Harmonic analysis, on $\mathbb{A}_k/k$, adelic Poisson summation.

Theorem: Fourier transform is a topological isomorphism $\mathscr{S}(k_v) \to \mathscr{S}(k_v)$ and $\mathscr{S}(\mathbb{A}_k) \to \mathscr{S}(\mathbb{A}_k)$.

Plancherel: Fourier transform is an L^2-isometry on Schwartz functions.

(big) Theorem: For a *compact abelian* group G, with total measure 1, the continuous group homomorphisms (*characters*) $\psi : G \to \mathbb{C}^\times$ form an orthonormal *Hilbert-space basis* for $L^2(G)$. That is,

$$L^2(G) \;=\; \text{completion of} \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

and

$$f \;=\; \sum_{\psi \in G^\vee} \langle f, \psi \rangle \cdot \psi \qquad\qquad (\text{for } f \in L^2(G), \text{ convergence in } L^2(G))$$

Proof of big Theorem: so far: orthonormality is immediate from *cancellation lemma.*

Completeness requires existence of sufficiently many *eigenvectors...* for the translation action of G on complex-valued functions

For G *finite*, by finite-dimensional spectral theory for *unitary* operators, [*we saw*]

$$L^2(G) \;=\; \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi \qquad\qquad (G \text{ finite abelian})$$

We did *not* use the structure theorem for finite abelian groups.

The best operators on infinite-dimensional Hilbert spaces are self-adjoint *compact* operators.

Compactness is that the image TB of the unit ball B has *compact closure.* Thus, the image $\{Tv_i\}$ of a *bounded* sequence $\{v_i\}$ has a *convergent subsequence* $\{Tv_{i_k}\}$.

One of the most useful theorems in the universe:

Theorem: Let R be a set of compact, self-adjoint, mutually commuting operators on a Hilbert space V. Suppose the action is *non-degenerate* in the sense that for $0 \neq v \in V$ there is $T \in R$ with $Tv \neq 0$. Then V has an *orthonormal* Hilbert-space basis of *simultaneous eigenvectors* for R. The joint eigenspaces are finite-dimensional.

[Simple proof below]

Mostly, compact operators come from *integral operators* attached to η in $C_c^o(G)$, acting on $L^2(G)$ by

$$(\eta \cdot f)(x) \;=\; \int_G \eta(g)\, f(xg)\, dg$$

A change of variables gives

$$(\alpha \cdot f)(x) \;=\; \int_G \alpha(y)\, f(xy)\, dy \;=\; \int_G \alpha(x^{-1}y)\; f(y)\, dy$$

Write $K(x,y) = \alpha(x^{-1}y)$. It defines a linear operator
$T : L^2(G) \to L^2(G)$ defined by

$$Tf(x) \;=\; (\alpha \cdot f)(x) \;=\; \int_G K(x,y)\, f(y)\, dy \qquad\qquad (\text{for } f \in L^2(G))$$

Claim: For locally compact Hausdorff topological spaces X, Y with nice measures, for $K(x,y) \in C_c^o(X \times Y)$, the linear operator $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) \;=\; \int_Y K(x,y)\, f(y)\, dy$$

is *compact*. For $X = Y$ and $K(y,x) = \overline{K(x,y)}$, T is *self-adjoint.*

Proof of spectral theorem for commuting compact self-adjoint operators: The key point is the spectral theorem for a *single* self-adjoint compact operator $T : V \to V$. We need

Slightly Clever Lemma: The *operator norm* $|T| = \sup_{|v|\leq 1} |Tv|$ of continuous *self-adjoint* operator T on a Hilbert space V is expressible as

$$|T| \;=\; \sup_{|v|\leq 1} |\langle Tv, v\rangle|$$

Key Lemma: A compact self-adjoint operator T has largest eigenvalue $\pm|T|$.

