Harmonic analysis, on $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{A}$, and $\mathbb{A}_k/k$, key ingredients in Iwasawa-Tate.

Need the abelian topological group analogue of *characters* $x \to e^{2\pi i x\xi}$ for $\xi \in \mathbb{R}$, $x \in \mathbb{R}$, and *Fourier transforms*

$$\widehat{f}(\xi) \;=\; \mathscr{F}f(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i x\xi}\, f(x)\, dx$$

and *Fourier inversion*

$$f(x) \;=\; \mathscr{F}^{-1}\widehat{f}(x) \;=\; \int_{\mathbb{R}} e^{2\pi i \xi x}\, \widehat{f}(\xi)\, d\xi$$

for nice functions f on $\mathbb{Q}_p$ and $\mathbb{A}$. Similarly for all completions k_v and adeles $\mathbb{A}_k$ of number fields. And *adelic Poisson summation*

$$\sum_{x\in k} f(x) \;=\; \sum_{x\in k} \widehat{f}(x) \qquad\qquad (\text{for suitable } f \text{ on } \mathbb{A}_k)$$

Recap:

No small subgroups: The circle S^1 *has no small subgroups*: there is a neighborhood U of the identity $1 \in S^1$ such that the only subgroup inside U is $\{1\}$. ///

Unitary duals of abelian topological groups: The unitary dual $G^\vee$ of an abelian topological group G is all continuous group homs $G \to S^1$. For example, $\mathbb{R}^\vee \approx \mathbb{R}$, by $\xi \to (x \to e^{i\xi x})$.

Theorem: $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$ and $\mathbb{A}^\vee \approx \mathbb{A}$. ///

Remark: $\mathbb{Z}_p$ as *limit* and $\mathbb{Q}_p$ as *colimit*, and $\mathfrak{o}_v$ and k_v similarly in general, are admirably adapted to determine these duals.

Remark: Since our model of the topological group $\mathbb{Q}_p$ implicitly specifies more information, namely, the subroup $\mathbb{Z}_p$, the isomorphisms *are* canonical. If we only gave the *isomorphism class* without specifying a compact-open subgroup, the isomorphism would *not* be canonical, just as the dual vector space to a finite-dimensional vector space V has the same dimension as V, but is not *canonically* isomorphic to V.

Corollary: Given *non-trivial* $\psi \in \mathbb{Q}_p^\vee$, every other element of $\mathbb{Q}_p^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{Q}_p$. Similarly, given *non-trivial* $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$.

Remark: This sort of result is already familiar from the analogue for $\mathbb{R}$, that $x \to e^{i\xi x}$ for $\xi \in \mathbb{R}$ are all the unitary characters of $\mathbb{R}$.

Proof: On one hand, it is clear that, for given continuous group hom $\psi : \mathbb{Q}_p \to S^1$ and $\xi \in \mathbb{Q}_p$, the character $x \to \psi(\xi \cdot x)$ is another. Thus, the dual is a $\mathbb{Q}_p$-vectorspace.

On the other hand, in the proof that $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$, we *chose* the *pairing* $\mathbb{Q}_p \times \mathbb{Q}_p^\vee \to \mathbb{C}^\times$, which would determine the isomorphism. Indeed, given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, the **standard character** is

$$\psi_1(x) \;=\; e^{-2\pi i x'} \qquad \text{(sign choice for later purposes)}$$

The character ψ_1 is trivial on $\mathbb{Z}_p$. For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) \;=\; \psi_1(\xi \cdot x) \qquad\qquad (\text{for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension k_v of $\mathbb{Q}_p$ (whether or not we know how k_v arises as a completion of a number field), the **standard character** is described as

$$\psi_\xi(x) \;=\; \psi_1\big(\mathrm{tr}^{k_v}_{\mathbb{Q}_p}(\xi \cdot x)\big) \qquad\qquad (\text{for } x, \xi \in k_v)$$

Since $\mathrm{tr}(\mathfrak{o}_v) \subset \mathbb{Z}_p$, certainly $\ker \psi_\xi \supset \xi^{-1}\mathfrak{o}_v$.

Occasionally, the kernel of ψ_ξ can be slightly larger than $\xi^{-1}\mathfrak{o}_v$.

Compact-discrete duality

For abelian topological groups G, pointwise multiplication makes $\widehat{G}$ an abelian group. A reasonable topology on $\widehat{G}$ is the *compact-open* topology, with a sub-basis

$$U = U_{C,E} = \{f \in \widehat{G} : f(C) \subset E\}$$

for compact $C \subset G$, open $E \subset S^1$.

