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Traces, Cauchy identity, Schur polynomials

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The unobvious *Cauchy identity*[1] is proven by taking a *trace* of a representation of $GL_n(\mathbb{C})$ in two different ways: one as the sum of traces of irreducible subrepresentations, the other as sum of weight spaces, that is, irreducible subrepresentations for a maximal torus.

The representation whose trace we take is the space of holomorphic polynomials on n-by-n complex matrices.

The holomorphic irreducibles occurring are identified by restriction to the realform $U(n)$, invoking the decomposition of the biregular representation of $U(n)$.

Such identities arise in Rankin-Selberg integral representations of L-functions. For GL_2, a naive, direct computation is sufficient. However, for general GL_n and other higher-rank groups, direct computation is inadequate. Further, connecting local Rankin-Selberg computations to Schur functions usefully connects these computations to the Shintani-Casselman-Shalika formulas for spherical p-adic Whittaker functions.

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1. *Example:* GL_2

First, we consider the smallest example of the Cauchy identity, namely, for $GL_2(\mathbb{C})$.

The identity [2]

$$\sum_{n=0}^{\infty} z^n \cdot \left(\frac{a^{n+1}-a^{-(n+1)}}{a-a^{-1}}\right)\left(\frac{b^{n+1}-b^{-(n+1)}}{b-b^{-1}}\right) \;=\; \frac{1-z^2}{(1-zab)\,(1-zab^{-1})\,(1-za^{-1}b)\,(1-za^{-1}b^{-1})}$$

is easily proven by elementary algebra and summing geometric series. Slightly more generally, absorbing z into the other indeterminates,

$$\sum_{n=0}^{\infty} \left(\frac{a^{n+1}-c^{n+1}}{a-c}\right)\left(\frac{b^{n+1}-d^{n+1}}{b-d}\right) \;=\; \frac{1-abcd}{(1-ab)\,(1-ad)\,(1-cb)\,(1-cd)}$$

for a, b, c, d small, or viewing everything as taking place in a formal power series ring. The above is the simplest instance of *Cauchy's identity*, involving the smallest *Schur polynomials*

$$\frac{a^{n+1}-c^{n+1}}{a-c} \;=\; a^n + a^{n-1}c + a^{n-2}c^2 + \ldots a^2c^{n-2} + ac^{n-1} + c^n$$

In fact, the above identities have meaning: we claim that the rearrangement

$$\sum_{m,n\geq 0} (ac)^m \left(\frac{a^{n+1}-c^{n+1}}{a-c}\right)\,(bd)^m \left(\frac{b^{n+1}-dn+1}{b-d}\right) \;=\; \frac{1}{(1-ab)\,(1-ad)\,(1-cb)\,(1-cd)}$$

[1] The Cauchy identity is often treated as a combinatorial result, in contexts where Schur functions are defined by *formulas*, rather than acknowledged to be *traces* of highest-weight *representations*.

[2] This identity also arises in the non-archimedean local factors of the GL_2 Rankin-Selberg integrals: the two factors in the infinite sum are values of spherical p-adic Whittaker functions.

computes the *trace* of the representation of $GL_2(\mathbb{C})\times GL_2(\mathbb{C})$ on the space $\mathbb{C}[E]$ of holomorphic polynomials on the set E of two-by-two complex matrices, in two ways. First, let $g\times h\in GL_2(\mathbb{C})\times GL_2(\mathbb{C})$ act on a polynomial f on E by

$$(g\times h)f(x) \;=\; f(g^\top xh)$$

Using transpose rather than inverse in the left action is a convenience: the function $g\to(g\times h)f(x)$ is thereby a *polynomial* function in g. On the left-hand side of the identity above,

$$\begin{pmatrix} a & 0\\ 0 & c\end{pmatrix} \longrightarrow (ac)^m\left(\frac{a^{n+1}-c^{n+1}}{a-c}\right)$$

