Iwasawa-Tate on ζ-functions and L-functions

1. Simplest case: Riemann's zeta
2. Dirichlet L-functions
3. Dedekind zetas of number fields
4. General case: Hecke L-functions

Proof of *analytic continuation* and *functional equation* of Riemann's zeta, in modern form.

Repeated for Dedekind zeta functions of number fields, noting complications.

Repeated for Hecke's (größencharakter) L-functions, noting complications.

Some issues are postponed: adelic Poisson summation, evaluation of local integrals, ...

A virtue of the modern (Tate-Iwasawa) viewpoint is that concern about *units* and *class numbers* evaporates completely, and all number fields are treated in a fashion scarcely different from Riemann's treatment of zeta.

Simplest case: Riemann's zeta

The modern argument is completely parallel to Riemann's.

Let $d^\times x$ be a Haar measure on $\mathbb{J}$. Define **global zeta integrals**

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \qquad\qquad (f \in \mathscr{S}(\mathbb{A}),\ s\in\mathbb{C},\ \mathrm{Re}\, s > 1)$$

We will see that, for suitable choice of f, the zeta integral is the zeta function with its gamma factor. We prove that *every* such global zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s,f) \;=\; Z(1-s,\widehat{f}) \qquad\qquad (\text{for arbitrary } f\in\mathscr{S}(\mathbb{A}))$$

Part of the point is that meromorphic continuation and functional equation of $Z(s,f)$ follow for *all* f, *without* worrying about best choice of Schwartz function f.

Euler products and local zeta integrals

Let $d_v^\times x$ be a Haar measure on $\mathbb{Q}_v^\times$ with $d^\times x = \prod_v d_v^\times x$. For *monomial* Schwartz functions $f = \bigotimes f_v$, for $\mathrm{Re}\, s > 1$, the zeta integral is

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \prod_v \int_{\mathbb{Q}_v^\times} |x|_v^s\; f_v(x)\; d_v^\times x$$

an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,f_v) \;=\; \int_{\mathbb{Q}_v^\times} |x|_v^s\; f_v(x)\; d_v^\times x$$

and

$$Z(s,f) \;=\; \prod_v Z_v(s,f_v) \qquad\qquad (\text{for Re } s>1, \text{ with } f=\textstyle\bigotimes_v f_v)$$

The usual Euler factors appear

We see later that a reasonable choice for f, with $\widehat{f} = f$ produces the standard factors:

$$Z_v(s, f_v) = \begin{cases} \dfrac{1}{1 - \dfrac{1}{p^s}} & \text{(for finite } v \sim p\text{)} \\ \\ \pi^{-\frac{s}{2}} \Gamma\left(\dfrac{s}{2}\right) & \text{(for } v = \infty\text{)} \end{cases}$$

That is, for reasonable choices, in this situation,

$$Z(s, f) = \xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

Functional equation of a theta function

The analogue of the *theta function* appearing in Riemann's and Hecke's classical arguments is

$$\theta_f(x) \;=\; \sum_{\alpha\in\mathbb{Q}} f(\alpha x) \qquad\qquad (\text{for } x \in \mathbb{J},\ f \in \mathscr{S}(\mathbb{A}))$$

Adelic Poisson summation will give the functional equation of the theta function. From the obvious change of variables,

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha)\; f(\alpha x)\; d\alpha \;=\; \int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x)$$

The adelic change of measure is the idele norm, and

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x) \;=\; \frac{1}{|x|}\int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d\alpha \;=\; \frac{1}{|x|}\widehat{f}\Big(\frac{\xi}{x}\Big)$$

Then Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in\mathbb{Q}} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in\mathbb{Q}} \widehat{f}\big(\frac{\alpha}{x}\big) \;=\; \frac{1}{|x|}\;\theta_{\widehat{f}}\big(\frac{1}{x}\big)$$

Main argument: analytic continuation and functional equation of global zeta integrals

The analytic continuation and functional equation arise from *winding up*, breaking the integral into two pieces, and applying the functional equation of θ's, as in the classical scenario. Let

$$\mathbb{J}^+ \;=\; \{x \in \mathbb{J} : |x| \geq 1\} \qquad\qquad \mathbb{J}^- \;=\; \{x \in \mathbb{J} : |x| \leq 1\}$$

and $\mathbb{J}^1 \;=\; \{x \in \mathbb{J} : |x| = 1\}$. Let

$$\theta^*_f(x) \;=\; \theta_f(x) \;-\; f(0) \;=\; \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) \qquad (x \in \mathbb{J} \text{ and } f \in \mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula, and break the integral into two pieces:

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha\in\mathbb{Q}^\times} |\alpha x|^s\, f(\alpha x)\, d^\times(\alpha x)$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\, d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\, \theta_f^*(x)\, d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s\, \theta_f^*(x)\, d^\times x \;+\; \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\, \theta_f^*(x)\, d^\times x$$

just like classical

$$\xi(s) \;=\; \int_1^\infty y^{s/2}\, \frac{\theta(iy)-1}{2}\, \frac{dy}{y} \;+\; \int_0^1 y^{s/2}\, \frac{\theta(iy)-1}{2}\, \frac{dy}{y}$$

The integral over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. (Proof!?!)

