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Spheres and hyperbolic spaces

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Basic examples of non-Euclidean geometries are best studied by studying the groups that preserve the geometries. In fact, rather than specifying the geometry, we specify the *group*.

The group-invariant geometry on spheres is the familiar *spherical geometry*, with a simple relation to the ambient Euclidean geometry.

The group-invariant geometry on real and complex n-balls is *hyperbolic* geometry, in the sense that there are infinitely many *straight lines* (geodesics) through a given point not on a given straight line, thus contravening the parallel postulate for *Euclidean* geometry. We will not directly consider geometric notions, since the transitive group action determines structure in a more useful form. Still, this explains the terminology.

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1. *Rotations of spheres*

The elementary ideas of this section are important enough to deserve a review.

Let $\langle,\rangle$ be the usual inner product on $\mathbb{R}^n$, namely

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i\, y_i \qquad \text{(where } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n))$$

The *distance* function is definable in terms of this, as usual, by

$$\text{distance from } x \text{ to } y = |x - y| \qquad \text{(where } |x| = \langle x, x\rangle^{1/2})$$

The standard $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ is

$$S^{n-1} = \{x \in \mathbb{R}^n \;:\; |x| = 1\}$$

As usual, the **general linear** and **special linear** groups of size n (over $\mathbb{R}$) are

$$\begin{array}{rcccl} GL_n(\mathbb{R}) & = & \{n\text{-by-}n \text{ invertible real matrices}\} & = & \text{general linear group} \\ SL_n(\mathbb{R}) & = & \{g \in GL_n(\mathbb{R}) : \det g = 1\} & = & \text{special linear group} \end{array}$$

The modifier *special* refers to the determinant-one condition.

Our definition of **rotation** [1] in $\mathbb{R}^n$ will be a *linear* map of $\mathbb{R}^n$ to itself which preserves *distances*, *angles*, and has *determinant one* (to preserve *orientation*). The condition that a linear map g preserves angles and

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[1] A direct way to define *rotation* in $\mathbb{R}^2$ is as a linear map with a matrix of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some real θ. This definition is deficient insofar as it depends on a choice of basis. A definition in $\mathbb{R}^3$ is that a rotation is a linear map g that has an *axis*, in the sense that there is a line L *fixed* by g, and on the orthogonal complement $L^\perp$ of L the restriction of g is a two-dimensional rotation. For this to make sense, one must have understood that the two-dimensional definition is independent of basis, and that g does stabilize the orthogonal complement of any line fixed by it. Indeed, there is no necessity of reference here to $\mathbb{R}^n$. We could instead use *any* $\mathbb{R}$ vector space with an inner product.

distances is exactly that the inner product is preserved, in the sense that

$$\langle gx,\, gy\rangle = \langle x,\, y\rangle$$

Define the standard **orthogonal group**

$$\begin{aligned} O_n(\mathbb{R}) &= \text{orthogonal group} \\ &= \text{angle-and-distance-preserving group} \\ &= \{g \in GL_n(\mathbb{R}) \;:\; \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\} \end{aligned}$$

Since distances are preserved, $O_n(\mathbb{R})$ stabilizes the (unit) sphere S^{n-1} in $\mathbb{R}^n$. The preservation of the inner product does *not* fully distinguish rotations, since it does *not* imply the orientation-preserving (determinant-one) condition, which has to be added explicitly. Thus, the standard [2] **special orthogonal group** [3] meant to be the group of *rotations* is

$$\begin{aligned} SO_n(\mathbb{R}) &= \text{special orthogonal group} \\ &= \text{rotation group} \\ &= \{g \in O_n(\mathbb{R}) \;:\; \det g = 1\} \end{aligned}$$

A common, expedient, but structurally unenlightening definition of the standard orthogonal group is that

$$O_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) \;:\; g^\top g = 1_n\} \qquad (g^\top \text{ is } g\text{-transpose, } 1_n \text{ is the } n\text{-by-}n \text{ identity})$$

and then the standard special orthogonal group is still

$$SO_n(\mathbb{R}) \;=\; \{g \in O_n(\mathbb{R}) : \det g = 1\}$$

It is important that these two definitions specify the same objects:

[1.0.1] Claim: The two definitions of *orthogonal group* are the same. That is,

$$\{g \in GL_n(\mathbb{R}) \;:\; \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\} \;=\; \{g \in GL_n(\mathbb{R}) \;:\; g^\top g = 1_n\}$$

Proof: The usual inner product is also expressible as

$$\langle x, y\rangle \;=\; \sum_i x_i y_i = y^\top x \qquad (\text{for column vectors, } y^\top = y\text{-transpose})$$

For $g^\top g = 1_n$, compute directly

$$\langle gx, gy\rangle \;=\; (gy)^\top (gx) \;=\; y^\top\,(g^\top g)\,x \;=\; y^\top x \;=\; \langle x, y\rangle$$

from which we conclude that $g^\top g = 1_n$ implies that $\langle,\rangle$ is preserved. On the other hand, suppose that g preserves $\langle,\rangle$. The main trick is that, for column vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, all of length n, inserted as the columns of two matrices V and W,

$$W^\top V \;=\; [\,w_1 \quad \ldots \quad w_n\,]^\top\,[\,v_1 \quad \ldots \quad v_n\,] \;=\; n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle w_i, v_j\rangle$$

[2] In fact, the form of these definitions did *not* use the explicit form of the inner product, so applies as well to other possible inner products, as well. While in the near future we care mostly about the standard one, it is wise to present things in a form which does not needlessly depend on irrelevant particulars.

[3] As with the *special linear group*, the modifier *special* on *special orthogonal group* refers to the determinant-one condition.