Spectral theorem: for a *single* self-adjoint compact operator T... the non-zero eigenvalues are *real*, have no accumulation point but $\{0\}$, and multiplicities are finite. For $0 \neq \lambda \in \mathbb{C}$ not among the eigenvalues, $T - \lambda$ is *invertible* (as continuous linear operator).

Proof of theorem for single operator: In part, this is similar to the proof for self-adjoint operators on *finite*-dimensional spaces.

If $|T| = 0$, then $T = 0$. Otherwise, the key lemma gives a non-zero eigenvalue. The orthogonal complement of the corresponding eigenvector v is T-stable: for $w \perp v$,

$$\langle v, Tw \rangle = \langle Tv, w \rangle = \lambda \langle v, w \rangle = 0 \qquad (\text{for } Tv = \lambda v \text{ and } \langle v, w \rangle = 0)$$

The restriction of T to that orthogonal complement is still compact (!), so unless that restriction is 0, T has a non-zero eigenvalue there, too. Continue...

For $\lambda \neq 0$, the λ-eigenspace being infinite-dimensional would contradict the compactness of T: the unit ball in an infinite-dimensional inner-product space is not compact, as any infinite orthonormal set is a sequence with no convergent subsequence.

Similarly, for $c > 0$, the set of eigenvalues (counting multiplicities) larger than c being infinite would contradict compactness.

Thus, 0 is the only limit-point of eigenvalues.

Finally, the restriction of T to the orthogonal complement of the sum of all its non-zero eigenspaces is still compact. If its operator norm were positive, there would be a further non-zero eigenvalue, contradiction. Thus, that restriction has 0 norm, so is 0. This proves the spectral theorem for a single self-adjoint compact operator. ///

For the commuting family of operators: as usual, the commutativity ensures that the operators stabilize each others' eigenspaces: for v a λ-eigenvalue for T, for another operator S,

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot Sv$$

The non-degeneracy ensures that the orthogonal complement of all the joint eigenspaces is $\{0\}$. ///

Remark: For proving existence of eigenfunctions, there really is no alternative to self-adjoint compact operators. Meanwhile, compact operators have been understood, in terms appropriate for the time, for at least 120 years.

Claim: *Hilbert-Schmidt* operators $K(x,y) \in C_c^o(X \times Y)$ give compact operators $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) = \int_Y K(x,y)\, f(y)\, dy$$

Remark: The class of *Hilbert-Schmidt* operators often is taken to include not only operators with kernels in $C^o(X \times Y)$, but also kernels in $L^2(X \times Y)$. In practice, usually kernels are in L^2 because they are in C^o_c.

Remark: In fact, the *Schwartz Kernel Theorem* shows that continuous operators from $C^\infty_c(\mathbb{R}^n)$ to *distributions* on $\mathbb{R}^n$ are given by kernels $K(x, y)$, themselves distributions on $\mathbb{R}^n \times \mathbb{R}^n$. *Pseudo-differential* operators, *singular integral operators*, and *Fourier integral operators* are important, non-trivial examples.

Proof: We show that T is an operator-norm limit of *finite-rank* operators, that is, operators with finite-dimensional images.

Fix $\varepsilon > 0$, find a *finite* collection of functions f_i, F_i such that

$$\sup_{x,y} \Big| K(x,y) - \sum_i f_i \otimes F_i \Big| \;<\; \varepsilon$$

For each (x,y) in the support of K, let $U_x \times V_y$ be a neighborhood of (x,y) such that $|K(x,y) - K(x',y')| < \varepsilon$ for $x' \in U_x$ and $y' \in V_y$, where U_x and V_y are neighborhoods of x, y.

By compactness of the support of $K(x,y)$, there are finitely-many x_j, y_j such that $U_j \times V_j$ (abbreviating $U_{x_j} \times V_{y_j}$) cover the support of $K(x,y)$. Let

$$\varphi_j \;=\; \text{char fcn } U_j \qquad \text{and} \qquad \Phi_j \;=\; K(x_j, y_j) \cdot (\text{char fcn } U_j)$$

The sets $U_j \times V_j$ *overlap*, so $K \neq \sum_j \varphi_j \otimes \Phi_j$, necessitating minor adjustments.

