Harmonic analysis, on $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{A}$, and $\mathbb{A}_k/k$, key ingredients in Iwasawa-Tate:

Characters and *Fourier transforms* on $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}$, and for completions and adeles of number fields.

Fourier inversion expresses nice functions as *superpositions* (integrals) of characters.

Schwartz spaces $\mathscr{S}(\mathbb{R})$, $\mathscr{S}(\mathbb{Q}_p)$, $\mathscr{S}(\mathbb{A})$ of very-nice functions on $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}$, and k_v, $\mathbb{A}_k$ for number fields k.

Adelic Poisson summation

$$\sum_{x\in k} f(x) \;=\; \sum_{x\in k} \widehat{f}(x) \qquad\qquad (\text{for } f \in \mathscr{S}\mathbb{A}_k)$$

Recap:

Corollary: Given *non-trivial* $\psi \in k_v^\vee$, every other element of $k_v^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in k_v$. Similarly, given *non-trivial* $\psi \in \mathbb{A}_k^\vee$, every other element of $\mathbb{A}_k^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}_k$.

The **standard character** ψ_1 on $\mathbb{Q}_p$ is as follows: given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, and

$$\psi_1(x) \;=\; e^{-2\pi i x'} \qquad\qquad \text{(sign choice for later)}$$

For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) \;=\; \psi_1(\xi \cdot x) \qquad\qquad (\text{for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension k_v of $\mathbb{Q}_p$, the **standard character** is

$$\psi_\xi(x) \;=\; \psi_1\big(\mathrm{tr}^{k_v}_{\mathbb{Q}_p}(\xi \cdot x)\big) \qquad\qquad (\text{for } x, \xi \in k_v)$$

Proposition (later): the compact-open topology makes $\widehat{G}$ an abelian (locally-compact, Hausdorf) topological group.

Theorem: Unitary dual of a *compact* abelian group is *discrete*, unitary dual of a *discrete* abelian group is *compact*. ///

Fourier transforms, Fourier inversion, Schwartz spaces of functions, adelic Poisson summation Unsurprisingly, the Fourier transform on k_v is

$$\mathscr{F} f(\xi) \;=\; \widehat{f}(\xi) \;=\; \int_{k_v} \overline{\psi}_\xi(x)\, f(x)\, dx$$

where, given the characters ψ_ξ (for example, the standard ones), the Haar measures are normalized so that *Fourier inversion* holds exactly:

$$f(x) \;=\; \int_{k_v} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \qquad\qquad \text{(for nice functions } f\text{)}$$

Why does Fourier inversion hold at all?

The usual space $\mathscr{S}(\mathbb{R})$ of *Schwartz functions* on $\mathbb{R}$ consists of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, decaying more rapidly at $\pm\infty$ than every $1/|x|^N$. Its topology is given by semi-norms

$$\nu_{k,N}(f) \;=\; \sup_{0\le i\le k}\,\sup_{x\in\mathbb{R}} \Big((1+|x|)^N \cdot \big|f^{(i)}(x)\big|\Big)$$

for $0 \le k \in \mathbb{Z}$ and $0 \le N \in \mathbb{Z}$.

There are countably-many associated (pseudo-) metrics $d_{k,N}(f,g) = \nu_{k,N}(f-g)$, so $\mathscr{S}(\mathbb{R})$ is naturally *metrizable.*

The usual two-or-three-epsilon arguments show that $\mathscr{S}(\mathbb{R})$ is *complete* metrizable.

Theorem: $\mathscr{F}$ is a topological isomorphism $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

Before describing the Schwartz space $\mathscr{S}(\mathbb{Q}_p)$ and proving Fourier inversion, sample computations of Fourier transforms are useful.

We need a simply-described function on $\mathbb{Q}_p$ which is its own Fourier transform, to play a role analogous to that of the Gaussian in the archimedean case.

Claim: With Fourier transform on $\mathbb{Q}_p$ defined via the standard character $\psi_1(x) = e^{-2\pi i x'}$ (where $x' \in p^{-\infty}\mathbb{Z}_p$ and $x - x' \in \mathbb{Z}_p$), the characteristic function of $\mathbb{Z}_p$ is its own Fourier transform.

Proof: Let f be the characteristic function of $\mathbb{Z}_p$. Then

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Q}_p} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{Z}_p} \overline{\psi}_1(\xi \cdot x)\, dx \;=\; \int_{\mathbb{Z}_p} \psi_1(-\xi \cdot x)\, dx$$

Recall a form of the *cancellation lemma:* (a tiny case of *Schur orthogonality...*)

Lemma: Let $\psi : K \to \mathbb{C}^\times$ be a continuous group homomorphism on a compact group K. Then

$$\int_K \psi(x)\,dx \;=\; \begin{cases} \text{meas}\,(K) & (\text{for } \psi = 1) \\ \\ 0 & (\text{for } \psi \neq 1) \end{cases}$$

Proof of Lemma: Yes, of course, the measure is a Haar measure on K. Since K is *compact*, it is *unimodular*.