is the *character* (that is, *trace*) $\chi_{m,n}$ of the representation $\pi_{m,n}=\det^m\otimes\mathrm{Sym}^n(\mathrm{std})$ of GL_2, where $\mathrm{Sym}^n(\mathrm{std})$ is the n^{th} symmetric power of the standard representation *std*. The representation is *polynomial* if and only if $m\geq 0$. The character of the representation $\pi_{m,n}\otimes\pi_{m,n}$ of $GL_2(\mathbb{C})\times GL_2(\mathbb{C})$ is the product:

$$\mathrm{trace}\,\big(\pi_{m,n}(g)\otimes\pi_{m,n}(h)\big) \;=\; \chi_{m,n}(g)\cdot\chi_{m,n}(h)$$

Thus, the left-hand side of the identity is

$$\mathrm{tr}\Big(\bigoplus_{m,n\geq 0}\pi_{m,n}\otimes\pi_{m,n}\Big) \;=\; \sum_{m,n\geq 0}\mathrm{tr}\Big(\pi_{m,n}\otimes\pi_{m,n}\Big) \;=\; \sum_{m,n\geq 0}\chi_{m,n}\cdot\chi_{m,n}$$

Thus, we must verify (below) that

$$\mathbb{C}[E] \;\approx\; \bigoplus_{m,n\geq 0}\pi_{m,n}\otimes\pi_{m,n} \qquad\qquad (\text{as } GL_2(\mathbb{C})\times GL_2(\mathbb{C}) \text{ representations})$$

On the other hand, we can look at the subgroups

$$M \;=\; \{\begin{pmatrix} a & 0\\ 0 & c\end{pmatrix}\}\subset GL_2(\mathbb{C}) \qquad \text{and} \qquad M\times M\subset GL_2(\mathbb{C})\times GL_2(\mathbb{C})$$

and decompose $\mathbb{C}[E]$ as a representation of $M\times M$. Monomials

$$f_{p,q,r,s}\begin{pmatrix} x & y\\ z & w\end{pmatrix} \;=\; x^p\,y^q\,z^r\,w^s$$

are obvious *weight vectors*, that is, vector in one-dimensional representations for the action of $M\times M$:

$$(\begin{pmatrix} a & 0\\ 0 & c\end{pmatrix}\times\begin{pmatrix} b & 0\\ 0 & d\end{pmatrix})f_{p,q,r,s}\begin{pmatrix} x & y\\ z & w\end{pmatrix} \;=\; a^{p+q}\,b^{p+r}\,c^{r+s}\,d^{q+s}\cdot f_{p,q,r,s}\begin{pmatrix} x & y\\ z & w\end{pmatrix}$$

Thus, it is easy to decompose $\mathbb{C}[E]$ into one-dimensional irreducibles over $M\times M$, and the trace is

$$\sum_{p,q,r,s\geq 0}a^{p+q}\,b^{p+r}\,c^{r+s}\,d^{q+s} \;=\; \sum_p(ab)^p\cdot\sum_q(ad)^q\cdot\sum_r(bc)^r\cdot\sum_s(cd)^s$$

$$=\;\frac{1}{(1-ab)\,(1-ad)\,(1-bc)\,(1-cd)} \;=\; \frac{1}{\det\big(1_4-\begin{pmatrix} a & 0\\ 0 & c\end{pmatrix}\otimes\begin{pmatrix} b & 0\\ 0 & d\end{pmatrix}\big)} \qquad (\text{for small } a,b,c,d)$$

Thus, if we verify the claim about decomposition of $\mathbb{C}[E]$ as an $GL_2(\mathbb{C})\times GL_2(\mathbb{C})$ representation, we will have

$$\sum_{m,n\geq 0}\chi_{m,n}(g)\otimes\chi_{m,n}(h) \;=\; \frac{1}{\det\big(1_4-g\otimes h\big)} \qquad (\text{for } g,h \text{ } \textit{diagonal}\text{, small})$$

Both sides are conjugation-invariant, so we have equality for small *diagonalizable* matrices. The density of diagonalizable matrices in $GL_2(\mathbb{C})$ and continuity imply that the identity holds for all small elements of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$.

