Context: Finiteness of class number and *Dirichlet's units theorem* are corollaries of *Fujisaki's lemma*, that $\mathbb{J}^1/k^\times$ is *compact*. ... a corollary of

Measure-theory pigeon-hole principle: for *discrete* subgroup Γ of a *unimodular* topological group G, with $\Gamma\backslash G$ of finite measure, if a set $E \subset G$ has measure strictly greater than $\Gamma\backslash G$, then there are $x \neq y \in E$ such that $x^{-1}y \in \Gamma$.

As expected, measure on Γ is *counting* measure, and

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg \;=\; \int_G f(g)\, dg \qquad\qquad (\text{for } f \in C_c^o(G))$$

Subsumes Minkowski's *Geometry of Numbers* proposition: for *lattices* L in $\mathbb{R}^n$, a *convex* subset C of $\mathbb{R}^n$, symmetric about 0, with measure strictly greater than 2^n times the measure of $\mathbb{R}^n/L$, contains a point of L other than 0.

Inspection of the arguments shows that we want very few things from (right-invariant) integrals on groups G, which *characterize* the integrals:

$$\begin{cases} f \to \int_G f(g)\,dg \text{ defined on } C_c^o(G) & \text{(functionals on } C_c^o(G)) \\ \int_G f(gh)\,dg = \int_G f(g)\,dg \text{ for } h \in G & \text{(right invariance)} \\ f \ge 0 \Longrightarrow \int_G f(g)\,dg \ge 0 & \text{(positivity)} \end{cases}$$

In fact, the positivity condition implies that $f \to \int_G f$ is a *continuous* linear functional on $C_c^o(G)$ in its natural topology, but the arguments here only use the positivity.

Recap of abstracted argument: with f the characteristic function of E, if there were *no* such x, y, then $\sum_{\gamma\in\Gamma} f(\gamma\cdot x) \le 1$. But then

$$\text{meas}\,(\Gamma\backslash G) \;<\int_G f(g)\,dg \;=\; \int_{\Gamma\backslash G} \Big(\sum_{\gamma\in\Gamma} f(\gamma\cdot g)\Big)\,dg \;\le\; \text{meas}\,(\Gamma\backslash G)$$

Impossible. So there *is* $1 \neq x^{-1}y \notin \Gamma$. ///

Existence of suitable measure on the quotient does not depend on discreteness of Γ, but on the condition $\Delta_G|_H = \Delta_H$, and then, as we proved, there exists a unique measure on $H\backslash G$ such that

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\,\dot{g})\,dh\Big)\,d\dot{g}$$

Another interruption! ... for context. Finite volume of $\mathbb{R}^n/\mathbb{Z}^n$ is familiar, but we have essentially *no* experience with discrete subgroups Γ in non-abelian G. The following is a prototype both for the assertion and for the proof mechanisms.

Claim: The quotient $\Gamma\backslash G = SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ has finite invariant volume (where $SL_n(R) = n \times n$ matrices with entries in ring R). In fact, in a natural normalization,

$$\mathrm{vol}\,(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})) \;=\; \zeta(2)\,\zeta(3)\,\zeta(4)\,\zeta(5)\ldots\zeta(n)$$

Remark: Mysterious ζ(odd) values appear.

Minkowski knew the finiteness, and Siegel computed the value. We grant the finiteness, and compute the volume *without* a *fundamental domain.*

Proof: (modernization of Siegel's argument) The point is

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg \;=\; \int_G f(g)\, dg$$

Treat $n = 2$, $G = SL(2,\mathbb{R})$, and $\Gamma = SL(2,\mathbb{Z})$. We showed that a right G-invariant measure on $\Gamma\backslash G$ is described by integrals of $C_c^o(\Gamma\backslash G)$. Every $F \in C_c^o(\Gamma\backslash G)$ is expressible as

$$F(g) \;=\; \sum_{\gamma\in\Gamma} f(\gamma\cdot g) \qquad\qquad (\text{for some } f \in C_c^o(G))$$

and the integral of F is sufficiently-defined and well-defined by

$$\int_{\Gamma\backslash G} F(g)\, dg \;=\; \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma\cdot g)\; dg \;=\; \int_G f(g)\, dg$$

Although we do *not* describe the geometry of $\Gamma\backslash G$, we *do* need details about the Haar measure on G, since a constant ambiguous by a constant is not interesting.