For ψ trivial, of course the integral is the total measure of K.

For ψ non-trivial, there is $y \in K$ such that $\psi(y) \neq 1$. Using the invariance of the measure, change variables by replacing x by xy:

$$\int_K \psi(x)\,dx = \int_K \psi(xy)\,d(xy) = \int_K \psi(x)\,\psi(y)\,dx = \psi(y)\int_K \psi(x)\,dx$$

Since $\psi(y) \neq 1$, the integral is 0. ///

Apply the lemma to the integrals computing the Fourier transform of the characteristic function f of $\mathbb{Z}_p$. Giving the compact group $\mathbb{Z}_p$ measure 1,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Z}_p} \psi_1(-\xi \cdot x)\, dx \;=\; \begin{cases} 1 & (\psi_1(-\xi x) = 1 \text{ for } x \in \mathbb{Z}_p) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

On one hand, for $\xi \in \mathbb{Z}_p$, certainly $\psi_1(\xi x) = 1$ for $x \in \mathbb{Z}_p$. On the other hand, for $\xi \notin \mathbb{Z}_p$, there is $x \in \mathbb{Z}_p$ such that, for example, $\xi \cdot x = 1/p$. Then

$$\psi_1(-\xi \cdot x) \;=\; \psi_1(\tfrac{-1}{p}) \;=\; e^{+2\pi i \cdot \frac{1}{p}} \;\neq\; 1$$

Thus, ψ_ξ is not trivial on $\mathbb{Z}_p$, so the integral is 0. Thus, the characteristic function of $\mathbb{Z}_p$ is its own Fourier transform. ///

Claim: With standard Fourier transform on $\mathbb{Q}_p$, the Fourier transform of the characteristic function of $p^k\mathbb{Z}_p$ is p^{-k} times the characteristic function of $p^{-k}\mathbb{Z}_p$.

Proof: Let f be the characteristic function of $p^k\mathbb{Z}_p$, so

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Q}_p} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{p^k\mathbb{Z}_p} \overline{\psi}_1(\xi\cdot x)\, dx$$

$$=\; |p^k|_p \cdot \int_{\mathbb{Z}_p} \psi_1(-\xi\cdot x/p^k)\, dx \;=\; p^{-k}\cdot \int_{\mathbb{Z}_p} \psi_1(-\xi\cdot x/p^k)\, dx$$

This reduces to the previous computation: by *cancellation*, for $\xi/p^k \notin \mathbb{Z}_p$ the character $x \to \psi_1(-\xi x/p^k)$ is non-trivial, so the integral is 0. Otherwise, the integral is 1. ///

Claim: With standard Fourier transform on $\mathbb{Q}_p$, the Fourier transform of the characteristic function of $\mathbb{Z}_p + y$ is ψ_y times the characteristic function of $\mathbb{Z}_p$.

Proof: Let f be the characteristic function of $\mathbb{Z}_p + y$, so

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Q}_p} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{Z}_p + y} \overline{\psi}_1(\xi \cdot x)\, dx$$

$$= \int_{\mathbb{Z}_p} \psi_1(-\xi \cdot (x+y))\, dx \;=\; \psi_1(-\xi \cdot y)\, dx \int_{\mathbb{Z}_p} \psi_1(-\xi \cdot x)\, dx$$

$$= \psi_1(-\xi \cdot y) \cdot f(\xi)$$

by the previous computation. ///

Combining the two computations above,

$$\mathscr{F}\Big(\text{char fcn } p^k\mathbb{Z}_p + y\Big) \;=\; \psi_y \cdot p^{-k} \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)$$

Conveniently, products $\psi_y \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)$ are in the same class of functions, since ψ_y has a kernel which is an open (and compact) neighborhood of 0, so we *this class of functions is mapped to itself under Fourier transform.*

Recall the earlier lemma proving that these *special* simple functions consisting of finite linear combinations of characteristic functions of sets $p^k\mathbb{Z}_p + y$ are *dense* in $C_c^o(\mathbb{Q}_p)$.

The space of **Schwartz functions** $\mathscr{S}(\mathbb{Q}_p)$ on $\mathbb{Q}_p$ is the vector space of these special simple functions, that is, finite linear combinations of characteristic functions of sets $p^k\mathbb{Z}_p + y$.