The functional equation of θ_f will transforms the integral over $\mathbb{J}^-/\mathbb{Q}^\times$ into an integral over $\mathbb{J}^+/\mathbb{Q}^\times$ plus two elementary terms describing the poles.

Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s \, \theta_f^*(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s \, \theta_f^*(1/x)\, d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s} \cdot \Big[|x|\theta_{\widehat{f}}(x) - f(0)\Big]\, d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s} \, \theta_{\widehat{f}}^*(x)\, d^\times x \;+\; \widehat{f}(0) \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\, d^\times x$$

$$- \, f(0) \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s}\, d^\times x$$

The integral of $\theta_{\widehat{f}}^*$ over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The elementary integrals can be evaluated:

$$\int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\, d^\times x \;=\; \big|J^1/\mathbb{Q}^\times\big| \cdot \int_1^\infty x^{1-s}\, \frac{dx}{x} \;=\; \frac{\big|J^1/\mathbb{Q}^\times\big|}{s-1}$$

In this case, the natural measure of $\mathbb{J}^1/\mathbb{Q}^\times$ is 1, so

$$Z(s,f) \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} \Big(|x|^s \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha\in\mathbb{Q}^\times} \widehat{f}(\alpha x)\Big)\, d^\times x$$

$$+\; \frac{\widehat{f}(0)}{s-1} \;-\; \frac{f(0)}{s}$$

The integral is entire, so the latter expression gives the *analytic continuation*. There is visible symmetry under $s \longleftrightarrow 1-s$ and $f \longleftrightarrow \widehat{f}$, so we have the *functional equation*

$$Z(s,f) \;=\; Z(1-s,\widehat{f})$$

Dirichlet L-functions

We adapt the argument to prove *analytic continuation* and *functional equation* for Dirichlet L-functions. One should observe how *few* changes are needed.

Dirichlet characters as idele-class characters For a Dirichlet character χ_d with conductor N. The main adaptation necessary is rewriting χ_d as a character χ on $\mathbb{J}/k^\times$.

Given idele α, by unique factorization in $\mathbb{Z}$, adjust α by $\mathbb{Q}^\times$ to put its local component inside $\mathbb{Z}_v^\times$ at all finite places. Adjust by ± 1 to make the archimedean component *positive*. Thus, an idele-class character is completely determined by its values on

$$U \;=\; \mathbb{R}^+ \cdot \prod_{v<\infty} \mathbb{Z}_v^\times$$

As the diagonal copy of $\mathbb{Q}^\times$ meets U just at $\{1\}$, there is no risk of ill-definedness. Continuity on U implies continuity on $\mathbb{J}$.

At finite places $v \sim p$ *not* dividing N, we declare χ to be trivial on the local units: $\chi(\mathbb{Z}_v^\times) = 1$ for $v \sim p$ not dividing N.

For $v \sim p$ with $N = p^e M$ and $p \not| M$, given $x \in \mathbb{Z}_v^\times$, let $n \in \mathbb{Z}$ such that $n = x \bmod p^e \mathbb{Z}_v$, and $n = 1 \bmod M$, and define $\chi(x) = \chi_d(n)$. Say χ is *unramified* at v when $\chi(\mathbb{Z}_v^\times) = 1$. At finite places v where χ is *non-trivial* on the local units, χ is *ramified*.