Running this backward, for $g \in GL_n(\mathbb{R})$

$$n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle gw_i, gv_j\rangle \;=\; [\,gw_1 \;\; \ldots \;\; gw_n\,]^\top [\,gv_1 \;\; \ldots \;\; gv_n\,]$$
$$=\; (gW)^\top (gV) \;=\; W^\top (g^\top g)\, V$$

For g preserving inner products, slightly cleverly taking $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ to be the standard basis $e_1, \ldots, e_n$, the previous relation becomes

$$n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle ge_i, ge_j\rangle \;=\; W^\top (g^\top g)\, V \;=\; 1_n^\top (g^\top g)\, 1_n \;=\; g^\top g$$

proving that $g^\top g = 1_n$. Thus, the two definitions of orthogonal groups agree. ///

Regarding possible values of determinants of elements of $O_n(\mathbb{R})$, observe that for any g such that $g^\top g = 1$

$$(\det g)^2 \;=\; \det g^\top \cdot \det g \;=\; \det(g^\top g) \;=\; \det 1_n \;=\; 1$$

so $\det g = \pm 1$.

[1.0.2] Claim: The action of $SO_n(\mathbb{R})$ on S^{n-1} is *transitive*, for $n \geq 2$.

Proof: We first show that, given $x \in S^{n-1}$, there is $g \in O_n(\mathbb{R})$ such that $ge_1 = x$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. That is, we construct $g \in O_n(\mathbb{R})$ such that the left column of g is x. Indeed, complete x to an $\mathbb{R}$-basis $x, x_2, x_3, \ldots, x_n$ for $\mathbb{R}^n$. Then apply the *Gram-Schmidt* process [4] to find an orthonormal (with respect to the standard inner product) basis $x, v_2, \ldots, v_n$ for $\mathbb{R}^n$. As observed in the previous proof, the condition $g^\top g = 1_n$, is exactly the assertion that the columns of g form an orthonormal basis. Thus, taking $x, v_2, \ldots, v_n$ as the columns of g gives $g \in O_n(\mathbb{R})$ such that $ge_1 = x$. As noted above, the determinant of this g is ± 1. To ensure that it is 1, replace v_n by $-v_n$ if necessary. This still gives $ge_1 = x$, giving the transitivity. ///

[1.0.3] Claim: The *isotropy group* $SO_n(\mathbb{R})_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$(\text{isotropy group}) \;=\; SO_n(\mathbb{R})_{e_n} \;=\; \{\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \;:\; A \in SO_{n-1}(\mathbb{R})\} \;\approx\; SO_{n-1}(\mathbb{R})$$

Thus, by transitivity, as $SO_n(\mathbb{R})$-spaces

$$S^{n-1} \;\approx\; SO_n(\mathbb{R}) \,/\, SO_{n-1}(\mathbb{R})$$

Proof: There are at least two ways to think about this. First, we can think of the geometry specified by $\langle,\rangle$. Second, we can manipulate matrices. Taking the first course first, we claim that for g fixing e_n, the orthogonal complement

$$e_n^\perp \;=\; \{v \in \mathbb{R}^n \;:\; \langle v, e_n\rangle = 0\}$$

is stabilized by g, and on $e_n^\perp$ the linear map preserves (the restriction to $e_n^\perp$ of) $\langle,\rangle$. Indeed, for $\langle v, e_n\rangle = 0$ and $ge_n = e_n$,

$$\langle gv, e_n\rangle \;=\; \langle gv, ge_n\rangle \;=\; \langle v, e_n\rangle \;=\; 0$$

[4] Recall that, given a basis $v_1, \ldots, v_n$ for a (real or complex) vector space with an inner product (real-symmetric or complex hermitian), the Gram-Schmidt process produces an *orthogonal* or *orthonormal* basis, as follows. Replace v_1 by $v_1/|v_1|$ to give it length 1. Then replace v_2 first by $v_2 - \langle v_2, v_1\rangle v_1$ to make it orthogonal to v_1 and then by $v_2/|v_2|$ to give it length 1. Then replace v_3 first by $v_3 - \langle v_3, v_1\rangle v_1$ to make it orthogonal to v_1, then by $v_3 - \langle v_3, v_2\rangle v_2$ to make it orthogonal to v_2, and then by $v_3/|v_3|$ to give it length 1. And so on.

since g preserves the inner product, showing that g stabilizes the orthogonal complement to e_n. Certainly g preserves the restriction to $e_n^\perp$ of $\langle,\rangle$, since it preserved $\langle,\rangle$ on the whole space $\mathbb{R}^n$. And, on the other hand, for g stabilizing $e_n^\perp$ and preserving the restriction of $\langle,\rangle$ to $v_o^\perp$, *define* (an extension of) g on e_n by $ge_n = e_n$. To check that this extended g is in $O_n(\mathbb{R})$, for general vectors $v = v' + ae_n$ and $w = w' + be_n$ with $v', w' \in e_n^\perp$, $a, b \in \mathbb{R}$, a natural computation gives

$$\langle gv, gw\rangle \;=\; \langle g(v' + ae_n), g(w' + be_n)\rangle \;=\; \langle gv' + age_n, gw' + bge_n\rangle \;=\; \langle gv' + ae_n, gw' + be_n\rangle$$

$$=\; \langle gv', gw'\rangle + \langle ae_n, be_n\rangle \;=\; \langle v', w'\rangle + \langle ae_n, be_n\rangle \;=\; \langle v' + ae_n, w' + be_n\rangle \;=\; \langle v, w\rangle$$

Thus, the extension does preserve $\langle,\rangle$ on the larger space, and we have proven that the stabilizer subgroup of e_n is a copy of $SO_{n-1}(\mathbb{R})$.

