi 1978,1985] have other requirements. See [Shahidi 2010] for a recent survey.

[3] In contrast to [Godement-Jacquet 1972], the *standard* L-function for f is *not* produced by this Hecke-type integral representation, except for $n = 2$. This was understood by Jacquet, Piatetski-Shapiro, and Shalika in the late 1970's, who developed the desired integral representations of a family of L-functions including the standard ones in the papers in the bibliography below. See also Cogdell's lecture notes in the bibliography.

1. *Fourier-Whittaker expansions of cuspforms on GL_r*

Some non-trivial aspects of the group structure of GL_r enters in the derivation of the Fourier expansion. The outcome is not obvious for $r > 2$.

Let $G = GL_r$, reserving the character n for elements of unipotent subgroups. Let

$$N^{\min} \;=\; \begin{pmatrix} 1 & * & \ldots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix} \;=\; \text{unipotent radical of standard minimal parabolic}$$

Fix a non-trivial additive character ψ_o on $k\backslash\mathbb{A}$, and let $\psi_{\rm std}$ be the corresponding standard character on the unipotent radical of the standard minimal parabolic, namely,

$$\psi_{\rm std}(u) \;=\; \psi_o(\text{sum super-diagonal entries}) \;=\; \psi_o(u_{12} + u_{23} + \ldots + u_{r-1,r}) \qquad (\text{non-trivial } \psi_o \text{ on } k\backslash\mathbb{A})$$

We obtain the Fourier expansion of a *cuspform* by an induction. First, a cuspform f has a Fourier expansion along the *abelian* unipotent radical

$$N \;=\; N^{r-1,1} \;=\; \{n_x = \begin{pmatrix} 1_{r-1} & x \\ 0 & 1 \end{pmatrix} \;:\; x = (r-1)\text{-by-}1\}$$

of the form

$$f(g) \;=\; \sum_\psi \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, f(ng)\; dn \qquad (\psi \text{ summed over characters on } N_k\backslash N_{\mathbb{A}})$$

The cuspform condition implies that the component for the *trivial* character on $N_k\backslash N_{\mathbb{A}}$ is 0. The fragment

$$H \;=\; H^{r-1} \;=\; \{\begin{pmatrix} GL_{r-1} & 0 \\ 0 & 1 \end{pmatrix}\}$$

of the Levi component of the parabolic $P^{r-1,1}$ acts *transitively* on the *non-trivial* characters on $N_k\backslash N_{\mathbb{A}}$. Letting

$$\psi_1(n_x) \;=\; \psi_o(x_{r-1})$$

the isotropy subgroup $\Theta = \Theta^{r-1}$ of ψ_1 in H is

$$\Theta \;=\; \{m \in H_k : \psi_1(mnm^{-1}) = \psi_1(n) \text{ for all } n \in N_{\mathbb{A}}\} \;=\; \{\begin{pmatrix} GL_{r-2} & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

Thus, for a cuspform f,

$$f(g) \;=\; \sum_{\gamma\in\Theta_k\backslash H_k} \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(\gamma n\gamma^{-1})\, f(ng)\; dn$$

Replacing n by $\gamma^{-1}n\gamma$ and using the left G_k-invariance of f, this is

$$f(g) \;=\; \sum_{\gamma\in\Theta_k\backslash H_k} \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(n\gamma g)\; dn$$

For the induction step, let

$$N' \;=\; \{u_x = \begin{pmatrix} 1_{r-2} & x & 0 \\ 0 & 1 & 0 \\ 0 & & 1 \end{pmatrix}\} \;\subset\; H^{r-1}$$

Note that N' normalizes N, and

$$\psi_1(unu^{-1}) \;=\; \psi_1(n) \qquad\qquad (\text{for all } n \in N_{\mathbb{A}} \text{ and } u \in N'_{\mathbb{A}})$$

