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S. Bernstein's proof of Weierstraß' approximation theorem

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Weierstraß proved that polynomials are dense in $C^o(\mathbb{R}^n)$. Decades later, Stone greatly abstracted this. Prior to Stone, S. Bernstein gave a memorable argument for Weierstraß' concrete case, with the additional virtue of suggesting similarly intuitive arguments for various function spaces on topological spaces with transitive group actions.

[0.0.1] Theorem: Given $f \in C^o(U)$ for U open in $\mathbb{R}^n$, for compact $C \subset U$ and $\varepsilon > 0$, there is a polynomial P such that

$$\sup_{x \in C} |f(x) - P(x)| \;<\; \varepsilon$$

Proof: The idea is to create a sequence P_ℓ of polynomials on $\mathbb{R}^n$ whose restrictions φ_ℓ to a fixed compact, such as the cube

$$C \;=\; \{x = (x_1, \ldots, x_n) \;:\; |x_i| \le 1, \text{ for all } i\}$$

form an *approximate identity*, in the sense that their masses bunch up at 0.

More precisely, we want the restrictions φ_ℓ to C to be non-negative, to have integrals 1 on the smaller cube $\frac{1}{2}C$, and to satisfy

$$\lim_{\ell \to \infty} \frac{\int_{\delta C} \varphi_\ell}{\int_{\frac{1}{2}C} \varphi_\ell} \;=\; 1 \qquad \text{(for each fixed positive } \delta \le \tfrac{1}{2}\text{)}$$

Granting all that, we can show that the *mollifications* of f by the φ_ℓ approach f in the sup norm:

$$\lim_{\ell \to \infty} \sup_{x \in \frac{1}{2}C} \left| f(x) - \int_{\frac{1}{2}C} f(x+y)\, \varphi_\ell(y)\, dy \right| \;=\; 0 \qquad (\text{as } \ell \to \infty)$$

Indeed,

$$\int_{\frac{1}{2}C} f(x+y)\, \varphi_\ell(y)\, dy \;=\; \int_{\delta C} f(x+y)\, \varphi_\ell(y)\, dy + \int_{\frac{1}{2}C - \delta C} f(x+y)\, \varphi_\ell(y)\, dy$$

$$=\; f(x) \int_{\delta C} \varphi_\ell(y)\, dy + \int_{\delta C} \big(f(x+y) - f(x)\big)\, \varphi_\ell(y)\, dy + \int_{\frac{1}{2}C - \delta C} f(x+y)\, \varphi_\ell(y)\, dy$$

The first integral goes to 1 as $\ell \to \infty$, for fixed $\delta > 0$, by the bunching-up property. The second integral goes to 0 uniformly in x as $\delta \to 0$, by the uniform continuity of f on C. The third integral goes to 0 as $\ell \to \infty$, since the masses of the φ_ℓ bunch up inside δC. Thus, assuming we have such polynomials,

$$\lim_{\ell \to \infty} \int_{\frac{1}{2}C} f(x+y)\, \varphi_\ell(y)\, dy \;=\; f(x) \qquad (\text{uniformly in } x \in \tfrac{1}{2}C)$$

At the same time,

$$\int_{\frac{1}{2}C} f(x+y)\, \varphi_\ell(y)\, dy \;=\; \int_{x + \frac{1}{2}C} f(y)\, \varphi_\ell(-x+y)\, dy$$

is a superposition of polynomials of degrees at most that of φ_ℓ. The space V of such polynomials is finite-dimensional. Thus, this integral of a compactly-supported continuous V-valued function lies in V. That is, this integral is equal to a *polynomial*, as a function. This would prove the theorem.

To make suitable polynomials P_ℓ, it suffices to treat the single-variable case. Let

$$P_\ell(x) \;=\; (1-x^2)^\ell \qquad\qquad (\text{for } x \in \mathbb{R})$$

First, determine where the second derivative vanishes: solve

$$0 \;=\; \frac{d}{dx}\Big(-2\ell x(1-x^2)^{\ell-1}\Big) \;=\; 4\ell(\ell-1)x^2(1-x^2)^{\ell-2} - 2\ell(1-x^2)^{\ell-1}$$

$$=\; 2\ell\cdot\Big((\ell-1)x^2 - (1-x^2)\Big)\cdot(1-x^2)^{\ell-2}$$

Thus, in the interior of $[-1,1]$, the second derivative vanishes at $\pm 1/\sqrt{\ell}$, so the curve bends downward in $[-1/\sqrt{\ell}, 1/\sqrt{\ell}]$, and bends upward outside that interval. In particular, the line segments from the points $(\pm 1/\sqrt{\ell}, 0)$ to $(0,1)$ are below the graph of P_ℓ, so

$$\int_{|x|\le 1/\sqrt{\ell}} P_\ell(x)\,dx \;\ge\; \frac{2}{\sqrt{\ell}}$$

On the other hand, the standard fact that

$$\lim_{\ell\to\infty}(1-\frac{x}{\ell})^\ell \;=\; e^{-x}$$

suggests a certain approach. For example,

$$P_\ell(\frac{\sqrt{\log\ell}}{\sqrt{\ell}}) \;=\; \Big(1-\frac{\log\ell}{\ell}\Big)^\ell$$

Since $\log(1-x) \le -x$ for $x \ge 0$,

$$\log P_\ell(\frac{\sqrt{\log\ell}}{\sqrt{\ell}}) \;\le\; -\log\ell$$

Thus,

$$P_\ell(\frac{\sqrt{\log\ell}}{\sqrt{\ell}}) \;\le\; \frac{1}{\ell}$$

Thus we have a sufficient bunching-up result: obviously $\frac{1}{\sqrt{\ell}} < \frac{\log\ell}{\sqrt{\ell}}$, so

$$\int_{|x|<\frac{\log\ell}{\sqrt{\ell}}} P_\ell(x)\,dx \;\ge\; \frac{2}{\sqrt{\ell}}$$

while

$$\int_{\frac{\log\ell}{\sqrt{\ell}}<|x|<1} P_\ell(x)\,dx \;\le\; \frac{2}{\ell}$$

That is, letting

$$\varphi_\ell(x) \;=\; \frac{P_\ell(x)}{\displaystyle\int_{|x|\le\frac{1}{2}} P_\ell(x)\,dx}$$

gives the single-variable approximate identity desired. The product

$$\varphi_\ell(x_1)\ldots\varphi_\ell(x_n)$$

is the desired collection for $\mathbb{R}^n$. ///

[0.0.2] Remark: Although it is unnecessary for the above, it is interesting to determine the integral of the single-variable P_ℓ over $[-1,1]$. Integrating by parts repeatedly, it is

$$\int_{-1}^1 (1-x)^\ell\cdot(1+x)^\ell\,dx \;=\; \frac{\ell}{\ell+1}\int_{-1}^1 (1-x)^{\ell-1}\cdot(1+x)^{\ell+1}\,dx \;=\; \frac{\ell(\ell-1)}{(\ell+1)(\ell+2)}\int_{-1}^1 (1-x)^{\ell-2}\cdot(1+x)^{\ell+2}\,dx$$

$$=\;\ldots\;=\;\frac{\ell!\,\ell!}{(2\ell)!}\int_{-1}^1 (1+x)^{2\ell}\,dx \;=\; \frac{\ell!\,\ell!}{(2\ell)!}\,\frac{2^{2\ell+1}}{2\ell+1}$$