(*March 18, 2011*)

[0.0.1] Claim: On a *simple* complex Lie algebra $\mathfrak{g}$, up to scalars there is at most one $\mathfrak{g}$-invariant non-degenerate bilinear form $\langle , \rangle$.

Proof: A non-degenerate $\mathfrak{g}$-equivariant bilinear form B gives an $\mathfrak{g}$-isomorphism φ_B of $\mathfrak{g}$ (under $\mathrm{ad}\,\mathfrak{g}$) to its dual $\mathfrak{g}^*$ (under the contragredient $\mathrm{ad}^*\mathfrak{g}$), and vice-versa, via

$$\varphi_B(x)(y) \;=\; \langle x, y\rangle$$

The equivariance property is readily verified:

$$\varphi_B(\mathrm{ad}\,z(x))(y) \;=\; B([z,x],y) \;=\; B(x,-[z,y]) \;=\; \varphi_B(x)(\mathrm{ad}\,(-z)(y)) \;=\; \Big((\mathrm{ad}^*z)\big(\varphi_B(x)\big)\Big)(y)$$

The non-degeneracy of B is equivalent to φ_B having trivial kernel, so by finite-dimensionality equivalent to its being an isomorphism.

Thus, given two $\mathfrak{g}$-invariant non-degenerate bilinear forms, B, C, the composite $\varphi = \varphi_B^{-1} \circ \varphi_C$ is a $\mathfrak{g}$-isomorphism $\mathfrak{g} \to \mathfrak{g}$.

That $\mathfrak{g}$ is *simple* is equivalent to the *irreducibility* of the $\mathfrak{g}$-module $\mathfrak{g}$ (under *ad*).

By Schur's lemma for finite-dimensional $\mathfrak{g}$-representations over an algebraically closed field, the only $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ are scalars. Thus, φ is a scalar, so φ_B and φ_C differ by a scalar. From this, $B(x,y)$ and $C(x,y)$ differ by a uniform scalar. ///

[0.0.2] Remark: $\mathfrak{g}$-invariance of a bilinear form $\langle , \rangle$ on $\mathfrak{g}$ is

$$\langle [z,x], y\rangle + \langle x, [z,y]\rangle \;=\; 0 \qquad\qquad (\text{for all } x,y,z \in \mathfrak{g})$$

[0.0.3] Remark: Complete reducibility of finite-dimensional $\mathfrak{g}$-modules is not used in the proof.

[0.0.4] Remark: The sign in

$$\varphi_B(\mathrm{ad}\,z(x))(y) \;=\; \varphi_B(x)(\mathrm{ad}\,(-z)(y))$$

is a special case: for *any* $\mathfrak{g}$-representation π on vector space V, the contragredient π^* on V^* is

$$(\pi^* z)(\lambda)(y) \;=\; \lambda(\pi(-z)(y)) \qquad\qquad (\text{for } z \in \mathfrak{g},\ \lambda \in V^*,\ y \in V)$$

That the involutive anti-automorphism $z \to -z$ appear in the contragredient is understandable and desirable for at least two reasons. For compatibility with formation of contragredients of *group* representations, appearance of this involution is necessary. In a different vein, for the contragredient to be in the same category of *left* $\mathfrak{g}$-modules, *some* involutory anti-automorphism is needed, to convert a *right* $\mathfrak{g}$-module to a *left* $\mathfrak{g}$-module. This is completely parallel to the fact that the construction of *contragredient* M^* of a *left* modules M over a *non-commutative* ring R is naturally a *right* R-module, by

$$(\lambda \cdot r)(m) \;=\; \lambda(r \cdot m) \qquad\qquad (\text{for } \lambda \in M^*,\ r \in R,\ m \in M)$$