Letting

$$H^{r-2} \;=\; \{\begin{pmatrix} GL_{r-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

we have $\Theta = N'H^{r-2} = H^{r-2}N'$. For each γ, the function

$$h \quad\longrightarrow\quad \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nh\gamma g)\, dn \qquad\qquad (\text{for } h \in H^{r-1})$$

is left N'_k-invariant, because

$$\int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(n\alpha h\gamma g)\, dn \;=\; \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(\alpha nh\gamma g)\, dn \;=\; \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nh\gamma g)\, dn \qquad (\text{for } \alpha \in U_k)$$

by replacing n by $\alpha n\alpha^{-1}$ and using the left G_k-invariance of f. Thus, for each γ, there is a Fourier expansion along N', namely,

$$\int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nh\gamma g)\, dn \;=\; \sum_{\psi'} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nuh\gamma g)\, dn\, du \qquad (\text{characters } \psi' \text{ of } N'_k\backslash N'_{\mathbb{A}})$$

In fact, we only need $h = 1$:

$$\int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(n\gamma g)\, dn \;=\; \sum_{\psi'} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nu\gamma g)\, dn\, du \qquad (\text{characters } \psi' \text{ of } N'_k\backslash N'_{\mathbb{A}})$$

The $\psi' = 1$ summand is 0, because f is cuspidal, since

$$N' \cdot \big(\ker\psi_1 \text{ on } N\big) \;\supset\; \{\begin{pmatrix} 1_{r-2} & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\} \;=\; \text{unipotent radical of } (r-2, 2) \text{ parabolic}$$

The action of H^{r-2} on non-trivial characters ψ' on N' is *transitive*. Let

$$\psi'_1 \begin{pmatrix} 1_{r-2} & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \psi_o(x_{r-2}) \qquad\qquad (\text{with } x \ (r-2)\text{-by-}1)$$

The isotropy group of ψ'_1 is

$$\Theta^{r-2} \;=\; \{\begin{pmatrix} GL_{r-3} & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\}$$

Thus,

$$\sum_{\psi'} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nu\gamma g)\, dn\, du \;=\; \sum_{\delta \in \Theta_k^{r-2}\backslash H_k^{r-2}} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'_1(\delta u\delta^{-1}) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nu\gamma g)\, dn\, du$$

$$=\; \sum_{\delta \in \Theta_k^{r-2}\backslash H_k^{r-2}} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'_1(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(n\delta^{-1}u\delta\gamma g)\, dn\, du$$

by replacing u by $\delta^{-1}u\delta$. We can also replace n by $\delta^{-1}n\delta$ without affecting ψ_1, so this becomes

$$\sum_{\delta\in\Theta_k^{r-2}\backslash H_k^{r-2}} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'_1(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nu\delta\gamma g)\, dn\, du$$

Altogether,

$$f(g) \;=\; \sum_{\gamma\in\Theta_k^{r-1}\backslash H_k^{r-1}} \sum_{\delta\in\Theta_k^{r-2}\backslash H_k^{r-2}} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'_1(u) \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}_1(n)\, f(nu\delta\gamma g)\, dn\, du$$

Since $\Theta^{r-1} = H^{r-2}N'$, the elements $\delta\gamma$ with $\gamma\in\Theta_k^{r-1}\backslash H_k^{r-1}$ and $\delta\in\Theta_k^{r-2}\backslash H_k^{r-2}$ are in natural bijection with $\Theta_k^{r-2}N'_k\backslash H_k^{r-1}$. Certainly $nu\to\psi'_1(u)\psi_1(n)$ gives a character ψ_2 on NN', which is the unipotent radical $N^{r-2,1,1}$ of the $(r-2,1,1)$ parabolic. Thus, so far,

$$f(g) \;=\; \sum_{\gamma\in\Theta_k^{r-2}N'_k\backslash H_k} \int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}} \overline{\psi}_2(n)\, f(n\gamma g)\, dn$$