Remark: The reasonable-ness of this topology is utilitarian. For a compact topological space X, $C^o(X)$ with the *sup-norm* is a *Banach space*. The compact-open topology is the analogue for $C^o(X, Y)$ when X, Y are topological groups. More aspects of this will become clear later.

Granting for now that the compact-open topology makes $\widehat{G}$ an abelian (locally-compact, Hausdorf) topological group,

Theorem: The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

Proof: Let G be compact. Let E be a small-enough open in S^1 so that E contains no non-trivial subgroups of G. Using the compactness of G itself, let $U \subset \widehat{G}$ be the open

$$U = \{f \in \widehat{G} : f(G) \subset E\}$$

Since E is *small*, $f(G) = \{1\}$. That is, f is the trivial homomorphism. This proves discreteness of $\widehat{G}$ for compact G.

For G discrete, *every* group homomorphism to S^1 is continuous. The space of *all* functions $G \to S^1$ is the cartesian product of copies of S^1 indexed by G. By Tychonoff's theorem, this product is *compact*. For *discrete* X, the compact-open topology on the space $C^o(X,Y)$ of continuous functions from $X \to Y$ *is* the product topology on copies of Y indexed by X.

The set of functions f satisfying the group homomorphism condition

$$f(gh) = f(g) \cdot f(h) \qquad \text{(for } g, h \in G\text{)}$$

is *closed*, since the group multiplication $f(g) \times f(h) \to f(g) \cdot f(h)$ in S^1 is continuous. Since the product is also *Hausdorff*, $\widehat{G}$ is also compact. ///

Theorem: $(\mathbb{A}/k)\widehat{\ } \approx k$. In particular, given any non-trivial character ψ on $\mathbb{A}/k$, *all* characters on $\mathbb{A}/k$ are of the form $x \to \psi(\alpha \cdot x)$ for some $\alpha \in k$.

Proof: For a (discretely topologized) number field k with adeles $\mathbb{A}$, $\mathbb{A}/k$ is *compact*, and $\mathbb{A}$ is *self-dual.*

Because $\mathbb{A}/k$ is compact, $(\mathbb{A}/k)\widehat{\ }$ is *discrete.* Since multiplication by elements of k respects cosets $x + k$ in $\mathbb{A}/k$, the unitary dual has a k-vectorspace structure given by

$$(\alpha \cdot \psi)(x) \;=\; \psi(\alpha \cdot x) \qquad\qquad (\text{for } \alpha \in k,\, x \in \mathbb{A}/k)$$

There is no topological issue in this k-vectorspace structure, because $(\mathbb{A}/k)\widehat{\ }$ is discrete. The quotient map $\mathbb{A} \to \mathbb{A}/k$ gives a natural *injection* $(\mathbb{A}/k)\widehat{\ } \to \widehat{\mathbb{A}}$.

Given non-trivial $\psi \in (\mathbb{A}/k)\widehat{}$, the k-vectorspace $k\cdot\psi$ inside $(\mathbb{A}/k)\widehat{}$ injects to a copy of $k \cdot \psi$ inside $\widehat{\mathbb{A}} \approx \mathbb{A}$. *Assuming* for a moment that the image in $\mathbb{A}$ is essentially the same as the diagonal copy of k, $(\mathbb{A}/k)\widehat{}/k$ injects to $\mathbb{A}/k$. The topology of $(\mathbb{A}/k)\widehat{}$ is discrete, and the quotient $(\mathbb{A}/k)\widehat{}/k$ is still discrete. These maps are continuous group homs, so the image of $(\mathbb{A}/k)\widehat{}/k$ in $\mathbb{A}/k$ is a discrete subgroup of a compact group, so is *finite*. Since $(\mathbb{A}/k)\widehat{}$ is a k-vectorspace, $(\mathbb{A}/k)\widehat{}/k$ is a singleton. Thus, $(\mathbb{A}/k)\widehat{} \approx k$, if the image of $k \cdot \psi$ in $\mathbb{A} \approx \widehat{\mathbb{A}}$ is the usual diagonal copy.

To see how $k \cdot \psi$ is imbedded in $\mathbb{A} \approx \widehat{\mathbb{A}}$, fix non-trivial ψ on $\mathbb{A}/k$, and let ψ be the corresponding character on $\mathbb{A}$. The self-duality of $\mathbb{A}$ is that the action of $\mathbb{A}$ on $\widehat{\mathbb{A}}$ by $(x \cdot \psi)(y) = \psi(xy)$ gives an *isomorphism*. The subgroup $x \cdot \psi$ with $x \in k$ is certainly the usual diagonal copy. ///

Next: Fourier transforms, Fourier inversion, Schwartz spaces of functions, adelic Poisson summation.