Recap: Finiteness of class number and *Dirichlet's units theorem* are corollaries of *Fujisaki's lemma*, that $\mathbb{J}^1/k^\times$ is *compact.* This is an almost-immediate corollary of

Measure-theory pigeon-hole principle: On $\mathbb{R}$, for example, $E \subset \mathbb{R}$ with measure > 1 contains $x \neq y$ such that $x - y \in \mathbb{Z}$.

This really about **integration on quotients:** for a *discrete* subgroup Γ of a *unimodular* topological group G, such that $\Gamma\backslash G$ has finite invariant measure, if a set $E \subset G$ has measure strictly greater than $\Gamma\backslash G$, then there are $x \neq y \in E$ such that $x^{-1}y \in \Gamma$.

As expected, the Haar measure on Γ is *counting* measure, and we normalize measures so that

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg \;=\; \int_G f(g)\, dg \qquad\qquad (\text{for } f \in C_c^o(G))$$

Minkowski formulated a version about *lattices* L in $V \approx \mathbb{R}^n$, that is, *discrete* subgroups L with V/L *compact*.

Minkowski showed that, a *convex* subset C of V, symmetric about 0, with measure strictly greater than 2^n times the measure of V/L, contains a point of L other than 0. This is a foundational element of his *Geometry of Numbers*.

This is a corollary of the measure-theoretic pigeon-hole principle: with $E = \frac{1}{2} \cdot C$, the measure of E is more than the measure of V/L, and we've shown that there are $x \neq y \in E$ such that $x - y \in L$. The condition $x \neq y$ gives $x - y \neq 0$. Evidently, we claime $E - E = C$...

One half of $E - E = C$ is easy: using the symmetry of C,

$$E - E = \tfrac{1}{2} \cdot C - \tfrac{1}{2} \cdot C = \tfrac{1}{2} \cdot C + \tfrac{1}{2} \cdot C \supset C$$

The other direction uses the convexity, also:

$$\tfrac{1}{2} \cdot C + \tfrac{1}{2} \cdot C = \{\frac{x+y}{2} : x, y \in C\} \subset C$$

Thus, $E - E = C$, and Minkowski's theorem follows from the measure-theoretic pigeon-hole principle. ///

Remark: ... but the convexity and symmetry and having the ambient group be $\mathbb{R}^n$ are misleading specifics, even though this is a very important application.

Interrupting the storyline to emphasize what we *are* doing, as opposed to *not*:

Inspection of the arguments shows that we want very few things from (right-invariant) integrals on groups G, which *characterize* the integrals:

$$\begin{cases} f \to \int_G f(g)\,dg \text{ defined on } C_c^o(G) & \text{(functionals on } C_c^o(G)) \\ \int_G f(gh)\,dg = \int_G f(g)\,dg \text{ for } h \in G & \text{(right invariance)} \\ f \geq 0 \Longrightarrow \int_G f(g)\,dg \geq 0 & \text{(positivity)} \end{cases}$$

In fact, the positivity condition implies that $f \to \int_G f$ is a *continuous* linear functional on $C_c^o(G)$ in its natural topology, but the arguments here only use the positivity.

For context: the usual *Riesz representation theorem* (*not* the more elementary Riesz-Fischer theorem about continuous functionals on Hilbert spaces), also uses only *positivity*, not giving any topology on $C_c^o(X)$, for X the locally compact, Hausdorff, preferably countably-based topological space in question.

Riesz' theorem asserts that, given a *positive* linear functional λ on $C_c^o(X)$, there is a positive Borel measure μ so that

$$\lambda(f) \;=\; \int_X f(x)\, d\mu(x)$$

The countably-based hypothesis promises that there is *regular* μ, meaning that $\mu(E)$ is both the *sup* of $\mu(C)$ for compact $C \subset E$, and the *inf* of $\mu(U)$ for open $U \supset E$.

Without this hypothesis, regularity is not guaranteed, ... but we are mostly interested in countably-based topological spaces, such as $\mathbb{R}, \mathbb{Q}_p, \mathbb{A}, \mathbb{J}$, ...

Returning to the main thread: as above, Minkowski's theorem about lattice-points in convex bodies in $\mathbb{R}^n$ abstracts to:

For *discrete* Γ in *unimodular* topological group G, such that $\Gamma\backslash G$ has finite invariant measure, if a set $E \subset G$ has measure strictly greater than $\Gamma\backslash G$, then there are $x \neq y \in E$ such that $x^{-1}y \in \Gamma$.

Recapitulating the argument: the modular-function condition for existence of measures is met. With f the characteristic function of E, if there were *no* such x, y, then $\sum_{\gamma\in\Gamma} f(\gamma \cdot x) \leq 1$. But then

$$\text{meas}\,(\Gamma\backslash G) \;<\int_G f(g)\,dg \;=\; \int_{\Gamma\backslash G} \Big(\sum_{\gamma\in\Gamma} f(\gamma\cdot g)\Big)\,dg \;\leq\; \text{meas}\,(\Gamma\backslash G)$$

Impossible. So there *is* $1 \neq x^{-1}y \notin \Gamma$. ///

Another interruption! ... for the sake of context.