We need a separate notation for unipotent radicals inside $H = H^{r-1}\approx GL_{r-1}$: let $U^{b_1,\ldots,b_m}$ be the unipotent radical of the standard parabolic of H with blocks of size $b_1,\ldots,b_m$ along the diagonal. Then

$$\Theta^{r-2}\cdot N' \;=\; H^{r-3}\cdot U^{r-3,1,1}$$

Thus,

$$f(g) \;=\; \sum_{\gamma\in H_k^{r-3}U_k^{r-3,1,1}\backslash H_k} \int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}} \overline{\psi}_2(n)\, f(n\gamma g)\, dn$$

We repeat the induction step once more. For each $\gamma\in H_k^{r-3}U_k^{r-3,1,1}\backslash H_k$, the function

$$h \;\longrightarrow\; \int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}} \overline{\psi}_2(n)\, f(nh\gamma g)\, dn \qquad (\text{for } h\in H^{r-2})$$

is left invariant under N'_k, where

$$N' \;=\; \{u_x = \begin{pmatrix} 1_{r-3} & x & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\ \text{with}\ \ x = (r-3)\text{-by-}1\}$$

For each γ, there is a Fourier expansion along N',

$$\int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}} \overline{\psi}_2(n)\, f(nh\gamma g)\, dn \;=\; \sum_{\psi'} \int_{N'_k\backslash N'_{\mathbb{A}}} \overline{\psi}'(u) \int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}} \overline{\psi}_2(n)\, f(nuh\gamma g)\, dn\, du$$

The summand for trivial ψ' is 0, because f is a cuspform, and

$$N'\cdot\big(\ker\psi_2 \text{ on } N^{r-2,1,1}\big) \;\supset\; \{\begin{pmatrix} 1_{r-3} & * & * & *\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\} \;=\; \text{unipotent radical of } (r-3,3) \text{ parabolic}$$

The rational points of

$$H^{r-3} \;=\; \{\begin{pmatrix} GL_{r-3} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\}$$

act *transitively* on non-trivial characters ψ' of $U_k\backslash U_{\mathbb{A}}$. Let $\psi_2'(u_x) = \psi_o(x_{r-3})$. The isotropy group of ψ_2' is

$$\Theta^{r-3} \;=\; \{\begin{pmatrix} GL_{r-4} & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\}$$

Note that

$$\Theta^{r-3}\cdot U^{r-3,1,1} \;=\; H^{r-4}\cdot U^{r-4,1,1,1}$$

Setting $h = 1$,

$$f(g) \;=\; \sum_{\gamma\in H_k^{r-3}U_k^{r-3,1,1}\backslash H_k}\;\sum_{\delta\in\Theta_k^{r-3}\backslash H_k^{r-3}}\int_{N_k'\backslash N_{\mathbb{A}}'}\overline{\psi}_2'(\delta u\delta^{-1})\int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}}\overline{\psi}_2(n)\,f(nu\gamma g)\,dn\,du$$

Replace u by $\delta^{-1}u\delta$, and n by $\delta^{-1}n\delta$, noting that conjugation by $N_{\mathbb{A}}'$ leaves ψ_2 invariant:

$$f(g) \;=\; \sum_{\gamma\in H_k^{r-3}U_k^{r-3,1,1}\backslash H_k}\;\sum_{\delta\in\Theta_k^{r-3}\backslash H_k^{r-3}}\int_{N_k'\backslash N_{\mathbb{A}}'}\overline{\psi}_2'(u)\int_{N_k^{r-2,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1}}\overline{\psi}_2(n)\,f(nu\delta\gamma g)\,dn\,du$$

The double sum over $\delta\gamma$ can be regrouped into a single sum of $\gamma\in H_k^{r-4}\cdot U_k^{r-4,1,1,1}\backslash H_k$. Let

$$\psi_3\begin{pmatrix} 1_{r-4} & 0 & * & * & * \\ 0 & 1 & x_{r-3,r-2} & * & * \\ 0 & 0 & 1 & x_{r-2,r-1} & * \\ 0 & 0 & 0 & 1 & x_{r-1,r} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;=\; \psi_o(x_{r-3,r-2}+x_{r-2,r-1}+x_{r-1,r})$$