One way to compensate for the overlaps is by subtracting two-fold overlaps, adding back three-fold overlaps, subtracting four-fold, and so on: let ...

$$Q = \sum_i \varphi_i \otimes \Phi_i - \sum_{i_1<i_2} \min(\varphi_{i_1}, \varphi_{i_2}) \otimes \min(\Phi_{i_1}, \Phi_{i_2})$$
$$+ \sum_{i_1<i_2<i_3} \min(\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_3}) \otimes \min(\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}) - \ldots$$

Because the subcover is finite, Q is a finite linear combination $Q = \sum_j f_j \otimes F_j$. By construction, $\sup_{x,y} \big|K(x,y) - Q(x,y\big| < \varepsilon$. The operator

$$f \longrightarrow \int_G Q(x,y)\, f(y)\, dy$$

is finite-rank, because the image is in the span of the finitely-many f_i appearing in the definition of $Q(x,y)$.

Let χ be the characteristic function of the closure $\overline{U}$ of a compact-closure open U containing the support of K. For every $\varepsilon > 0$, the opens U_x and U_y can be chosen inside U. Then

$$\Big| \int_G Q(x,y)\, f(y)\, dy \; - \; \int_G K(x,y)\, f(y)\, dy \Big|$$

$$\le \; \int_G |Q(x,y) - K(x,y)| \cdot |f(y)|\, dy$$

$$< \; \varepsilon \int_G |\chi(x,y)| \cdot |f(y)|\, dy \; \le \; \varepsilon \cdot |\chi|_{L^2} \cdot |f|_{L^2}$$

Thus, the operator norm of the difference can be made arbitrarily small, proving that the operator T given by $K(x,y) \in C_c^o(X \times Y)$ is an operator-norm limit of finite-rank operators. ///

Prove operator-norm limits of finite-rank operators are compact:

Remark: on Hilbert spaces, the converse is true, that compact operators are operator-norm limits of finite-rank ones. On Banach spaces, the converse is false, by counter-examples due to P. Enflo.

Let $T = \lim_i T_i$, where $T_i : X \to Y$ is finite-rank $X \to Y$. Let B be the unit ball in X. We show that TB has compact closure by showing that it is *totally bounded*, that is, for every $\varepsilon > 0$ it can be covered by finitely-many ε-balls.

Given $\varepsilon > 0$, let i be large-enough so that $|T - T_i| < \varepsilon$. Since T_i is finite-rank, $T_i B$ is covered by finitely-many ε-balls $B_1, \ldots, B_n$ in Y with respective centers $y_1, \ldots, y_n$. For $x \in B$, with $T_i x \in B_j$,

$$|Tx - y_j| \;\le\; |Tx - T_i x| + |T_i x - y_j| \;<\; \varepsilon + \varepsilon$$

Thus, TB is covered by a finite number of 2ε-balls. This holds for every $\varepsilon > 0$, so TB is *totally bounded.* ///

Recall the proof that *total boundedness* of a set E in a complete metric space implies compact closure:

Since metric spaces have countable local bases, it suffices to show *sequential* compactness. That is, a sequence $\{v_i\}$ in E, exhibit a convergent subsequence.

Cover E by finitely-many 2^{-1}-balls, choose one, call it B_1, with infinitely-many v_i in $E \cap B_1$, and let w_1 be one of those infinitely-many v_i.

Next, cover E by finitely-many 2^{-2}-balls. Certainly $E \cap B_1$ is covered by these, and $E \cap B_1 \cap B_2$ contains infinitely-many v_i for at least one of these, call it B_2. Let $w_2 \in E \cap B_1 \cap B_2$ be one of these v_i, other than w_1.

Inductively, find an infinite subsequence w_n of distinct points, with $w_n \in E \cap B_1 \cap \ldots \cap B_n$, where B_n is of radius 2^{-n}. The sequence w_i is Cauchy. ///