At heart, the matrix argument does the same things, but mutely. Let

$$g \;=\; \begin{bmatrix} A & b \\ c & d \end{bmatrix} \in O_n(\mathbb{R})$$

be a block decomposition with A of size $n-1$, etc., with $g^\top g = 1_n$ and $ge_n = e_n$. Also write

$$e_n \;=\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad\qquad \text{(where the 0 is } (n-1)\text{-by-1)}$$

In terms of the blocks, the condition $ge_n = e_n$ is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;=\; e_n \;=\; ge_n \;=\; \begin{bmatrix} A & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} b \\ d \end{bmatrix}$$

Thus, to fix e_n we have $b = 0$ and $d = 1$. In terms of these blocks, the condition $g^\top g = 1_n$ is

$$\begin{bmatrix} 1_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \;=\; 1_n \;=\; g^\top g \;=\; \begin{bmatrix} A^\top & c^\top \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & 0 \\ c & 1 \end{bmatrix} \;=\; \begin{bmatrix} A^\top A + cc^\top & c^\top \\ 0 & 1 \end{bmatrix}$$

Thus, $c = 0$, and $A^\top A = 1_{n-1}$, as claimed. Conversely, with $A^\top A = 1_{n-1}$, we can use A in the upper-left corner of such a block-decomposed n-by-n matrix, making an element of the orthogonal group which also fixes e_n. ///

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2. *Holomorphic rotations*

For several reasons, we may view $\mathbb{R}^{2n}$ as being $\mathbb{C}^n$, to look at spheres $S^{2n-1} \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$, and rotations which are $\mathbb{C}$-linear on $\mathbb{C}^n$, not merely $\mathbb{R}$-linear. These would be *holomorphic* rotations. [5]

Let $\langle,\rangle$ be the standard *hermitian* inner product [6] The (unit) sphere in $\mathbb{C}^n$ is

$$S^{2n-1} \;=\; \{z \in \mathbb{C}^n \;:\; \langle z, z\rangle = 1\}$$

[5] This use of *holomorphic* is a bit grandiose, since differentiability of *linear* maps is not subtle. Still, in a larger context, it is worthwhile to realize that among *all* rotations the holomorphic ones are the $\mathbb{C}$-linear ones, and these are a proper subgroup of the whole, but still transitive on the sphere.

[6] For $z, w \in \mathbb{C}^n$, the standard real-valued inner product on $\mathbb{R}^{2n}$ is simply the *real part* of this hermitian inner product on $\mathbb{C}^n$. Further, for $z = (x_1 + iy_1, \ldots, x_n + iy_n)$ with $x_j, y_j \in \mathbb{R}$, the value of $\langle z, z\rangle$ is the same with either inner product, namely

$$(x_1^2 + y_1^2) + \ldots + (x_n^2 + y_n^2) = (x_1^2 + \ldots + x_n^2) + (y_1^2 + \ldots + y_n^2)$$

The angle-preserving group of $\mathbb{C}$-linear (not merely $\mathbb{R}$-linear) maps is the **unitary group**

$$\begin{aligned} U(n) &= \text{unitary group} \\ &= \mathbb{C}\text{-linear angle-preserving group} \\ &= \{g \in GL_n(\mathbb{C}) \;:\; \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{C}^n\} \end{aligned}$$

As with orthogonal groups, this unitary group contains more than *rotations*. We need a determinant-one condition, and define the standard **special unitary group**

$$\begin{aligned} SU(n) &= \text{special unitary group} \\ &= \mathbb{C}\text{-linear rotation group} \\ &= \{g \in U(n) \;:\; \det g = 1\} \end{aligned}$$

As with orthogonal groups, there is also a direct, matrix definition of the standard unitary group, namely

$$U(n) \;=\; \{g \in GL_n(\mathbb{C}) \;:\; g^* \, g = 1_n\} \qquad\qquad (\text{where } g^* \text{ is } g\text{-conjugate-transpose})$$

And as with the orthogonal groups, the two definitions of unitary groups specify the same objects:

[2.0.1] Claim: The two definitions of *unitary group* are the same. That is,

$$\{g \in GL_n(\mathbb{C}) \;:\; \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{C}^n\} \;=\; \{g \in GL_n(\mathbb{C}) \;:\; g^* \, g = 1_n\}$$

Proof: The usual hermitian inner product is

$$\langle x, y\rangle = \sum_i x_i \overline{y}_i = y^* x \qquad\qquad (\text{for column vectors})$$

For $g^* g = 1_n$, compute directly

$$\langle gx, gy\rangle \;=\; (gy)^*(gx) \;=\; y^*\,(g^*g)\,x \;=\; y^* x \;=\; \langle x, y\rangle$$

from which $g^*g = 1_n$ implies that $\langle,\rangle$ is preserved. On the other hand, suppose that g preserves $\langle,\rangle$. For column vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, all of length n, inserted as the columns of two matrices V and W,

$$W^*V \;=\; [\,w_1 \quad \ldots \quad w_n\,]^*\,[\,v_1 \quad \ldots \quad v_n\,] \;=\; n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle w_i, v_j\rangle$$

Running this backward, for $g \in GL_n(\mathbb{C})$

$$n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle gw_i, gv_j\rangle \;=\; [\,gw_1 \quad \ldots \quad gw_n\,]^*\,[\,gv_1 \quad \ldots \quad gv_n\,]$$

$$=\; (gW)^*(gV) \;=\; W^*\,(g^*g)\,V$$

For g preserving $\langle,\rangle$, taking $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ to be the standard basis $e_1, \ldots, e_n$, the previous relation becomes

$$n\text{-by-}n \text{ matrix with } (i,j)^{th} \text{ entry } \langle ge_i, ge_j\rangle \;=\; W^*\,(g^*g)\,V \;=\; 1_n^*\,(g^*g)\,1_n \;=\; g^*g$$

proving that $g^*g = 1_n$. Thus, the two definitions of unitary groups agree. ///

Regarding possibly values of determinants of elements of unitary groups, note that

$$1 \;=\; \det 1_n = \det(g^*\,g) = \overline{\det g}\,\det g = |\det g|^2$$

Thus, $|\det g| = 1$.