Then

$$f(g) \;=\; \sum_{\gamma\in H_k^{r-4}U_k^{r-4,1,1,1}\backslash H_k}\int_{N_k^{r-3,1,1,1}\backslash N_{\mathbb{A}}^{r-2,1,1,1}}\overline{\psi}_3(n)\,f(n\gamma g)\,dn$$

By induction, with $U^{\min}$ the unipotent radical of the standard minimal parabolic in H, and $N^{\min}$ the unipotent radical of the standard minimal parabolic in G,

$$f(g) \;=\; \sum_{\gamma\in U_k^{\min}\backslash H_k}\int_{N_k^{\min}\backslash N_{\mathbb{A}}^{\min}}\overline{\psi}_{\rm std}(n)\,f(n\gamma g)\,dn$$

Letting the Whittaker function attached to f be

$$W_f(g) \;=\; \int_{N_k^{\min}\backslash N_{\mathbb{A}}^{\min}}\overline{\psi}_{\rm std}(n)\,f(ng)\,dn$$

the Fourier expansion is

$$f(g) \;=\; \sum_{\gamma\in U_k^{\min}\backslash H_k} W_f(\gamma g)$$

2. *The Hecke-type integral representation:* $GL_n \times GL_{n-1}$

Still let H denote the copy of GL_{r-1} in the standard Levi component of the standard $(r-1,1)$ parabolic of $G = GL_r$. For cuspform f on GL_r and cuspform F on $H \approx GL_{r-1}$, the Hecke-type integral representation

$$\int_{H_k\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}}\, f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\, F(h)\, dh$$

produces the L-function $\Lambda(s, f\times F)$, up to normalization, as follows. Expressing f in its Fourier expansion, as above, unwind:

$$\int_{H_k\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}} \sum_{\gamma\in U_k^{\min}\backslash H_k} W_f\Big(\gamma\cdot\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\Big)\, F(h)\, dh \;=\; \int_{U_k^{\min}\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\, F(h)\, dh$$

The function $|\det h|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is left $\psi_{\rm std}$-equivariant under $U_{\mathbb{A}}^{\min}$, so rewrite

$$\int_{U_k^{\min}\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\, F(h)\, dh$$

$$= \int_{U_{\mathbb{A}}^{\min}\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \Big(\int_{U_k^{\min}\backslash U_{\mathbb{A}}^{\min}} \psi_{\rm std}(u)\, F(uh)\, du\Big)\, dh$$

$$= \int_{U_{\mathbb{A}}^{\min}\backslash H_{\mathbb{A}}} |\det h|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\, W_F(h)\, dh$$

where we note that the Whittaker function for F is formed with the complex-conjugated character $\psi_{\rm std}$ restricted from $N^{\min}$ to $U^{\min}$.

Under various hypotheses on f, F, the Whittaker functions factor over primes, and, then the zeta integral factors over primes, giving an *Euler product*

$$Z(s, f\times F) \;=\; \prod_v \Big(\int_{U_v^{\min}\backslash H_v} |\det h_v|^{s-\frac{1}{2}} W_{f,v}\begin{pmatrix} h_v & 0 \\ 0 & 1 \end{pmatrix}\, W_{F,v}(h_v)\, dh_v\Big)$$

Further, at places v where f and F are *spherical*, via an Iwasawa decomposition $H = U_v M_v^{\min} K_v$ with $M^{\min}$ the standard Levi component of the minimal parabolic in H, the v^{th} local integral becomes a much smaller, $(r-1)$-dimensional integral:

$$\int_{M_v^{\min}} |\det m_v|^{s-\frac{1}{2}} W_{f,v}\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\, W_{F,v}(m)\, \frac{dm}{\delta(m)} \qquad \Big(\text{with } m = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_{r-1} \end{pmatrix} \in GL_{r-1}\Big)$$

where $\delta(m)$ is the modular function of $M_v^{\min}$ on $U_v^{\min}$.