It is clear that only non-negative weights can appear in a decomposition of the space of polynomials $\mathbb{C}[E]$ as $M \times M$ representation space, but it is less clear that *all* holomorphic irreducibles of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ with exclusively non-negative weights *do* appear. This will be proven after some preparation.

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2. $GL_n(\mathbb{C})$ and $U(n)$

The group $K = U(n)$ is a maximal compact subgroup of $G = GL_n(\mathbb{C})$. However, for present purposes, the critical relationship is that any holomorphic function on G vanishing identically when restricted to K is already identically 0 on G. This will imply that the restriction from G to K of an irreducible *holomorphic* representation of G remains irreducible. Further, we will see that *every* irreducible of K occurs as the restriction from G of an irreducible holomorphic representation.

[2.1] Realforms The conclusions about restriction of holomorphic functions will follow from the fact that the Lie algebra $\mathfrak{k} = \mathfrak{u}(n)$ of K is a *realform* [3] of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. To see that $\mathfrak{k}$ is a realform of $\mathfrak{g}$, observe that $\mathfrak{k}$ consists of skew-hermitian matrices. The $\mathbb{R}$-dimension of $\mathfrak{k}$ is equal to the $\mathbb{C}$-dimension of $\mathfrak{g}$, and any complex n-by-n matrix z can be written as a $\mathbb{C}$-linear combination of elements of $\mathfrak{k}$, from the familiar identity

$$z = \frac{z+z^*}{2} + i \cdot \frac{z-z^*}{2i} = i \cdot \frac{z+z^*}{2i} + \frac{z-z^*}{2}$$

where $z \to z^*$ is conjugate-transpose. This proves the realform property.

[2.2] Restrictions of holomorphic functions We claim that a holomorphic function on G vanishes identically on K if and only if it is identically zero on G. Subordinate to this, we claim that a holomorphic function on $\mathfrak{g}$ vanishes identically on $\mathfrak{k}$ if and only if it is identically zero on $\mathfrak{g}$.

Since G is *connected*, a holomorphic function on it is completely determined by its power series expansion at the identity $e \in G$. The exponential map $\mathfrak{g} \to G$ is holomorphic, and is a local isomorphism near $0 \in \mathfrak{g}$ and $e \in G$. Thus, a holomorphic function f near $e \in G$ gives a holomorphic function $F(x) = f(e^x)$ for x near 0 in $\mathfrak{g}$, and vice-versa. That is, f is identically 0 if and only if F is identically 0. This reduces the first claim about restriction from G to K to the second claim about restriction from $\mathfrak{g}$ to $\mathfrak{k}$.

The coefficients of the complex power series expansion at 0 of a holomorphic function on a complex vector space V are essentially complex derivatives of the function. All these complex derivatives can be computed along a realform V_o of V, exactly because the realform spans the complex vectorspace. Thus, if the restriction of a holomorphic function to V_o is identically 0, all its real derivatives along V_o are 0, so its complex derivatives at 0 are 0. Thus, its complex power series expansion at 0 is identically 0. Since V is connected, the function is identically 0.

[2.3] Irreducibility criterion via coefficient functions We say that a representation π of $G = GL_n(\mathbb{C})$ is **holomorphic** when $g \to g \cdot v$ is a holomorphic π-valued function on G, for every $v \in \pi$. For finite-dimensional π this property is immediately equivalent [4] to holomorphy of the $\mathbb{C}$-valued coefficient functions

$$c^{\pi}_{v,\mu}(g) = \langle g \cdot v, \mu \rangle = \mu(gv) \qquad (\text{for } v \in \pi \text{ and } \mu \in \pi^*)$$

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[3] To assert that a real vector subspace V of a complex vector space W is a *real form* of W is to assert that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} W$ and that the *complex* span of V is all of W. Equivalently, the natural complex-linear map $V \otimes_{\mathbb{R}} \mathbb{C} \to W$ is an isomorphism.