Since our intuition is based on the two-sphere $S^2 \subset \mathbb{R}^3$, and $\mathbb{R}$-linear rotations in any case, we might be too timid to hope that

[2.0.2] Claim: The special unitary group $SU(n)$ is *transitive* on the sphere S^{2n-1} in $\mathbb{C}^n$, for $n \geq 2$.

Proof: As usual, it suffices to show that $SU(n)$ maps $e_1 = (1, 0, \ldots, 0)$ to any other vector v_1 of length 1 in $\mathbb{C}^n$. We first show that, given $x \in S^{2n-1}$ there is $g \in U(n)$ such that $ge_1 = x$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{C}^n$. That is, we construct $g \in U(n)$ such that the left column of g is x. Indeed, complete x to an $\mathbb{C}$-basis $x, x_2, x_3, \ldots, x_n$ for $\mathbb{C}^n$. Then apply the *Gram-Schmidt* process [7] to find an orthonormal (with respect to the standard hermitian inner product) basis $x, v_2, \ldots, v_n$ for $\mathbb{C}^n$. The condition $g^* g = 1_n$, is the assertion that the columns of g form an orthonormal basis. Thus, taking $x, v_2, \ldots, v_n$ as the columns of g gives $g \in U(n)$ such that $ge_1 = x$. To make $\det g = 1$, replace v_n by $(\det g)^{-1} v_n$. Since $|\det g| = 1$, this change does not harm the orthonormality. ///

As with orthogonal groups, with essentially the same proof, the sphere S^{2n-1} is a quotient of $SU(n)$:

[2.0.3] Claim: The *isotropy group* $SU(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\text{(isotropy group)} \;=\; SU(n)_{e_n} \;=\; \{\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \;:\; A \in SU(n-1)\} \;\approx\; SU(n-1)$$

Thus,

$$S^{2n-1} \;\approx\; SU(n) \,/\, SU(n-1)$$

by transitivity, as $SU(n)$-spaces. ///

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3. *Action of $GL_{n+1}(\mathbb{C})$ on projective space $\mathbb{P}^n$*

For a positive integer n, both the group $\mathbb{C}^\times$ and the **general linear group**

$$GL(n+1, \mathbb{C}) \;=\; \{(n+1)\text{-by-}(n+1) \text{ invertible complex matrices}\}$$

act on $\mathbb{C}^{n+1}$, which we view as *column vectors*. The scalars $\mathbb{C}^\times$ act by scalar multiplication, and $GL(n+1, \mathbb{C})$ by matrix multiplication (on the left). The action of $GL(n+1, \mathbb{C})$ is linear, which is exactly that this action commutes with the action of $\mathbb{C}^\times$. Indeed, the scalar matrices in $GL(n+1, \mathbb{C})$ duplicate the action of $\mathbb{C}^\times$. But we want to keep track of the separate copy of $\mathbb{C}^\times$ as well.

There are exactly two orbits of $GL(n+1, \mathbb{C})$ on C^{n+1}, namely $\{0\}$ and $\mathbb{C}^{n+1} - 0$. Complex **projective n-space** is the quotient

$$\mathbb{P}^n \;=\; (\mathbb{C}^{n+1} - 0)/\mathbb{C}^\times$$

As a *set*, this is the collection of lines through 0 in $\mathbb{C}^{n+1}$, which has some intuitive appeal, but presenting $\mathbb{P}^n$ as a quotient gives it a topology and other more refined structures.

Since the actions of $\mathbb{C}^\times$ and $GL(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1} - 0$ commute, the action of $GL(n+1, \mathbb{C})$ respects $\mathbb{C}^\times$ orbits, which is to say that the action of $GL(n_1, \mathbb{C})$ descends [8] to the quotient $\mathbb{P}^n$. In symbols,

$$g \cdot (v \cdot \mathbb{C}^\times) \;=\; gv \cdot \mathbb{C}^\times$$

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[7] The Gram-Schmidt process works as well with a hermitian $\mathbb{C}$-valued inner product as with a $\mathbb{R}$-valued inner product.

[8] Since the scalar action of $\mathbb{C}^\times$ is also given by the action of the (normal) subgroup of $GL(n+1, \mathbb{C})$ consisting of scalar matrices, in fact the quotient $PGL(n+1, \mathbb{C})$ of $GL(n+1, \mathbb{C})$ by scalar matrices has a well-defined action on $\mathbb{P}^n$. This group $PGL(n+1, \mathbb{C})$ is the *projective linear group.*

There is a fairly obvious copy of $\mathbb{C}^n$ sitting inside $\mathbb{P}^n$, namely

$$\mathbb{C}^n \ni v \to \begin{bmatrix} v \\ 1 \end{bmatrix} \cdot \mathbb{C}^\times$$

where $v \in \mathbb{C}^n$ is a column vector of length n and $\begin{bmatrix} v \\ 1 \end{bmatrix}$ is a column vector of length $n+1$, in a block decomposition. The part of $\mathbb{P}^n$ *not* hit by this map is

$$\{\begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathbb{C}^n - 0\}/\mathbb{C}^\times \;\approx\; \mathbb{P}^{n-1}$$

Repeating, as a *set*

$$\mathbb{P}^n \;=\; \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{C}^{n-2} \sqcup \ldots \sqcup \mathbb{C}^1 \sqcup \{\infty\}$$

where ∞ is the traditional name for the single point added to $\mathbb{C}$ to make $\mathbb{P}^1$.

