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Number theory exercises 10

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Due Friday, 06 April 2012, preferably as PDF emailed to me.

[number theory 10.1] Show that the Euclidean Laplacian is rotation-invariant.

[number theory 10.2] Show that the rotation group $SO(n,\mathbb{R})$ is *transitive* on the $(n-1)$-sphere.

[number theory 10.3] Show that the z and $\overline{z}$ calculus works as claimed, that is, for example, for a *rational function* f in two variables,

$$\frac{\partial}{\partial z} f(z,\overline{z}) \;=\; f_1(z,\overline{z})$$

where f_1 is the partial derivative of f with respect to its first argument.

[number theory 10.4] Up to essentially irrelevant normalization constants, the n^{th} *Hermite polynomial* is

$$h_n(x) \;=\; e^{x^2} \cdot \Big(\frac{d^n}{dx^n} e^{-x^2} \Big)$$

Show that the degree of h_n is n. Show that $h_0, h_1, h_2, \ldots$ are *orthogonal* with respect to the inner product

$$\langle f, g\rangle \;=\; \int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\; e^{-x^2}\; dx$$

[number theory 10.5] * *[Starred problems are optional]* Show that the Euler product for the Dedekind zeta function $\zeta_k(s)$ of a number field k of degree n over $\mathbb{Q}$ is absolutely convergent in $\mathrm{Re}(s) > 1$, by comparing it prime-wise to $\zeta_{\mathbb{Q}}(s)^n$, by grouping together all primes of $\mathfrak{o}_k$ lying over a prime p in $\mathbb{Z}$. That is, show that

$$\prod_{\mathfrak{p}|p} \frac{1}{1 - N\mathfrak{p}^{-\sigma}} \;\leq\; \Big(\frac{1}{1-p^{-\sigma}}\Big)^n \qquad \text{(for } \sigma > 0\text{, primes } \mathfrak{p} \text{ in } \mathfrak{o}_k\text{)}$$