[4] Holomorphy of coefficient functions implies holomorphy of the representation for many infinite-dimensional representations, as well, simply because *weak* holomorphy implies (strong) holomorphy for a broad class of topological

By the above discussion, for $v \neq 0$ and $\mu \neq 0$, the non-zero coefficient function $c^\pi_{v,\mu}$ on G restricts to a non-zero function on K. This restriction is obviously the v, μ coefficient function of the restriction $\mathrm{Res}^G_K \pi$ of π to K.

We claim that a representation τ is irreducible if and only if no one of its coefficients $c^\tau_{v,\mu}$ vanishes identically, of course for $v \neq 0$ and $\mu \neq 0$.

Indeed, for such v, μ, if $c^\tau_{v,\mu}$ were identically 0, then the span of images gv would be inside the kernel of μ, a *proper* closed subspace of τ, contradicting the irreducibility of τ. Conversely, if τ had a proper closed subspace, there would be a continuous linear functional [5] μ whose kernel contains that subspace, so $c^\tau_{v,\mu}$ would be identically 0 for v in that subspace. This proves the irreducibility criterion.

[2.4] Restriction of holomorphic representations from G to K

Thus, since non-zero coefficient functions of irreducible holomorphic representations π of G restrict to non-zero functions on K, the restriction $\mathrm{Res}^G_K \pi$ of π to K remains irreducible.

We will show just below that *every* irreducible of K occurs as the restriction from G of a holomorphic representation. [6]

[2.5] Highest weights

The irreducibles of $K = U(n)$ are parametrized by *highest weights* λ, in the following sense. Specifically, let $\mathfrak{a}$ be the standard Cartan subalgebra of diagonal elements of the complexification $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ of $\mathfrak{k}$, and $\mathfrak{n}$ the strictly upper-triangular elements of $\mathfrak{g}$. The irreducible complex representations of K are *finite*-dimensional, so $\mathfrak{k}$ and $\mathfrak{g}$ act on them [7] by taking derivatives: for $\theta \in \mathfrak{k}$, $v \in \tau$,

$$\theta \cdot v \;=\; \frac{\partial}{\partial t}\Big|_{t=0} e^{t\theta} \cdot v$$

The action of $\mathfrak{g}$ is the $\mathbb{C}$-linear extension.

In an irreducible τ of K, the space $\tau^{\mathfrak{n}}$ of vectors annihilated by $\mathfrak{n}$ is *one-dimensional.* Necessarily $\mathfrak{a}$ acts on $\tau^{\mathfrak{n}}$ by a character $\lambda = \lambda_\tau : \mathfrak{a} \to \mathbb{C}$. The subspace $\tau^{\mathfrak{n}}$ is the space of *highest weight vectors* in τ, and λ is the *highest weight.*

The map $\tau \to \lambda$ is *injective*: the highest weight of an irreducible determines the isomorphism class of the irreducible.

Not every $\lambda \in \mathfrak{a}^*$ appears as a highest weight of an irreducible of K. There are two conditions necessary (and sufficient) for λ to appear as a highest weight. Any $\lambda \in \mathfrak{a}^*$ is of the form

$$\lambda \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \;=\; \lambda_1\alpha_1 + \ldots + \lambda_n\alpha_n \qquad\qquad (\text{with } \lambda_i \in \mathbb{C})$$

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vector spaces. Namely, weak holomorphy implies holomorphy when for functions taking values in a locally convex and quasi-complete topological vector space. The argument uses a variant of the Banach-Steinhaus theorem, and Gelfand-Pettis integrals.

[5] For finite-dimensional π this is easy, and the same conclusion holds for locally convex representation spaces, by Hahn-Banach.

[6] Of course, the holomorphic representations of G that restrict to irreducibles of K are finite-dimensional and irreducible, as representations of G, since they are finite-dimensional and irreducible as representations of K.

[7] In infinite-dimensional representations of Lie groups, the collection of vectors upon which the Lie algebra can act is *dense*, but typically *not* the whole representation space. For finite-dimensional spaces this issue does not arise.