[2.0.1] Remark: The most important normalization constants are $\rho_f(1)$ and $\rho_F(1)$, which are the (higher-rank analogues of) leading Fourier coefficients of f and F, when f and F are normalized to have L^2-norm 1. Further, it is less clear that the archimedean local zeta integral is the correct gamma factor. Thus, even with everywhere-spherical f and F,

$$\text{zeta-integral } Z(s, f\times F) \;=\; \rho_f(1)\cdot\rho_{\overline{F}}(1)\cdot(\text{archimedean integrals})\cdot L(s, f\times F)$$

[2.0.2] Remark: The analytic continuation of this Euler product follows from the original integral representation, with relatively straightforward estimates on the cuspforms f, F. The functional equation comes essentially from replacing h by h-transpose-inverse in the integral. The effect of transpose-inverse on the local representations, hence, on the Euler factors, requires some further attention.

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3. *The Rankin-Selberg case* $GL_n \times GL_n$

Now we need an auxiliary (degenerate) Eisenstein series

$$E_s(g) \;=\; \sum_{\gamma \in P_k^{n-1,1} \backslash GL_n(k)} \varphi_s(\gamma \cdot g)$$

where $P^{n-1,1}$ has Levi component $M \approx GL_{n-1} \times GL_1$, and

$$\varphi_s \;=\; \bigotimes_v \varphi_{s,v}$$

with $\varphi_{s,v}$ in a (degenerate) induced representation from $P_v^{n-1,1}$. Specifically, the representation induced from M should be of the form

$$m \;=\; \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \;\longrightarrow\; \Big|\frac{\det A}{d^{n-1}}\Big|^s \cdot \chi\big(\frac{\det A}{d^{n-1}}\big) \qquad\qquad (\text{for } A \in GL_{n-1} \text{ and } d \in GL_1)$$

where $s \in \mathbb{C}$ and χ is a Hecke character. [4] We suppress reference to other data specifying the vector φ_s in the induced representation, although in practice this data must be chosen to accommodate bad-prime aspects of f, F, for example. The character χ must be chosen so that the central character of $E_s \cdot f \cdot \overline{F}$ is trivial, or the following integral is not well-defined.

The Rankin-Selberg zeta integral attached to two cuspforms f, F on GL_n is [5]

$$Z(s, f \otimes \overline{F}) \;=\; \int_{Z_{\mathbb{A}} GL_n(k) \backslash GL_n(\mathbb{A})} E_s(g) \cdot f(g) \cdot \overline{F}(g)\; dg$$

Let H be the GL_{n-1} factor of the Levi component M. In the region of convergence, unwind the zeta integral by unwinding the Eisenstein series:

$$Z(s, f \otimes F) \;=\; \int_{Z_{\mathbb{A}} H_k N_k^{n-1,1} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot f(g) \cdot \overline{F}(g)\; dh$$

Let $U^{\min}$ be the unipotent radical of the standard minimal parabolic in H. Expanding f in its Fourier-Whittaker expansion and unwinding, using the H_k-invariance of φ_s,

$$Z(s, f \otimes F) \;=\; \int_{Z_{\mathbb{A}} H_k N_k^{n-1,1} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot \sum_{\gamma \in U_k \backslash H_k} W_f(\gamma g) \cdot \overline{F}(g)\; dg$$

$$= \int_{Z_{\mathbb{A}} U^{\min}{}_k N_k^{n-1,1} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot W_f(g) \cdot \overline{F}(g)\; dh \;=\; \int_{Z_{\mathbb{A}} N_k^{\min} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot W_f(g) \cdot \overline{F}(g)\; dh$$

[4] The case of trivial χ is already useful. On the other hand, the character $|\cdot|^s$ can be incorporated into χ, if desired. Nevertheless, for analytical purposes, it is often convenient to separate the continuous parameter s from a parametrization of the compact part of the idele-class group: $\mathbb{J}^1/k^\times$ where $\mathbb{J}^1$ is ideles of idele-norm 1.

