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Number theory discussion 07

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[number theory 07.1] We first give a formulaic verification that $dg = dx\,\frac{dy}{y}$ in coordinates $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ is a right Haar measure on the group

$$G \;=\; \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;:\; 0 < y \in \mathbb{R},\; x \in \mathbb{R}\}$$

First say what is intended by the notation:

$$\int_G f(g)\,dg \;=\; \int_0^\infty \int_{-\infty}^\infty f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} dx\,\frac{dy}{y} \qquad\qquad (\text{for } f \in C_c^o(G))$$

Let $h = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix}$. Then

$$\int_G f(gh)\,dg \;=\; \int_0^\infty \int_{-\infty}^\infty f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix} dx\,\frac{dy}{y} \;=\; \int_0^\infty \int_{-\infty}^\infty f\begin{pmatrix} y\eta & x + y\xi \\ 0 & 1 \end{pmatrix} dx\,\frac{dy}{y}$$

Replacing y by y/η leaves dy/y invariant, stabilizes the set $(0, +infty)$, and the integral becomes

$$\int_0^\infty \int_{-\infty}^\infty f\begin{pmatrix} y & x + \frac{y\xi}{\eta} \\ 0 & 1 \end{pmatrix} dx\,\frac{dy}{y}$$

Replacing x by $x - y\xi/\eta$ leaves dx invariant, stabilizes $(-\infty, +\infty)$, and the integral becomes

$$\int_0^\infty \int_{-\infty}^\infty f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} dx\,\frac{dy}{y} \;=\; \int_G f(g)\,dg$$

This proves the invariance by computation. ///

[0.0.1] Remark: As discussed in the context of Siegel's computation of volumes of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$, when a group G is expressible as $G = PK$ the Haar measure on G is expressible in terms of those on P and K.

Indeed, G is a $P \times K$ space, with action $(p,k)(g) = p^{-1}gk$, with the inverse for associativity, as usual. The isotropy group of the point $1 \in G$ is $P \cap K$. Thus, as $P \times K$ -spaces, $G \approx (P \cap K)\backslash(P \times K)$. As proven earlier, there is a (unique!) $P \times K$-invariant measure on that quotient if and only if the modular function condition is met, namely,

$$\Delta_{P\times K}\Big|_{P\cap K} \;=\; \Delta_{P\cap K}$$

When this condition is met, the right Haar measure on $P \times K$ gives rise to a (unique!) right $P \times K$-invariant measure on $(P \cap K)\backslash(P \times K)$.

At the same time, the right Haar measure on G is certainly right K-invariant. The left invariance by P requires $d(p^{-1}g) = dg$. For a right Haar measure dg, the *modular function* Δ_G is $d(hg) = \Delta(h)\cdot dg$. Thus, for the right Haar measure on G to give a right $P \times K$-invariant measure at all, we need $\Delta_G\big|_P = 1$.

In summary, when $\Delta_{P\times K}\Big|_{P\cap K} = \Delta_{P\cap K}$ and $\Delta_G\big|_P = 1$, the $P \times K$-invariant measure must be equal to (a constant multiple of) the Haar measure on G.

In particular, when G is a *product* $G = N \times A$, necessarily $N \cap A = \{1\}$, and the only condition is $\Delta_G\big|_N = 1$. When this is met,

$$dg \;=\; dn\,da \qquad\qquad \text{(right Haar measures)}$$

[number theory 07.2] Verify that $dg = dx\,dy\,dz$ in coordinates $g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ is a right Haar measure on the group

$$G \;=\; \{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;:\; x, y, z \in \mathbb{R}\}$$

This means that the intended integral is

$$\int_G f(g)\,dg \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_R f\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} dx\,dy\,dz \qquad\qquad (\text{for } f \in C_c^o(G))$$

For $h = \begin{pmatrix} 1 & \xi & \eta \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix}$,

$$\int_G f(gh)\,dg \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \xi & \eta \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix} dx\,dy\,dz$$

$$=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f\begin{pmatrix} 1 & x+\xi & y+\eta+x\zeta \\ 0 & 1 & z+\zeta \\ 0 & 0 & 1 \end{pmatrix} dx\,dy\,dz$$

The additive Haar measures dx, dy, dz are invariant under translations, as are the three copies of $\mathbb{R}$. First replacing y by $y - \eta - x\zeta$, then x by $x - \xi$ and z by $z - \zeta$ does not change the *value* of the integral, and puts it back in the form

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_R f\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} dx\,dy\,dz \;=\; \int_G f(g)\,dg$$

This gives the invariance. ///

[0.0.2] Remark: The latter example illustrates another class wherein the Haar measure on a larger group is built up from smaller groups. That example presents the whole group G as fitting into a short exact sequence in which the outer groups are more elementary: with $Z = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \mathbb{R}$ the center of G, $G/Z \approx \mathbb{R}^2$ by $\begin{pmatrix} 1 & u & * \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (u, v)$. That is, we have a short exact sequence

$$1 \longrightarrow \mathbb{R} \longrightarrow G \longrightarrow \mathbb{R}^2 \longrightarrow 1$$

Generally, consider $1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$ where N is a closed normal subgroup. We *claim* that

$$\int_G f(g)\,dg \;=\; \int_{N\backslash G}\Big(\int_N f(n\dot{g})\,dn\Big)d\dot{g} \qquad\qquad (\text{for } f \in C_c^o(\mathcal{G}))$$

where $d\dot{g}$ is Haar measure on $N\backslash G$ and dn is Haar measure on N. Indeed, we have already treated a more general version of this, in which N need not be normal. The condition for success is the usual $\Delta_G\big|_N = \Delta_N$.

[number theory 07.3] Verify that $G = SL_2(k)$, the two-by-two matrices with entries in a field k with more than 2 elements, with determinant 1, has the property $G = [G, G]$.

Hint: To get started, note that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2x \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x(a^2-1) \\ 0 & 1 \end{pmatrix}$$

This shows that all unipotent upper-triangular matrices are commutators. Similarly for lower-triangular unipotent.

Various further bits of fooling around suffice to express the general invertible matrix in terms of upper-triangular or lower-triangular unipotent matrices.
