Recap:

Idele norm is $|\{x_v\}| = \prod_{v\le\infty} |x_v|_v$ and $\mathbb{J}^1 = \{x \in \mathbb{J} : |x| = 1\}$

Fujisaki's lemma: $\mathbb{J}^1/k^\times$ is *compact*. (via a measure-theory *pigeon-hole* principle)

Corollary: Ideal class groups are finite.

Let $k \otimes_\mathbb{Q} \mathbb{R} \approx \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. That is, k has r_1 *real* archimedean completions, and r_2 *complex* archimedean completions. The global degree is the sum of the local degrees: $[k : \mathbb{Q}] = r_1 + 2r_2$.

Corollary: *(Dirichlet's Units Theorem)* The unit group $\mathfrak{o}^\times$, modulo roots of unity, is a free $\mathbb{Z}$-module of rank $r_1 + r_2 - 1$. Generally, S-units $\mathfrak{o}_S^\times$ mod roots of unity are rank $|S| - 1$.

Generalized ideal class groups: The class number above is the *absolute* class number. The *narrow* class number is ideals modulo principal ideals generated by *totally positive* elements.

For non-zero ideal $\mathfrak{a}$, the *narrow ray class group* mod $\mathfrak{a}$ is fractional ideals *prime to* $\mathfrak{a}$ modulo principal ideals $\alpha\mathfrak{o}$ generated by *totally positive* $\alpha = 1 \bmod \mathfrak{a}$.

Lemma: Generalized ideal class groups are *idele* class groups, quotients of the compact group $\mathbb{J}^1/k^\times$ by *open* subgroups. ///

Corollary: Generalized ideal class groups are *finite.* ///

Generalized units: Let S be a finite collection of places of k, including all archimedean places. The S-integers $\mathfrak{o}_S$ in k are

$$\mathfrak{o}_S \;=\; k \cap \Big(\prod_{v \in S} k_v \times \prod_{v \notin S} \mathfrak{o}_v \Big) \;=\; \{\alpha \in k : \alpha \text{ is } v\text{-integral for } v \notin S\}$$

The group of S-units is $\mathfrak{o}_S^\times = k^\times \cap \big(\prod_{v \in S} k_v^\times \times \prod_{v \notin S} \mathfrak{o}_v^\times \big)$

Theorem: *(Units)* $\mathfrak{o}_S^\times$ mod roots of unity is free rank $|S| - 1$. ///

Theorem: (*Kronecker)* For $\alpha \in \mathfrak{o}$, if $|\alpha|_v = 1$ for all $v|\infty$ then α is a root of unity. ///

Closed subgroups of $\mathbb{R}^n$: The closed subgroups H of $\mathbb{R}^n$ are: for a *vector subspace* W of $\mathbb{R}^n$, and *discrete* subgroup Γ of $\mathbb{R}^n/W$,

$$H \;=\; q^{-1}(\Gamma) \qquad\qquad \text{(with } q : \mathbb{R}^n \to \mathbb{R}^n/W \text{ the quotient map)}$$

The *discrete* subgroups Γ of $\mathbb{R}^n$ are free $\mathbb{Z}$-modules $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ on $\mathbb{R}$-linearly-independent vectors $v_j \in V$. ///

Measure-theory pigeon-hole principle: [*recap*] Again, the adelic version is the obvious extension of...

Proposition: $E \subset \mathbb{R}$ with measure > 1 contains $x \neq y$ such that $x - y \in \mathbb{Z}$.

Proof: [*again*] With f the characteristic function of E, if no two points of E differ by an integer, $0 \leq \sum_{n \in \mathbb{Z}} f(x+n) \leq 1$. Thus,

$$1 \ < \ \int_{-\infty}^{\infty} f(x)\, dx \ = \ \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n)\, dx \ \leq \ 1$$

Impossible. Thus, there are $x \neq y \in E$ with $x - y \in \mathbb{Z}$. ///

Remark: We exploited the convenient obvious *fundamental domain* for the action of $\mathbb{Z}$ on $\mathbb{R}$, that is, the subset $[0,1]$ of $\mathbb{R}$ whose translates by $\mathbb{Z}$ fill out $\mathbb{R}$ with overlaps of measure 0. This was unnecessary and misleading. This is rectified below.

Numerous remaining supporting details:

Integration on quotients: Quotients $\Gamma\backslash G$ such as $\mathbb{R}/\mathbb{Z}$ have a reasonable integration theory *without* finding/constructing/using a so-called *fundamental domain.* Intrinsic integration on quotients is essential for situations $\Gamma\backslash G$ where determination of a fundamental domain is complicated or impossible.

Example: We *want* a continuous linear map (integral!) $F \to \int_{\mathbb{R}/\mathbb{Z}} F(x)\,dx$ on $C_c^o(\mathbb{R}/\mathbb{Z})$ (think of the Riesz representation theorem), translation-invariant, non-negative for non-negative F, and with the essential compatibility

$$\int_{\mathbb{R}/\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} f(x+n) \Big)\,dx \;=\; \int_{\mathbb{R}} f(x)\,dx \qquad (\text{for } f \in C_c^o(\mathbb{R}))$$

Try to *define* the integral on $\mathbb{R}/\mathbb{Z}$ by this relation...!?!