Yes, unlike the archimedean case, p-adic Schwartz functions are *compactly supported.*

p-adic Fourier inversion is the claim that

$$f(x) \;=\; \int_{\mathbb{Q}_p} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{Q}_p))$$

Proof: We have essentially proven this in the computations above, if we keep track, as follows. Let f^o be the characteristic function of $\mathbb{Z}_p$. We computed $\widehat{f^o} = f$. Let δ_t be the dilation operator $\delta_t f(x) = f(t \cdot x)$ for $t \in \mathbb{Q}_p^\times$. We computed, by changing variables in the integral defining the Fourier transform, that

$$\mathscr{F}(\delta_t f) \;=\; \frac{1}{|t|_p} \cdot \delta_{1/t}(\mathscr{F} f)$$

Let τ_y be the translation operator $\tau_y f(x) = f(x+y)$. By changing variables,

$$\mathscr{F}(\tau_y f) \;=\; \psi_y \cdot (\mathscr{F} f)$$

It is convenient to also compute that

$$\mathscr{F}(\psi_y \cdot f)(\xi) \;=\; \int_{\mathbb{Q}_p} \overline{\psi}_\xi(x) \cdot \psi_y(x)\, f(x)\, dx$$

$$=\; \int_{\mathbb{Q}_p} \overline{\psi}_{\xi - y}(x)\, f(x)\, dx \;=\; \widehat{f}(\xi - y) \;=\; \tau_{-y}(\mathscr{F} f)$$

Let $\mathscr{F}^*$ be the integral for Fourier inversion, namely,

$$\mathscr{F}^* f(x) \;=\; \int_{\mathbb{Q}_p} \psi_\xi(x)\; f(\xi)\; d\xi$$

Similar computations give

$$\mathscr{F}^*(\delta_t f) \;=\; \frac{1}{|t|_p}\delta_{1/t}(\mathscr{F}^* f) \qquad\qquad \mathscr{F}^*(\tau_y f) \;=\; \psi_{-y}(\mathscr{F}^* f)$$

and

$$\mathscr{F}^*(\psi_y f) \;=\; \tau_y(\mathscr{F}^* f)$$

Since every element of $\mathscr{S}(\mathbb{Q}_p)$ is a linear combination of images of f^o under dilation and translation, it suffices to give a sort of inductive proof of Fourier inversion:

$$\mathscr{F}^*\mathscr{F}(\tau_y f) \;=\; = \;\mathscr{F}^*\psi_y\mathscr{F}f \;=\; \tau_y\mathscr{F}^*\mathscr{F}f$$

$$\mathscr{F}^*\mathscr{F}(\delta_t f) \;=\; \mathscr{F}^*\frac{1}{|t|_p}\delta_{1/t}\mathscr{F}f \;=\; \frac{1}{|t|_p}\frac{1}{|1/t|_p}\delta_t\mathscr{F}^*\mathscr{F}f \;=\; \delta_t\mathscr{F}^*\mathscr{F}f$$

Similarly for multiplication by ψ_y. Since $\mathscr{F}^*\mathscr{F}f^o = \mathscr{F}^*f^o = f^o$, we have Fourier inversion on $\mathscr{S}(\mathbb{Q}_p)$. ///

Remark: p-adic Fourier inversion is much easier than on $\mathbb{R}$.

The space $\mathscr{S}(\mathbb{A})$ of Schwartz functions on the adeles is finite linear combinations of *monomial* functions

$$\Big(\bigotimes_{v\le\infty} f_v\Big)(\{x_v\}) \;=\; \prod_v f_v(x_v)$$

with $f_v \in \mathscr{S}(\mathbb{Q}_v)$, and where *for all but finitely-many* v the local function f_v is the characteristic function of $\mathbb{Z}_v$.

Fourier transform on $\mathscr{S}(\mathbb{A})$ is the product of all the local Fourier transforms, and Fourier inversion follows for $\mathscr{S}(\mathbb{A})$ because it holds for each $\mathscr{S}(\mathbb{Q}_v)$.

Remark: We do not directly need it, but one might reflect on what the natural topology is on $\mathscr{S}(\mathbb{Q}_p)$, especially to have it be *complete*.

Fourier series on $\mathbb{A}/k$: For a unimodular topological group G, let $L^2(G)$ be the *completion* of $C_c^o(G)$ with respect to the usual L^2-norm given by

$$|f|^2 \;=\; \int_G |f(g)|^2\,dg \qquad\qquad (\text{for } f \in C_c^o(G))$$

Remark: The measurable-function $L^2(G)$ contains this completion, and is provably equal, but we only need integrals of continuous compactly-supported functions.

Theorem: For a compact abelian group G, with total measure 1, the continuous group homomorphisms (*characters*) $\psi : G \to \mathbb{C}^\times$ form an orthonormal *Hilbert-space basis* for $L^2(G)$. That is,

$$L^2(G) \;=\; \text{completion of} \bigoplus_{\psi\in G^\vee} \mathbb{C}\cdot\psi$$

Remark: A *Hilbert-space basis* of a Hilbert space V is not a *vector-space* basis for V, but only for a dense subspace.