This action of $GL(n+1,\mathbb{C})$ on $\mathbb{P}^n$ *almost* gives a **linear fractional action** on the copy of $\mathbb{C}^n$ sitting inside $\mathbb{P}^n$, although the transitivity of $GL(n+1,\mathbb{C})$ on $\mathbb{P}^n$ means that $\mathbb{C}^n$ is *not* actually stabilized by the action. Despite this literal failure, the formulas obtained are useful, if interpreted properly. That is, using a block decomposition

$$g \;=\; \begin{bmatrix} A & b \\ c & d \end{bmatrix} \quad (A \text{ is } n\text{-by-}n,\ b \text{ is } n\text{-by-}1,\ c \text{ is } 1\text{-by-}n, \text{ and } d \text{ is } 1\text{-by-}1)$$

and $v \in \mathbb{C}^n$,

$$g \cdot v \;=\; g \cdot \begin{bmatrix} v \\ 1 \end{bmatrix} \cdot \mathbb{C}^\times \;=\; \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix} \cdot \mathbb{C}^\times \;=\; \begin{bmatrix} Av+b \\ cv+d \end{bmatrix} \cdot \mathbb{C}^\times$$
$$=\; \begin{bmatrix} (Av+b)(cv+d)^{-1} \\ 1 \end{bmatrix} \cdot \mathbb{C}^\times \;=\; (Av+b)(cv+d)^{-1} \in \mathbb{C}^n$$

with the problem being that $cv+d$ can be 0, so the image is *not* in the copy of $\mathbb{C}^n$, but in some other part of $\mathbb{P}^n$.

Even though this formula is not literally correct, it shows how *fractional* actions arise.

[3.0.1] Remark: The *associativity* and other useful properties of such actions are immediate when we realize that it is induced from multiplication of matrices, that is, from composition of linear endomorphisms of a vector space.

[3.0.2] Remark: None of the above depends in any way on the fact that the underlying field was $\mathbb{C}$, apart from the specific topology inherited from it. The same discussion constructs projective spaces over any *field*, and inherits its topology, if any, from that field. Thus, for example, we have also *real* projective spaces $\mathbb{RP}^n$.

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4. *Real hyperbolic n-space*

We show that a standard *special orthogonal* group $SO(n,1)$ (introduced below) with indefinite signature $(n,1)$ acts *transitively* on the (open) unit ball in $\mathbb{R}^n$ by generalized linear fractional transformations. The isotropy group of the origin is a copy of $O(n)$, so the unit ball is essentially $SO(n,1)/O(n)$. The geometry implied by this is *hyperbolic*. [9]

[9] As noted in the introduction, a reason for the name *hyperbolic* (as opposed to *Euclidean* or *elliptic*) is that there are infinitely-many straight lines (geodesics) through a point not on a given straight line. In the Euclidean case there should be a *unique* such line, while in the elliptic case there should be none. In the $SO(n,1)/O(n)$ model, the maximal totally geodesic subspaces are the intersections of the ball with *hyperplanes* in $\mathbb{R}^n$. It is not hard to prove that such intersections are stable under the action of $SO(n,1)$, so at least in this sense the geometry is preserved by the action of $SO(n,1)$.

As the (open) unit disk in $\mathbb{C}$ sits inside $\mathbb{P}$, the (open) real n-ball in $\mathbb{R}^n$ sits inside the *real* projective space

$$\mathbb{RP}^n \;=\; (\mathbb{R}^{n+1} - 0)/\mathbb{R}^\times$$

The goal is to see that certain *orthogonal groups* (described below) stabilize and act transitively on the real n-ball B, just as the group

$$SU(1,1) \;=\; \{\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \;:\; |\alpha|^2 - |\beta|^2 = 1\}$$

acts transitively on the unit disk in $\mathbb{C} \approx \mathbb{R}^2$. [10] This combines structures from earlier discussion of orthogonal and unitary groups acting on spheres and linear groups acting on projective spaces.

As observed just above, *linear fractional transformations* are artifacts of trying to restrict to $\mathbb{R}^n$ the natural linear action of $GL_{n+1}(\mathbb{R})$ on $\mathbb{P}^n$. This suggests that it is better to describe the unit ball

$$B \;=\; \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 < 1\} \;\subset\; \mathbb{R}^n$$

in $(n+1)$-by-1 *homogeneous* coordinates v instead, with the map $\mathbb{R}^n \to \mathbb{P}^n$ given (as earlier) by

$$x \;\longrightarrow\; \begin{bmatrix} x \\ 1 \end{bmatrix} \cdot \mathbb{R}^\times$$

The desired modification of the presentation of the ball is easy, in effect replacing the 1 in the definition of the n-ball by v_{n+1}^2,

$$B \;=\; \{v \cdot \mathbb{R}^\times \in \mathbb{P}^n \;:\; v_1^2 + \ldots + v_n^2 - v_{n+1}^2 < 0\} \;\subset\; \mathbb{P}^n$$

The re-expression of B in homogeneous coordinates uses a *homogeneous* condition, stable under the action of scalars, and, thus, *well-defined on cosets*:

$$v_1^2 + \ldots + v_n^2 - v_{n+1}^2 < 0 \quad \text{if and only if} \quad (\lambda\, v_1)^2 + \ldots + (\lambda\, v_n)^2 - (\lambda\, v_{n+1})^2 < 0\} \quad (\text{for any } \lambda \in \mathbb{R}^\times)$$

To meet the condition $v_1^2 + \ldots + v_n^2 - v_{n+1}^2 < 0$ the component v_{n+1} must be non-zero, confirming that the defined set lies in the image of $\mathbb{R}^n$ inside $\mathbb{P}^n$. And, indeed, dividing through by v_{n+1}, we recover the condition

$$v_1/v_{n+1}^2 + \ldots v_n/v_{n+1}^2 - 1 \;<\; 0$$

defining the unit ball.

The homogeneous-coordinate description can be improved for compatibility with the action of $GL_{n+1}(\mathbb{R})$. Define an *indefinite*[11] symmetric form

$$\langle (x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \rangle \;=\; x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

In projective coordinates,

$$B \;=\; \{v \cdot \mathbb{R}^\times \;:\; \langle v, v \rangle < 0\}$$

Another standard **orthogonal group** $O(n,1)$ is definable via this *indefinite* $\langle , \rangle$, as

$$O(n,1) \;=\; \{g \in GL_{n+1}(\mathbb{R}) \;:\; \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w\}$$

[10] This description of the group $SU(1,1)$ is less than ideal, since, for example, it is not clear that it is a *group*. Nevertheless, this presentation is common in contexts where a misguided sense of expediency prevails.