[5] The complex conjugation on F avoids certain uninteresting technicalities, as will become apparent.

where $N^{\min} = U^{\min} N^{n-1,1}$ is the unipotent radical of the standard minimal parabolic $P^{\min}$ in GL_n. Since φ_s is left $N_{\mathbb{A}}$-invariant, and the Whittaker function W_f is left $N_{\mathbb{A}}, \psi_{\mathrm{std}}$-equivariant, with the standard character ψ_{std} on $N_{\mathbb{A}}$,

$$\int_{Z_{\mathbb{A}} N_k^{\min} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot W_f(g) \cdot \overline{F}(g)\, dh \;=\; \int_{Z_{\mathbb{A}} N_{\mathbb{A}}^{\min} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot W_f(g) \Big(\int_{N_k^{\min} \backslash N_{\mathbb{A}}^{\min}} \psi_{\mathrm{std}}(u) \cdot \overline{F}(ug)\, du \Big)\, dh$$

$$= \int_{Z_{\mathbb{A}} N_{\mathbb{A}}^{\min} \backslash GL_n(\mathbb{A})} \varphi_s(g) \cdot W_f(g) \cdot \overline{W_F}(g)\, dg$$

It is here that the pre-emptive complex-conjugation of F gives the Whittaker function of F with respect to ψ_{std}, rather than with respect to its complex conjugate.

Under various hypotheses, the Whittaker functions factor over primes. When we take φ_s to be a monomial tensor, we have an Euler product

$$Z(s, f \otimes \overline{F}) \;=\; \prod_v \int_{Z_v N_v^{\min} \backslash GL_n(k_v)} \varphi_s(g) \cdot W_{f,v}(g) \cdot \overline{W_{F,v}}(g)\, dg \qquad \text{(with inducing data suppressed)}$$

[3.0.1] Remark: The most important normalization constants are $\rho_f(1)$ and $\rho_F(1)$, the (higher-rank analogues of) leading Fourier coefficients of f and F, when f and F are normalized to have L^2-norm 1. It is less clear that the archimedean local zeta integral is the correct gamma factor. Thus, even with everywhere-spherical f and F,

$$\text{zeta-integral } Z(s, f \times F) \;=\; \rho_f(1) \cdot \rho_{\overline{F}}(1) \cdot (\text{archimedean integrals}) \cdot L(s, f \times F)$$

[3.0.2] Remark: The analytic continuation of the zeta integral follows from the original integral representation, with relatively straightforward estimates on the cuspforms f, F, and from the meromorphic continuation and function equation of E_s. For this very degenerate Eisenstein series, the analytic continuation and functional equation follow from Poisson summation.

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4. *Comments on $GL_m \times GL_n$ with $m \leq n-2$*

As Jacquet, Piatetski-Shapiro, and Shalike found, for $m < n-1$ an intermediate integration is necessary. In the literature, such auxiliary integrations are often called *unipotent* integrations, and occur in other situations, as well. In the end, one has a zeta integral

$$Z(s, f \otimes F) \;=\; \int_{GL_m(k) \backslash GL_m(\mathbb{A})} |\det h|^{s-\frac{1}{2}} \cdot (\mathrm{proj}_m^n f) \begin{pmatrix} h & 0 \\ 0 & 1_{n-m} \end{pmatrix} \cdot F(h)\, dh$$

where the projection operator proj_m^n is the identity map for $m = n-1$, but non-trivial otherwise, described as follows. Let

$$N \;=\; N_m^r \;=\; \begin{pmatrix} 1_{m+1} & * & \ldots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \;=\; \text{unipotent radical of } (m+1, 1, 1, \ldots, 1) \text{ parabolic}$$

The proper definition of the projection turns out to be

$$(\mathrm{proj}_m^r f)(g) \;=\; |\det h|^{\frac{r-(m+1)}{2}} \int_{N_k \backslash N_{\mathbb{A}}} \overline{\psi}(n)\, f(ng)\, dn$$

For example, for $r = 3$ and $m = 1$, with F trivial on GL_1, the standard L-function attached to f is essentially the zeta integral

$$Z(s, f) \;=\; \int_{GL_1(k)\backslash GL_1(\mathbb{A})} |h|^{s-\frac{1}{2}+\frac{3-(1+1)}{2}}\, \mathrm{proj}_1^3 f \begin{pmatrix} h & & \\ & 1 & \\ & & 1 \end{pmatrix} dh$$

$$= \int_{GL_1(k)\backslash GL_1(\mathbb{A})} |h|^s \int_{k\backslash\mathbb{A}} \int_{k\backslash\mathbb{A}} \overline{\psi}_o(y)\, f\Big(\begin{pmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} h & & \\ & 1 & \\ & & 1 \end{pmatrix} \Big)\, dx\, dy\;\, dh$$

See [Cogdell 2003,07,08] for many further details about this general case.