Then λ appears as a highest weight of an irreducible of K if and only if

$$\begin{cases} \text{all } \lambda_j \in \mathbb{Z} & \text{(Integrality)} \\ \\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n & \text{(Dominance)} \end{cases}$$

[2.6] Surjectivity of restriction from G to K Now we can show that *every* irreducible τ of K is the restriction of a holomorphic representation π of G, and uniquely so. Specifically, given a *dominant*, *integral* highest weight λ, we construct a finite-dimensional holomorphic representation π_λ of G which restricts to the irreducible τ_λ of K having highest weight λ.

Since λ is *integral*, we can tensor τ_λ with an integer power of *det* to assume that $\lambda_n \geq 0$.

Let G act on holomorphic polynomials f on n-by-n complex matrices x by right translation:

$$(g \cdot f)(x) \;=\; f(xg)$$

Letting $x^{(j)}$ be the upper left j-by-j minor of x, let

$$f_\lambda(x) \;=\; \big(\det x^{(1)}\big)^{\lambda_1-\lambda_2}\big(\det x^{(2)}\big)^{\lambda_2-\lambda_3}\big(\det x^{(3)}\big)^{\lambda_3-\lambda_2}\ldots\big(\det x^{(n-1)}\big)^{\lambda_{n-1}-\lambda_n}\big(\det x^{(n)}\big)^{\lambda_n}$$

Note that the integrality of λ, and $\lambda_n \geq 0$, makes f_λ a genuine polynomial. By design, for p in the standard minimal parabolic P of upper-triangular matrices,

$$(p \cdot f_\lambda)(x) \;=\; f_\lambda(xp) \;=\; p_1^{\lambda_1}\, p_2^{\lambda_2} \ldots p_n^{\lambda_n} \cdot f_\lambda(x)$$

where p_j is the j^{th} diagonal entry of p. Thus, $\mathfrak{n}$ annihilates f_λ, and $\mathfrak{a}$ acts on f_λ by the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Take π_λ to be the G-subrepresentation of the space of holomorphic polynomials generated by f_λ.

Thus, by construction, $\mathrm{Res}^G_K \pi_\lambda$ contains a copy of the irreducible τ_λ of K with highest weight λ, with highest weight vector f_λ. We claim that the restriction is *exactly* τ_λ.

Since the whole space of holomorphic polynomials in $x \in E$ of a given total degree is finite-dimensional, certainly π_λ is finite-dimensional. Use the Iwasawa decomposition

$$G \;=\; K \cdot A^+ \cdot N$$

where A^+ is positive real diagonal elements of G and N is upper-triangular unipotent elements. The exponential map from the Lie algebra of A^+ to A^+ is obviously surjective. It is less obvious, but an easy induction, that the exponential map from the Lie algebra $\mathfrak{n}$ to N surjects. Thus, [8] the G-orbit of v_o is equal to the K-orbit of v_o:

$$G \cdot v_o \;=\; K \cdot A^+ \cdot N \cdot v_o \;=\; K \cdot \mathbb{C}v_o$$

By *complete reducibility* for finite-dimensional representations of compact groups, any K-representation generated by a highest weight vector is irreducible. [9] Thus, the holomorphic G-representation π_λ so-

[8] For $x \in \mathfrak{g}$ and v a smooth vector in a representation space π, the smooth π-valued function $F(t) = e^{tx} \cdot v$ satisfies the differential equation $F' = x \cdot F$, with initial condition $F(0) = v$. The Mean Value Theorem proves uniqueness of the solution. Thus, assuming existence of the *group* representation π, the *algebra* action of $x \in \mathfrak{g}$ determines the *group* action of e^{tx} completely.