G is *unimodular*, since $G = [G,G]$. (!) To describe the measure on G usefully, we do need coordinates on G, but not the naive $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let K be the usual *special orthogonal group*

$$K = SO(2) = \{g \in G : g^\top g = 1_2\} \;=\; \{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\}$$

and

$$P^+ = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0,\; b \in \mathbb{R}\}$$

Compact K is unimodular, while P^+ is *not*. The *Iwasawa decomposition* (not too hard an exercise, in this example!) is

$$G \;=\; P^+ \cdot K \;\approx\; P^+ \times K$$

Lemma: Haar measure on G is $d(pk) = dp \cdot dk$, where dp is *left* Haar measure on P^+, and dk is *right* Haar on K. That is,

$$\int_G \varphi(g)\, dg \;=\; \int_{P^+} \int_K \varphi(pk)\, dk\, dp \qquad\qquad (\text{for } \varphi \in C_c^o(G))$$

Proof: Let the group $P^+ \times K$ act on G by $(p \times k)(g) = p^{-1}gk$. (The inverse is for associativity!) The isotropy subgroup in $P^+ \times K$ of $1 \in G$ is $\{p \times k : p^{-1} \cdot 1 \cdot k = 1\} = P^+ \cap K = \{1\}$. Thus, there is a unique $P^+ \times K$-invariant measure on G, and it fits into $\int_G = \int_P \int_K$. The Haar measure on G gives such a thing, as does a Haar measure on G. ///

Now completely specify the Haar measure on G. Normalize the Haar measure on the circle (!) K to have total measure 2π. Normalize the left Haar measure dp on P^+ to (!)

$$d\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \frac{1}{t^2}\,dx\,\frac{dt}{t} \qquad (x \in \mathbb{R} \text{ and } t > 0)$$

Corresponding to a nice (Schwartz?) function f on $\mathbb{R}^2$, let F on G be

$$F(g) = \sum_{v\in\mathbb{Z}^2} f(vg)$$

By design, this function F is left Γ-invariant. Eevaluating

$$\int_{\Gamma\backslash G} F(g)\,dg$$

in two different ways will determine the volume of $\Gamma\backslash G$.

Lemma: Given *coprime* $c, d \in \mathbb{Z}$, there exists $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$.

Proof: The ideal $\mathbb{Z}c + \mathbb{Z}d$ is $\mathbb{Z}$, so there are $a, b \in \mathbb{Z}$ such that $ad + bc = 1$. Then $\begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \Gamma$. ///

Thus, for a fixed positive integer ℓ, the set $\{(c,d) : \gcd(c,d) = \ell\}$ is an *orbit* of Γ in $\mathbb{Z}^2$. Take $(0,1)$ as convenient base point and observe that

$$\mathbb{Z}^2 - \{0\} \;=\; \{\ell \cdot (0,1) \cdot \gamma \;:\; \text{for } \gamma \in \Gamma,\; 0 < \ell \in \mathbb{Z}\}$$

Let

$$N \;=\; \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G\} \qquad\qquad N_{\mathbb{Z}} \;=\; N \cap \Gamma$$

The stabilizer of $(0,1)$ in Γ is $N_{\mathbb{Z}}$, and there is a bijection

$$\mathbb{Z}^2 - \{0\} \longleftrightarrow \{\ell > 0\} \times N_{\mathbb{Z}} \backslash \Gamma$$

by

$$\ell(0,1)\gamma \;\leftarrow\; \ell \times N_{\mathbb{Z}}\gamma$$

Then

$$\int_{\Gamma\backslash G} F(g)\, dg = \int_{\Gamma\backslash G} f(0)\, dg + \int_{\Gamma\backslash G} \sum_{x\neq 0} f(xg)\, dg$$

$$= \int_{\Gamma\backslash G} f(0)\, dg + \sum_{\ell>0} \int_{N_{\mathbb{Z}}\backslash G} f(\ell\cdot(0,1)g)\, dg$$