[11] A symmetric bilinear form $\langle , \rangle$ is *indefinite* if it can happen that $\langle v, v \rangle = 0$ without v being 0. It is easy to (correctly) anticipate that the geometric aspects of an indefinite form diverge from those of *definite* ones.

Alternatively, as usual, these objects and conditions can also be expressed in terms of matrices and column and row vectors. Let

$$S = \begin{bmatrix} 1 & 0 & 0 & \ldots & & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & & \ddots & & \vdots \\ & & & & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix}$$

be an $(n+1)$-by-$(n+1)$ diagonal matrix with n 1's and one -1 on the diagonal. Then, treating $v \in \mathbb{R}^{n+1}$ as *column* vectors, and letting $v \to v^\top$ be transpose,

$$v^\top S v = (\overline{v}_1 \ \ldots \ \overline{v}_n \ \overline{v}_{n+1}) \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ -v_{n+1} \end{bmatrix} = v_1^2 + \ldots + v_n^2 - v_{n+1}^2 = \langle v, v \rangle$$

Therefore,

$$B = \{v \cdot \mathbb{R}^\times \ : \ v^\top S v < 0\}$$

The standard **orthogonal group** $O(n,1)$ of **signature**[12] $O(n,1)$ is definable using S, as

$$O(n,1) = \{g \in GL_{n+1}(\mathbb{R}) \ : \ g^\top S g = S\}$$

The same proof used in the case of the positive-definite hermitian inner product used earlier to define $S(n)$ shows that

[4.0.1] Claim: These two descriptions of B and $S(n,1)$ yield the same objects. ///

It is not surprising that

[4.0.2] Claim: The (standard) orthogonal group $O(n,1)$ stabilizes the unit ball in $\mathbb{R}^n$ under the action by linear fractional transformations.

Proof: Let $g \in O(n,1)$ and let v be a homogeneous-coordinate representative for a point in the unit ball. That is, $\langle v, v \rangle < 0$. Then we test the sign of $\langle gv, gv \rangle$

$$\langle gv, gv \rangle = \langle v, v \rangle < 0$$

so gv is again in the unit ball. ///

[4.0.3] Remark: We might want to see, explicitly, that the denominators in the linear fractional transformation action of $O(n,1)$ on the unit ball do not vanish. Indeed, the denominator of the image gv is the $(n+1)^{th}$ component w_{n+1} of $w = gv \in \mathbb{R}^{n+1}$. Since

$$w_1^2 + \ldots + w_n^2 - w_{n+1}^2 < 0$$

it must be that $w_{n+1} \neq 0$, and we can indeed divide through by w_{n+1} if we want.

Now we prove that the *special* orthogonal unitary group

$$SO(n,1) = \{g \in O(n,1) \ : \ \det g = 1\}$$

[12] The *signature* in this case just tells the number of $+1$'s and the number of -1's on the diagonal, assuming all off-diagonal entries to be 0. That this is an isomorphism-class invariant of *symmetric forms* $v \to v^\top S v = \langle v, v \rangle$ is the content of the *Inertia Theorem*. We do not need to invoke the Inertia Theorem here.

acts *transitively* on the unit ball B in $\mathbb{R}^n$.

We will see readily that the isotropy group in $SO(n,1)$ of $0 \in \mathbb{R}^n$ is

$$K \;=\; \{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in SO(n,1) \;:\; a^\top a = 1_n, \quad d^2 = 1, \quad d \cdot \det a = 1\} \;\approx\; O(n)$$

so we will have

[4.0.4] Claim: The special orthogonal group $SO(n,1)$ is *transitive* on the unit ball B in $\mathbb{R}^n$, and as an $SO(n,1)$ space

$$B \;\approx\; SO(n,1)/O(n)$$

Proof: As usual, to prove transitivity, it suffices to show that $0 \in \mathbb{R}^n$ can be mapped to any other point x in the real n-ball.

First, determine the isotropy group of 0. Using a block decomposition $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of an element g in $GL_{n+1}(\mathbb{R})$ with a being n-by-n, d being 1-by-1, etc., the condition for fixing 0 is

$$0 \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}(0) \;=\; bd^{-1}$$

which requires that b is the n-by-1 matrix of 0's. Then the condition that the matrix be in $O(n,1)$ is

$$S \;=\; g^\top S g \;=\; \begin{bmatrix} a^\top a - c^\top c & -c^\top d \\ -d^\top c & -d^\top d \end{bmatrix}$$

From the off-diagonal entries, $c = 0$. Then $a^\top a = 1_n$ and $d^\top d = 1$. The further condition that the determinant be 1 requires that $d = (\det a)^{-1}$. Thus, any $a \in O(n)$ gives an element of the isotropy subgroup in $SO(n,1)$, and vice-versa.