[9] Let σ be a finite-dimensional K-representation generated by a highest weight vector v_o. By complete reducibility, σ is a direct sum of K-irreducibles σ_j. Each σ_j has a unique (up to scalars) highest weight vector w_j with weight λ_j. Thus, v_o is a linear combination of those w_j's such that $\lambda_j = \lambda$. That is, since v_o generates σ, in fact, *all* w_j's have $\lambda_j = \lambda$, and the σ_j's are mutually isomorphic. Scaling the w_j as necessary, we can write $v_o = \sum_j w_j$. Let $\varphi_j : \sigma_1 \to \sigma_j$ be the unique K-isomorphism taking w_1 to w_j. The K-orbit of v_o consists of elements

$$k \cdot v_o \;=\; \sum_j k \cdot w_j \;=\; \sum_j k \cdot \varphi_j(w_1) \;=\; \Big(\sum_j \varphi_j\Big)(k \cdot w_1)$$

That is, v_o generates a K-homomorphic image of σ_1, so is irreducible. Thus, in fact, there is just one summand σ_j.

constructed restricts to τ_λ on K.

That is, given the irreducible τ_λ of K with (dominant integral) highest weight λ, with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, the holomorphic polynomial representation π_λ constructed above restricts to τ_λ.

Dropping the assumption $\lambda_n \geq 0$, a similar conclusion follows by tensoring with integer powers of determinant, but the *polynomial* character of π_λ seems to be lost when $\lambda_n < 0$, whether or not λ is dominant integral.

That is, whether or not $\lambda_n \geq 0$, every irreducible τ_λ of K is the restriction of a holomorphic irreducible π_λ of G. If $\lambda_n \geq 0$, then π_λ occurs as a subrepresentation of holomorphic polynomials $\mathbb{C}[E]$ on n-by-n complex matrices E.

[2.7] Extendability of representations of G Since $G = GL_n(\mathbb{C})$ is a dense open subset of the complex vector space E of n-by-n complex matrices, for holomorphic (or other) representations π of G it is reasonable to ask whether π *extends* continuously to a holomorphic $\mathrm{End}_{\mathbb{C}}(\pi)$-valued function on E.

For *finite-dimensional* holomorphic representations, extendability of π is equivalent to extendability of all coefficient functions of π.

In fact, for finite-dimensional holomorphic representations, the highest-weight classification above shows that for any finite-dimensional holomorphic π, there is an integer ℓ such that $\det^\ell \otimes \pi$ is *polynomial*, so certainly extends to a holomorphic function on E.

Specifically, the finite-dimensional holomorphic representations π_λ with dominant integral highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_n \geq 0$ are polynomial, and extend to E.

We claim that π_λ does *not* extend when $\lambda_n < 0$. Recall that the rational function

$$f_\lambda(x) \;=\; \big(\det x^{(1)}\big)^{\lambda_1-\lambda_2} \big(\det x^{(2)}\big)^{\lambda_2-\lambda_3} \big(\det x^{(3)}\big)^{\lambda_3-\lambda_2} \ldots \big(\det x^{(n-1)}\big)^{\lambda_{n-1}-\lambda_n} \big(\det x^{(n)}\big)^{\lambda_n}$$

generates π_λ under the right action of G, regardless of the sign of λ_n. Specifically, for upper-triangular p,

$$f_\lambda(xp) \;=\; p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_n^{\lambda_n} \cdot f_\lambda(x)$$

where p_i is the i^{th} diagonal entry of p. For $\lambda_n < 0$, this function cannot extend to E, because for fixed non-zero $p_1, \ldots, p_{n-1}$ it blows up as p_n goes to 0. This proves the claim.

Thus, further, if a single (non-zero) coefficient function of π^λ extends, then *all* coefficient functions extend. Indeed, a coefficient function $c^\pi_{v\mu}(g)$ that does *not* extend is not polynomial, so $\lambda_n < 0$, and π_λ does not extend.

3. *Decomposing holomorphic polynomials over* $GL_n \times GL_n$

The Cauchy identity is obtained by taking the trace of a representation in two ways. The analogue of the Cauchy identity for bigger groups $GL_n(\mathbb{C})$ cannot use explicit elementary expressions for characters of finite-dimensional irreducibles, as did the $GL_2(\mathbb{C})$ example.

