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Algebraic Number Theory Exercises 01

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Due Wed, 21 Sept 2011, preferably as PDF emailed to me.

[number theory 01.1] Prove the Euler product expansion of the zeta function, namely, for $\mathrm{Re}(s) > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

A useful point is that

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + (p^2)^{-s} + (p^3)^{-s} + \ldots$$

Often this Euler product expansion is interpreted as a slightly analytic manifestation of the *unique factorization* in $\mathbb{Z}$.

Proper care for *convergence* is a non-trivial task, but worth doing once in one's life.

Part of the burden is merely notational, but the risks of bad notation are considerable.

[number theory 01.2] Prove that a prime p is expressible as $p = a^2 + ab + b^2$ for integers a, b if and only if $p = 1 \bmod 3$ (or $p = 3$).

[number theory 01.3] Let ω be a primitive 7^{th} root of unity, and let $\xi = \omega + \omega^{-1}$. Observe that $\xi^3 + \xi^2 - 2\xi - 1 = 0$. Find the precise congruence relation on primes p for there to be a solution of $x^3 + x^2 - 2x - 1 = 0$ in $\mathbb{Z}/p$.