To prove transitivity, prove that 0 can be mapped by $SO(n,1)$ to any other point x in the ball. Simplify the problem by reducing to the case $n = 1$, by using the isotropy group $K \approx O(n)$ of 0 to rotate the given x to a special form. The transitivity of $SO(n)$ on the *sphere* of a fixed radius r in $\mathbb{R}^n$ assures that there is $a \in SO(n)$ such that

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}(x) \;=\; \begin{bmatrix} r \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad\qquad \text{(where } r = |x| \text{ is usual length)}$$

In effect, the real number r is in the one-ball, which is just the closed interval $[-1,1]$. There is a corresponding copy of $SO(1,1)$ inside $SO(n,1)$, given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1_{n-1} & 0 \\ c & 0 & d \end{bmatrix}$$

which acts conveniently (via linear fractional transformations) by

$$\begin{bmatrix} a & 0 & b \\ 0 & 1_{n-1} & 0 \\ c & 0 & d \end{bmatrix} \begin{bmatrix} r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} (ar+b)/(cr+d) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The question of transitivity is reduced to finding $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(1,1)$ to map 0 to $r = |x|$ in the interval $[-1,1]$ in $\mathbb{R}^1$. A convenient special class of matrices suffices. For example, the condition that $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ be in $SO(1,1)$, with a, b is just that $a^2 - b^2 = 1$. The condition that this matrix map 0 to r (with $0 \le r < 1$) is $b/a = r$. Substituting $b = ra$ into the first relation gives $a^2(1-r^2) = 1$. This gives a choice for a, and then for b. ///

[4.0.5] Remark: Note that $SO(1,1)$ is essentially $GL_1(\mathbb{R})$! The isomorphism is given by an analogue of the *Cayley element* that maps the upper half-plane to the disk, with attention to using *real* numbers: here, the conjugating map sends the interval $(-1,1)$ to the interval $(0,\infty)$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \tfrac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{t+\frac{1}{t}}{2} & \frac{-t+\frac{1}{t}}{2} \\ \frac{-t+\frac{1}{t}}{2} & \frac{t+\frac{1}{t}}{2} \end{pmatrix}$$

For $t > 0$, with $u = -\log t$, the latter matrix can be rewritten as

$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \qquad \text{(with } u \in \mathbb{R}\text{)}$$

The latter expression is a common parametrization of the connected component of the identity of the standard $SO(1,1)$.

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5. *Complex hyperbolic* n*-space*

This section is a variant of the previous computations. A much smaller group is shown to still act transitively on an open ball. We consider only *even* dimensions, so the ball has a complex structure. The smaller group acts by *holomorphic* maps of the ball to itself.

As the (open) unit disk in $\mathbb{C}$ sits inside the projective space $\mathbb{P}^1$, the (open) complex n-ball in $\mathbb{C}^n$ sits inside $\mathbb{P}^n$. The goal of this section is to see that certain *unitary groups* (described below) stabilize and act transitively on the complex n-ball B, just as the group

$$SU(1,1) = \{\begin{bmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{bmatrix} : |\alpha|^2 - |\beta|^2 = 1\}$$

acts transitively on the unit disk in $\mathbb{C}$. [13] This combines structures from the earlier discussion of unitary groups acting on spheres and linear groups acting on projective spaces.

As above, the so-called *linear fractional transformations* are artifacts of trying to restrict to $\mathbb{C}^n$ the natural linear action of $GL_{n+1}(\mathbb{C})$ on $\mathbb{P}^n$. This suggests that it is better to describe the unit ball

$$B = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_n|^2 < 1\} \subset \mathbb{C}^n$$

in $(n+1)$-by-1 homogeneous coordinates v instead, with the map $\mathbb{C}^n \to \mathbb{P}^n$ given (as earlier) by

$$z \to \begin{bmatrix} z \\ 1 \end{bmatrix} \cdot \mathbb{C}^\times$$

[13] This description of the group $SU(1,1)$ is not ideal, since, for example, it is not immediately clear why it is a *group*. Nevertheless, this presentation is common.

The desired modification of the presentation of the ball is easy, in effect replacing the 1 in the definition of the n-ball by $|v_{n+1}|^2$,

$$B \;=\; \{v \cdot \mathbb{C}^\times \in \mathbb{P}^n \;:\; |v_1|^2 + \ldots + |v_n|^2 - |v_{n+1}|^2 < 0\} \subset \mathbb{P}^n$$

The re-expression of B in homogeneous coordinates uses a *homogeneous* condition, stable under the action of scalars, and, thus, *well-defined on cosets*:

$$|v_1|^2 + \ldots + |v_n|^2 - |v_{n+1}|^2 < 0 \quad \text{if and only if} \quad |\lambda\, v_1|^2 + \ldots + |\lambda\, v_n|^2 - |\lambda\, v_{n+1}|^2 < 0\} \quad (\text{for any } \lambda \in \mathbb{C}^\times)$$

To meet the condition $|v_1|^2 + \ldots + |v_n|^2 - |v_{n+1}|^2 < 0$ the component v_{n+1} must be non-zero, confirming that the defined set lies in the image of $\mathbb{C}^n$ inside $\mathbb{P}^n$. Indeed, dividing through by v_{n+1}, we recover the condition

$$|v_1/v_{n+1}|^2 + \ldots |v_n/v_{n+1}|^2 - 1 \;<\; 0$$

which defines the unit ball.

The homogeneous-coordinate description can be further improved for compatibility with the action of $GL_{n+1}(\mathbb{C})$. Define an *indefinite*[14] hermitian form

$$\langle (z_1, \ldots, z_{n+1}), (w_1, \ldots, w_{n+1}) \rangle \;=\; z_1 \overline{w}_1 + \ldots - z_{n+1} \overline{w}_{n+1}$$

In projective coordinates,

$$B \;=\; \{v \cdot \mathbb{C}^\times \;:\; \langle v, v\rangle < 0\}$$

Another standard **unitary group** $U(n,1)$ is definable via this *indefinite* $\langle , \rangle$, as

$$U(n,1) \;=\; \{g \in GL_{n+1}(\mathbb{C}) \;:\; \langle gv, gw\rangle = \langle v, w\rangle \text{ for all } v, w\}$$