[3.1] Schur polynomials Without trying to write an explicit expression [10] for it, take the λ^{th} **Schur polynomial** $s_\lambda(t_1, \ldots, t_n)$ to be the *character*

$$s_\lambda(t_1, \ldots, t_n) \;=\; \mathrm{trace}\,\Big(\pi_\lambda \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\Big)$$

[10] The Weyl character formula gives a *sort* of explicit expression for the Schur polynomials, but that expression is not useful in proving the Cauchy identity here.

of the holomorphic irreducible π_λ of $GL_n(\mathbb{C})$ with highest weight [11] λ, evaluated on diagonal matrices with entries $t_1, \ldots, t_n$.

[3.2] The polynomial representation Let $G \times G$ act on holomorphic polynomials $\mathbb{C}[E]$ on the collection E of complex n-by-n matrices, by

$$(g \times h)f(x) \;=\; f(g^\top \cdot x \cdot h)$$

Note that the left action has *transpose* in it, rather than *inverse*. We do this to have the special polynomial

$$f_\lambda(x) \;=\; \big(\det x^{(1)}\big)^{\lambda_1-\lambda_2}\big(\det x^{(2)}\big)^{\lambda_2-\lambda_3}\big(\det x^{(3)}\big)^{\lambda_3-\lambda_2}\ldots\big(\det x^{(n-1)}\big)^{\lambda_{n-1}-\lambda_n}\big(\det x^{(n)}\big)^{\lambda_n}$$

be acted upon on the *left* by P by

$$f_\lambda(p^\top x) \;=\; p_1^{\lambda_1}\, p_2^{\lambda_2} \ldots p_n^{\lambda_n} \cdot f_\lambda(x)$$

in addition to the corresponding action of P on the right noted above. Thus, the representation of $G \times G$ generated by f_λ restricts to the irreducible representation of $K \times K$ with highest weight $\lambda \oplus \lambda$.

In particular, the representation $\pi_\lambda \otimes \pi_\lambda$ occurs in $\mathbb{C}[E]$ at least once. This is a *lower* bound on the decomposition of $\mathbb{C}[E]$ into irreducibles.

To obtain an *upper* bound on $\mathbb{C}[E]$, observe that the restriction of functions from G to K is *injective* on *holomorphic* functions. Thus, under restriction of functions, $\mathbb{C}[E]$ *injects* to smooth functions on K. Further, since $\mathbb{C}[E]$ consists of *polynomials*, every vector in it is $G \times G$-*finite* [12] and, hence, $K \times K$-finite. In the biregular representation of $K \times K$ on functions on K, the collection of $K \times K$-finite vectors is exactly the algebraic direct sum

$$\sum_\lambda \tau_\lambda \otimes \tau_\lambda^\vee \qquad\qquad \text{(summed over dominant integral highest weights)}$$

where τ_λ is the irreducible of K with highest weight λ.

Since the left action of G on $\mathbb{C}[E]$ is not $(g \cdot f)(x) = f(g^{-1}x)$, but $(g \cdot f)(x) = f(g^{-1}x)$, we might want to reconcile this with the convention for the biregular representation of $K \times K$. We claim that for an irreducible τ of K, replacing $k \cdot v = \tau(g)(v)$ by $\tau((k^\top)^{-1})(v)$ yields the *contragredient* $\tau^\vee$ to τ. To see this, use the classification by highest weights. Indeed, taking transpose converts a *highest* weight to a *lowest* weight, and vice-versa. The theory of highest weights shows that *finite-dimensional* highest-weight representations have lowest weight equal to the negative of the highest weight of the *contragredient*. Thus, taking inverse (after taking tranpose) gives a representation of K with the same highest weight as the contragredient $\tau^\vee$, hence, isomorphic to $\tau^\vee$.