Bibliography

[Cogdell 2003] J. Cogdell, *Analytic theory of L-functions for GL_n; Langlands conjectures for GL_n; Dual Groups and Langlands Functoriality*, in *An introduction to the Langlands program*, ed. J. Bernstein, S. Gelbart, Birkhauser, Boston, 2003, 197-268.

[Cogdell 2007] J.W. Cogdell, *L-functions and converse theorems for GL_n*, in *Automorphic forms and applications*, ed. P. Sarnak, F. Shahidi, Park City/IAS Math. Series **12**, AMS, 2007, 97-177.

[Cogdell 2008] J. Cogdell, *Notes on L-functions for GL_n*, in *School on Automorphic Forms on GL_n*, ICTP Lecture Notes Series, vol. 21, June 2008, 75-157.

[Cogdell-PS 2003] J. Cogdell, I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions*, in *Contributions to Automorphic Forms, Geometry, and Number Theory (Shalikafest 2002)*, H. Hida, D. Ramakrishnan, and F. Shahidi eds., Johns Hopkins University Press, Baltimore, 2005.

[Godement-Jacquet 1972] R. Godement, H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math. **260**, Springer-Verlag, Berlin-New York, 1972.

[Hecke 1918] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen der Verteilung der Primzahlen*, Math. Z. **1** no. 4 (1918), 357-376.

[Hecke 1920] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen der Verteilung der Primzahlen*, Math. Z. **6** no. 1-2 (1920), 11-51.

[Hecke 1937a] E. Hecke, *Über Modulfunctionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, I*, Math. Ann. **114** (1937) no. 1, 1-28.

[Hecke 1937b] E. Hecke, *Über Modulfunctionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, II*, Math. Ann. **114** (1937) no. 1, 316-351.

[Iwasawa 1952] K. Iwasawa, *Letter to J. Dieudonné*, dated April 8, 1952, in *Zeta Functions in Geometry*, ed. N. Kurokawa and T. Sunada, Advanced Studies in Pure Mathematics **21** (1992), 445-450.

[Jacquet-PS-Shalika 1979] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on $GL(3)$*, Annals of Math. **109** (1979), 169-258.

[Jacquet-PS-Shalika 1981] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981),199-214.

[Jacquet-PS-Shalika 1983] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367-464.

[Jacquet-Shalika 1981] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations, I*, Amer. J. Math. **103** (1981), 499-558.

[Jacquet-Shalika 1981b] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations, II*, Amer. J. Math. **103** (1981), 777-815.

[Jacquet-Shalika 1990] H. Jacquet, J. Shalika, *Rankin-Selberg convolutions: archimedean theory*, in *Festschrift in Honor of I.I. Piatetski-Shapiro, I*, Weizmann Science Press, Jerusalem, 1990, 125-207.

[Langlands 1967/1976] R.P.Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, Berlin and New York, 1976.

[Langlands 1971] R. Langlands, *Euler Products*, Yale Univ. Press, New Haven, 1971.

[Shahidi 1978] F. Shahidi, *Functional equation satisfied by certain L-functions*, Comp. Math. **37** (1978), 171-208.

[Shahidi 1985] F. Shahidi, *Local coefficients as Artin fators for real groups*, Duke Math. J. **52** (1985), 973-1007.

[Tate 1950] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, thesis, Princeton (1950), in *Algebraic Number Theory*, ed. J. Cassels and J. Frölich, Thompson Book Co., 1967.