We can understand that quotients or real vector spaces by lattices, such as $\mathbb{R}^n/\mathbb{Z}^n$, have finite volume, but we have much less experience with discrete subgroups Γ in non-abelian G.

Among others, the exemplar of a finite-volume, but non-compact, quotient, is

$$SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R}) \qquad\qquad (SL_n(R) = n \times n \text{ matrices, entries in } R)$$

Minkowski and Siegel knew that this quotient had finite volume long ago. It is not at all obvious that this volume is finite, but let's not worry just now.

Back to the main existence theorem: given right-translation-invariant measures on $H \subset G$, and assuming the compatibility

$$\Delta_G \text{ restricted to } H \quad = \quad \Delta_H$$

there is a unique measure $d\dot{g}$ on $H\backslash G$ such that

$$\int_{H\backslash G} \int_H f(h\dot{g})\, dh\, d\dot{g} \;=\; \int_G f(g)\, dg$$

As in the prototypical case of $H = \mathbb{Z}$ and $G = \mathbb{R}$, the idea is to *define* the integral on $H\backslash G$ by this condition, and show that it is sufficiently-defined, and well-defined.

We (re-) prove the sufficiency starting from the existence of Haar measures on G and on H. First suppose that both are *unimodular*. With averaging map $\alpha : C_c^o(G) \to C_c^o(H\backslash G)$

$$\alpha f(g) \;=\; \int_H f(hg)\, dh \qquad\qquad (\text{for } f \in C_c^o(G))$$

attempt to define an integral on $C_c^o(H\backslash G)$ by

$$\int_{H\backslash G} \alpha f(\dot g)\, d\dot g \;=\; \int_G f(g)\, dg$$

We (re-) prove surjectivity of the averaging map α. Let q be the quotient map $q : G \to H\backslash G$.

Given $F \in C_c^o(H\backslash G)$, we need a compact subset C' of G such that $q(C') \supset \mathrm{spt}(F)$. By *local compactness* of G, there is open $U \ni 1$ with compact closure $\overline{U}$. Quotient maps are *open*, so $q(U)$ is open in $H\backslash G$, as are $q(U) \cdot G$ for $g \in G$. Certainly

$$\mathrm{spt}F \;\subset\; \bigcup_{g \in G} q(U) \cdot g$$

so by compactness of sptFC there is a finite subcover $\bigcup_i q(U) \cdot g_i$. The set $\bigcup_i \overline{U} \cdot g_i$ is compact in G, and its image under q contains spt$F$.

Let φ be in $C_c^o(G)$ identically 1 on a neighborhood of C', by Urysohn.

[Recall that, for open set U containing compact C in a locally-compact Hausdorff topologicaly space X, Urysohn's Lemma constructs $f \in C_c^o(X)$ which is identically 1 on C, and identically 0 off U.]

Let

$$g(x) \;=\; \varphi(x) \cdot F(x) \in C_c^o(G)$$

Since F is already left H-invariant

$$\alpha(g) \;=\; \alpha(\varphi \cdot F) \;=\; \alpha\varphi \cdot F$$

Since $\alpha(\varphi) > 0$ on an open containing the support of F,

$$\alpha(F/\alpha\varphi) = \alpha\varphi \cdot F/\alpha\varphi = F$$

and the quotient $F/\alpha(\varphi)$ is continuous. This gives surjectivity.

Now (re-) prove *well-definedness*: if $\alpha f = 0$, then $\int_G f(g)\,dg = 0$. Suppose $\alpha f = 0$. For all $F \in C_c^o(G)$, the integral of F against αf is certainly 0, and we rearrange

$$0 \;=\; \int_G F(g)\,\alpha f(g)\,dg \;=\; \int_G \int_H F(g)\,f(hg)\,dh\,dg$$
$$= \int_H \int_G F(h^{-1}g)\,f(g)\,dg\,dh$$

replacing g by $h^{-1}g$. Replace h by h^{-1}, so $0 = \displaystyle\int_G \alpha F(g)\,f(g)\,dg$

Surjectivity of α gives F with αF is identically 1 on the support of f. Thus, the integral of f is 0, proving the well-definedness for unimodular H and G. ///

Remark: We did *not* use formulas for the integrals.

Next: More about Haar measure...

Change-of-measure and Haar measure on $\mathbb{A}$ and k_v:

Another thing used in the proof of Fujisaki's lemma was that, for *idele* α, the change-of-measure on $\mathbb{A}$ is

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad (\text{for measurable } E \subset \mathbb{A})$$

Naturally, this should be examined...