Writing the integral on G as an iterated integral on P^+ and K, $\int_{\Gamma\backslash G} F$ is

$$\int_{\Gamma\backslash G} f(0)\,dg + \sum_{\ell>0} \int_{N_{\mathbb{Z}}\backslash P} \int_K f(\ell\cdot(0,1)pk)\,dg$$

With f *rotation invariant*, so $f(\ell(0,1)pk) = f(\ell(0,1)p)$, the integral is

$$\int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_{\ell>0} \int_{N_{\mathbb{Z}}\backslash P} f(\ell(0,1)p)\,dp$$

since the total measure of K is 2π. Expressing the Haar measure on P^+ in coordinates as above, the integral is

$$\int_{\Gamma\backslash G} f(0)\,dg + 2\pi \sum_\ell \int_0^\infty \int_{\mathbb{Z}\backslash\mathbb{R}} f(\ell(0,1)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\,dx\,\frac{dt}{t^2}$$

Note that N fixes $(0,1)$, so the integral over $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is $\int_{\mathbb{Z}\backslash\mathbb{R}} 1\,dx = 1$, and the whole integral is

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_{\Gamma\backslash G} f(0)\,dg + 2\pi \sum_\ell \int_M f(\ell(0,1)\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\,\frac{1}{t^2}\,\frac{dt}{t}$$

$$= \int_{\Gamma\backslash G} f(0)\,dg + 2\pi \sum_\ell \int_0^\infty f(\ell(0,t^{-1}))\,\frac{1}{t^2}\,\frac{dt}{t}$$

$$= f(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + 2\pi \sum_\ell \int_0^\infty f(0,\ell t)\,t^2\,\frac{dt}{t}$$

replacing t by t^{-1}.

Replacing t by t/ℓ gives

$$\int_{\Gamma\backslash G} F(g)\,dg = f(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + 2\pi \cdot \sum_\ell \ell^{-2} \int_0^\infty f(0,t)\,t^2\,\frac{dt}{t}$$

$$= f(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + 2\pi\,\zeta(2) \cdot \int_0^\infty f(0,t)\,t^2\,\frac{dt}{t}$$

Using the rotation invariance of f,

$$\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t} = \int_0^\infty f(0,t)\,t\,dt = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)\,dx = \frac{1}{2\pi}\,\hat{f}(0)$$

The 2π's cancel, and

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_{\Gamma\backslash G} \sum_{x\in\mathbb{Z}^2} f(xg)\,dg = f(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0)$$

On the other hand, by Poisson summation,

$$\sum_{x\in\mathbb{Z}^2} f(xg) = \frac{1}{|\det g|} \sum_{x\in\mathbb{Z}^2} \hat{f}(x^{\top} g^{-1}) = \sum_{x\in\mathbb{Z}^2} \hat{f}(x^{\top} g^{-1})$$

(since $\det g = 1$). Γ is stable under transpose-inverse, allowing an analogous computation with the roles of f and $\hat{f}$ reversed, obtaining

$$f(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0) = \int_{\Gamma\backslash G} F(g)\,dg$$

$$= \hat{f}(0)\cdot \mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,f(0)$$

from which

$$(f(0) - \hat{f}(0))\cdot \mathrm{vol}\,(\Gamma\backslash G) = (f(0) - \hat{f}(0))\cdot \zeta(2)$$

With $f(0) \neq \hat{f}(0)$, $\mathrm{vol}\,(\Gamma\backslash G) = \zeta(2)$. ///

Next: More about Haar measure...

Change-of-measure and Haar measure on $\mathbb{A}$ and k_v:

Another thing used in the proof of Fujisaki's lemma was that, for *idele* α, the change-of-measure on $\mathbb{A}$ is

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad \text{(for measurable } E \subset \mathbb{A})$$

Naturally, this should be examined...