Well-definedness is an issue, since the same $F(x) = \sum_n f(x+n)$ in $C_c^o(\mathbb{R}/\mathbb{Z})$ can arise by *periodicizing* two functions f in $C_c^o(\mathbb{R})$.

The complementary question is whether every $F \in C_c^o(\mathbb{R}/\mathbb{Z})$ is obtained by periodicizing *some* $f \in C_c^o(\mathbb{R})$.

We prove that this succeeds even in very general circumstances.

Let $\alpha : C_c^o(\mathbb{R}) \to C_c^o(\mathbb{R}/\mathbb{Z})$ be the *averaging map*

$$\alpha f(x) \;=\; \sum_{n\in\mathbb{Z}} f(x+n)$$

Lemma: The averaging map is *surjective*.

Proof: Let q be the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

Given $F \in C_c^o(\mathbb{R}/\mathbb{Z})$, let C' be a compact subset of $\mathbb{R}$ such that $q(C') \supset \text{spt}(F)$. [Here, this is trivial.] Let φ be in $C_c^o(\mathbb{R})$ identically 1 on a neighborhood of C'. [Urysohn, in general.] Let

$$g(x) \;=\; \varphi(x) \cdot F(x) \in C_c^o(\mathbb{R})$$

Since F is already left $\mathbb{Z}$-invariant

$$\alpha(g) \;=\; \alpha(\varphi \cdot F) \;=\; \alpha\varphi \cdot F$$

Since $\alpha(\varphi) \equiv 1$ on an open containing the support of F,

$$\alpha(g/\alpha\varphi) = \alpha\varphi \cdot F/\alpha\varphi = F$$

and the quotient $g/\alpha(\varphi)$ is continuous. This gives surjectivity.

///

For *well-definedness*, it suffices to prove that $\alpha f = 0$ implies $\int_{\mathbb{R}} f(x)\,dx = 0$. Suppose $\alpha f = 0$. For all $F \in C_c^o(\mathbb{R})$, the integral of F against αf is certainly 0, and we rearrange

$$0 \;=\; \int_{\mathbb{R}} F(x)\,\alpha f(x)\,dx \;=\; \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} F(x)\,f(x+n)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} F(x)\,f(x+n)\,dx \;=\; \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} F(x-n)\,f(x)\,dx$$

Replace n by $-n$, giving

$$0 \;=\; \int_{\mathbb{R}} \alpha F(x)\,f(x)\,dx$$

By surjectivity of α, there is F with $\alpha F = 1$ on the support of f. Then the integral of f is 0, proving the well-definedness. ///

More generally, replace $\mathbb{R}$ by a topological group G, and $\mathbb{Z}$ by a closed subgroup H. Given right-translation-invariant measures on G and H, we want a unique measure $d\dot{g}$ on $H\backslash G$ such that

$$\int_{H\backslash G} \int_H f(h\dot{g})\, dh\, d\dot{g} \;=\; \int_G f(g)\, dg$$

The same proof almost works.

However, when H and G are non-abelian and non-compact, a technical issue can arise: *left* translation produces another *right* translation-invariant measure. By uniqueness of Haar measure, this translated measure differs by a constant from the given Haar measure.

In general, left translation *does* change the right translation-invariant measure by a constant, called the *modular function*

$$d(xg) = \Delta_G(x) \cdot dg \qquad\qquad d(yh) = \Delta_H(y) \cdot dh$$

For straightforward reasons, the condition for existence of a right G-invariant measure on $H\backslash G$ is that

$$\Delta_G \text{ restricted to } H \quad = \quad \Delta_H$$

This modular function condition is obtained from

$$\int_{H\backslash G} \int_H f(hg)\, dh\, dg \;=\; \int_G f(g)\, dg$$

by change of variables: replace h by hx for $x \in H$, and g by $x^{-1}g$.

Having non-trivial modular function is not a pathology, but very reasonable in certain circumstances. Nevertheless, it is convenient that $\Delta_G \equiv 1$ for many G. Such G are called *unimodular*.

$\Delta_G \equiv 1$ for *abelian* G, because $d(xg) = d(gx)$.

Below, we show that Δ_G is a *continuous group homomorphism* to $(0, +\infty)$ with multiplication.

Since $(0, +\infty)$ has no proper compact subgroups, $\Delta_G \equiv 1$ for *compact* G.

Since $(0, +\infty)$ is *abelian*, Δ_G is 1 on the commutator subgroup $[G, G]$ of G, generated by all $[g, h] = ghg^{-1}h^{-1}$. Thus, G is unimodular when $G = [G, G]$ or even when $G/[G, G]$ is *compact*.

Examples:

$G = SL_2(\mathbb{R})$, the group of two-by-two real matrices with determinant 1, has $[G,G] = G$ (!), so is unimodular.

A non-pathological *not*-unimodular example is

$$G = \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y > 0,\ x \in \mathbb{R}\}$$

In those coordinates, *right* Haar measure is (!) $dg = dx\,\dfrac{dy}{y}$ with Lebesgue measures on $\mathbb{R}$. *Left* multiplication by $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ changes the measure by t, so $\Delta_G \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = t$.