Thus, with inverse replaced by transpose for the left action, the restriction of the representation $\mathbb{C}[E]$ of $G \times G$ to a representation of $K \times K$ necessarily gives a subrepresentation of

$$\sum_\lambda \tau_\lambda \otimes \tau_\lambda$$

[11] This use of *highest weight* is a potentially confusing slight abuse of language, since the *true* weights for $GL_n(\mathbb{C})$ would necessarily include *all* characters of the Cartan subgroup (diagonal matrices), not merely holomorphic ones. But restriction to $U(n)$ sends irreducibles to irreducibles, and it is *literally* correct to say that the restriction of π_λ to $U(n)$ has highest weight λ.

[12] As usual, a vector v in a representation π of a group G is *G-finite* when the span of the images $g \cdot v$ is finite-dimensional.

Since π_λ restricts to τ_λ, this implies that $\pi_\lambda \otimes \pi_\lambda$ occurs *at most* once in $\mathbb{C}[E]$, giving the desired upper bound on multiplicities.

Finally, since the representation of $G \times G$ on $\mathbb{C}[E]$ is polynomial, it certainly *extends* to $E \times E$, as do coefficient functions of it irreducible subrepresentations. From the earlier discussion of extendability, the only representations $\pi_\lambda \otimes \pi_\lambda$ appearing must have $\lambda_n \geq 0$. We have already observed that for $\lambda_n \geq 0$, the special polynomial f_λ certainly is in $\mathbb{C}[E]$, and generates a copy of $\pi_\lambda \otimes \pi_\lambda$. Thus,

$$\mathbb{C}[E] \;\approx\; \sum_{\lambda\,:\,\lambda_n \geq 0} \pi_\lambda \otimes \pi_\lambda$$

[3.3] Taking trace: the Cauchy identity Though $\mathbb{C}[E]$ is an infinite-dimensional representation of $G \times G$, it is an ascending union, or colimit, of finite-dimensional representations, so has a *trace*.

On one hand, expressing the trace in terms of the irreducible summands $\pi_\lambda \otimes \pi_\lambda$ gives

$$\mathrm{trace}\,\mathbb{C}[E] \;=\; \sum_{\lambda\,:\,\lambda_n \geq 0} \mathrm{tr}\big(\pi_\lambda \otimes \pi_\lambda\big) \;=\; \sum_{\lambda\,:\,\lambda_n \geq 0} \mathrm{tr}\pi_\lambda \cdot \mathrm{tr}\pi_\lambda \;=\; \sum_{\lambda\,:\,\lambda_n \geq 0} s_\lambda \cdot s_\lambda$$

On the other hand, the *monomials* in $\mathbb{C}[E]$ are weight vectors for the subgroup $M \times M$ of diagonal matrices in $G \times G$. Thus, evaluating the trace on diagonal elements

$$a \times b \;=\; \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \times \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

$$\big(\mathrm{trace}\,\mathbb{C}[E]\big)(a \times b) \;=\; \sum_{n_{ij} \geq 0} \prod_{i,j=1}^{n} \big(a_i b_j\big)^{n_{ij}} \;=\; \prod_{i,j=1}^{n} \frac{1}{1 - a_i b_j} \;=\; \frac{1}{\det\big(1_{n^2} - a \otimes b\big)}$$

Thus, we have Cauchy's identity for diagonal elements:

$$\sum_{\lambda\,:\,\lambda_n \geq 0} s_\lambda(a) \cdot s_\lambda(b) \;=\; \frac{1}{\det\big(1_{n^2} - a \otimes b\big)}$$

either as a formal power series identity or for convergent power series with a, b small. Both sides are conjugation-invariant, so the identity holds for *diagonalizable* elements of $G \times G$. These are *dense*, so the identity holds throughout:

$$\sum_{\lambda\,:\,\lambda_n \geq 0} s_\lambda(g) \cdot s_\lambda(h) \;=\; \big(\mathrm{tr}\mathbb{C}[E]\big)(g \times h) \;=\; \frac{1}{\det\big(1_{n^2} - g \otimes h\big)} \qquad\qquad (\text{for } g, h \in G)$$

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