As usual, alternatively, these objects and conditions can be expressed in terms of matrices and column and row vectors. Let

$$H \;=\; \begin{bmatrix} 1 & 0 & 0 & \ldots & & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & & \ddots & & \vdots \\ & & & & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix}$$

be an $(n+1)$-by-$(n+1)$ diagonal matrix with n 1's and one -1 on the diagonal. Treating $v \in \mathbb{C}^{n+1}$ as *column* vectors, and letting $v \to v^*$ be conjugate-transpose,

$$v^* H v \;=\; (\overline{v}_1 \;\ldots\; \overline{v}_n \; \overline{v}_{n+1}) \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ -v_{n+1} \end{bmatrix} \;=\; |v_1|^2 + \ldots + |v_n|^2 - |v_{n+1}|^2 \;=\; \langle v, v\rangle$$

Therefore,

$$B \;=\; \{v \cdot \mathbb{C}^\times \;:\; v^* H v < 0\}$$

And the standard **unitary group** $U(n,1)$ of **signature**[15] $(n,1)$ is definable using H, as

$$U(n,1) \;=\; \{g \in GL_{n+1}(\mathbb{C}) \;:\; g^* H g = H\}$$

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[14] A hermitian form $\langle , \rangle$ is *indefinite* if it can happen that $\langle v, v\rangle = 0$ without v being 0. It is easy to (correctly) anticipate that the geometric aspects of an indefinite form diverge from those of *definite* ones.

[15] The *signature* in this case tells the number of $+1$'s and the number of -1's on the diagonal, assuming all off-diagonal entries 0. That this is an isomorphism-class invariant of *hermitian forms* $v \to v^* H v = \langle v, v\rangle$ is the content of the *Inertia Theorem*. We do not need to invoke the Inertia Theorem here.

The same proof used in the case of the positive-definite hermitian inner product defining $U(n)$ shows that

[5.0.1] Claim: These two descriptions of B and $U(n,1)$ yield the same objects. ///

It is not surprising that we have

[5.0.2] Claim: The (standard) unitary group $U(n,1)$ stabilizes the unit ball in $\mathbb{C}^n$ under the action by linear fractional transformations.

Proof: Let $g \in U(n,1)$ and let v be a homogeneous-coordinate representative for a point in the unit ball. That is, $\langle v,v\rangle < 0$. Then we test the sign of $\langle gv, gv\rangle$

$$\langle gv, gv\rangle \;=\; \langle v,v\rangle \;<\; 0$$

so gv is again in the unit ball. ///

[5.0.3] Remark: We might want to see that the denominators in the linear fractional transformation action of $U(n,1)$ on the unit ball do not vanish. Indeed, the denominator of the image gv is the $(n+1)^{th}$ component w_{n+1} of $w = gv \in \mathbb{C}^{n+1}$. Since

$$|w_1|^2 + \ldots + |w_n|^2 - |w_{n+1}|^2 \;<\; 0$$

it must be that $w_{n+1} \neq 0$, and we can divide through by w_{n+1} if we want.

Now we prove that the special unitary group

$$SU(n,1) \;=\; \{g \in U(n,1) \;:\; \det g = 1\}$$

acts *transitively* on the unit ball B in $\mathbb{C}^n$.

We will see that the isotropy group in $SU(n,1)$ of $0 \in \mathbb{C}^n$ is

$$K \;=\; \{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in SU(n,1) \;:\; a^*a = 1_n,\;\; |d|^2 = 1,\;\; d\cdot\det a = 1\} \;\approx\; U(n)$$

so we will have

[5.0.4] Claim: The special unitary group $SU(n,1)$ is *transitive* on the unit ball B in $\mathbb{C}^n$, and as an $SU(n,1)$ space

$$B \;\approx\; SU(n,1)/U(n)$$

Proof: As usual, to prove transitivity, it suffices to show that $0 \in \mathbb{C}^n$ can be mapped to any other point z in the complex n-ball.

First, we determine the isotropy group of 0. Using a block decomposition $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of an element g in $GL_{n+1}(\mathbb{C})$ with a being n-by-n, d being 1-by-1, etc., the condition for fixing 0 is

$$0 \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}(0) \;=\; bd^{-1}$$

which requires that b is the n-by-1 matrix of 0's. The condition that the matrix be in $U(n,1)$ is

$$H \;=\; g^*Hg \;=\; \begin{bmatrix} a^*a - c^*c & -c^*d \\ -d^*c & -d^*d \end{bmatrix}$$

From the off-diagonal entries, $c = 0$. Then $a^*a = 1_n$ and $d^*d = 1$. The condition that the determinant be 1 requires $d = (\det a)^{-1}$. Thus, any $a \in U(n)$ gives an element of the isotropy subgroup in $SU(n,1)$, and vice-versa.

Now prove transitivity, by proving that 0 can be mapped by $SU(n,1)$ to any other point z in the ball. We simplify the problem, reducing to the case $n = 1$, by using the isotropy group $K \approx U(n)$ of 0 to rotate the given z to a special form. The transitivity of $SU(n)$ on the *sphere* of a fixed radius r in $\mathbb{C}^n$ assures that there is $a \in SU(n)$ such that

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}(z) = \begin{bmatrix} r \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \text{(where } r = |z| \text{ is usual length)}$$

There is a corresponding copy of $SU(1,1)$ inside $SU(n,1)$, given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1_{n-1} & 0 \\ c & 0 & d \end{bmatrix}$$

which acts conveniently (via linear fractional transformations) by

$$\begin{bmatrix} a & 0 & b \\ 0 & 1_{n-1} & 0 \\ c & 0 & d \end{bmatrix} \begin{bmatrix} r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} (ar+b)/(cr+d) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus, the question of transitivity is reduced to finding $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(1,1)$ to map 0 to $r = |z|$ in the disk in $\mathbb{C}^1$. We can hope that a conveniently special class of matrices suffices. For example, the condition that $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ be in $SU(1,1)$, with a, b *real*, is just that $a^2 - b^2 = 1$. The condition that this matrix map 0 to r (with $0 \le r < 1$) is $b/a = r$. Substituting $b = ra$ into the first relation gives $a^2(1 - r^2) = 1$. This gives a choice for a, and then for b. ///