Global zeta integrals We consider only idele-class characters χ *trivial* on the copy $\{(t, 1, 1, \ldots, 1) : t > 0\}$ of positive reals inside $\mathbb{J}$. Define **global zeta integrals**

$$Z(s, \chi, f) \;=\; \int_{\mathbb{J}} |x|^s \, \chi(x) \, f(x) \, d^\times x \qquad (f \in \mathscr{S}(\mathbb{A}), \; s \in \mathbb{C}, \; \mathrm{Re}\, s > 1)$$

For suitable f, $Z(s, \chi, f)$ is the Dedekind zeta function with its gamma factor, except for complications at ramified primes. *Every* zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s, \chi, f) \;=\; Z(1 - s, \chi^{-1}, \widehat{f}) \qquad \text{(for arbitrary } f \in \mathscr{S}(\mathbb{A}))$$

Euler products and local zeta integrals

For *monomial* Schwartz functions $f = \bigotimes f_v$, for Re $s > 1$,

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s \,\chi(x)\, f(x)\, d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s \,\chi_v(x)\, f_v(x)\, d_v^\times x$$

with χ_v the restriction of χ to $\mathbb{Q}_v^\times$. That is, $Z(s,f)$ is an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions, in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,\chi_v,f_v) \;=\; \int_{k_v^\times} |x|_v^s \,\chi_v(x)\; f_v(x)\; d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,f) \;=\; \prod_v Z_v(s,\chi_v,f_v) \qquad\qquad (\text{Re } s > 1, \text{ with } f = \textstyle\bigotimes_v f_v)$$

Usual Euler factors, with a complication

We see later that a reasonable choice of f produces the standard factors:

$$Z_v(s,\chi_v,f_v) = \begin{cases} \dfrac{1}{1-\dfrac{\chi(p)}{p^s}} & (v \sim p,\ p \not| N) \\ \pi^{-\frac{s}{2}}\Gamma\left(\dfrac{s}{2}\right) & (\ v \approx \mathbb{R} \text{ and } \chi_d(-1)=1) \\ \pi^{-\frac{s+1}{2}}\Gamma\left(\dfrac{s+1}{2}\right) & (\ v \approx \mathbb{R} \text{ and } \chi_d(-1)=-1) \end{cases}$$

There is a complication at finite $v \sim$ with $p|N$: typically there is no Schwartz function f recovering the factor $N^{-s/2}$ in the known functional equations

$$N^{\frac{s}{2}}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) = \varepsilon(\chi)\,N^{(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1})$$

for χ even, and for χ odd

$$N^{\frac{s}{2}}\pi^{-\frac{(s+1)}{2}}\Gamma(\tfrac{s+1}{2})\,L(s,\chi) = \varepsilon(\chi)N^{\frac{(1-s)}{2}}\pi^{-\frac{(2-s)}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1})$$

Nevertheless, a reasonable choice will produce $Z(s, \chi, f)$ and $Z(s, \chi^{-1}, \widehat{f})$ such that, letting $\Lambda(s, \chi)$ be the L-function with its gamma factor and *with* factor of $N^{s/2}$,

$$\begin{aligned} Z(s, \chi, f) &= N^{-s/2} \cdot \Lambda(s, \chi) \\ Z(1-s, \chi^{-1}, \widehat{f}) &= \varepsilon \cdot N^{-s/2} \cdot \Lambda(1-s, \chi^{-1}) \end{aligned}$$

with $|\varepsilon| = 1$. Thus, from $Z(s, \chi, f) = Z(1-s, \chi^{-1}, f)$ the symmetrical functional equation can be obtained.

Functional equation of a theta function As before, the *theta function* attached to a Schwartz function f is

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) \qquad (\text{for } x \in \mathbb{J},\ f \in \mathscr{S}(\mathbb{A}))$$

and Poisson summation gives the functional equation

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in k} \widehat{f}\big(\frac{\alpha}{x}\big) = \frac{1}{|x|}\, \theta_{\widehat{f}}\big(\frac{1}{x}\big)$$

Main argument: analytic continuation and functional equation of global zeta integrals

Again, analytic continuation and functional equation arise from *winding up*, breaking the integral into two pieces, and applying the functional equation of θ, as in the classical scenario.

For non-trivial χ, the Schwartz function f can be taken so that

$$f(0) = 0 \qquad \text{and} \qquad \widehat{f}(0) = 0$$

relieving us of tracking those values, and giving the simpler presentation

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \qquad\qquad (\text{for } x \in \mathbb{J} \text{ and } f \in \mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula and $\mathbb{Q}^\times$-invariance of χ, and break the integral into two pieces:

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha\in k^\times} |\alpha x|^s\,\chi(\alpha x)\,f(\alpha x)\,d^\times(\alpha x)$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x) \sum_{\alpha\in k^\times} f(\alpha x)\,d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x \;+\; \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

The integral over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/\mathbb{Q}^\times$ into an integral over $\mathbb{J}^+/\mathbb{Q}^\times$. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s\,\chi(1/x)\,\theta_f(1/x)\,d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s}\,\chi^{-1}(x)|x|\theta_{\widehat{f}}(x)\,d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\,\chi^{-1}(x)\,\theta_{\widehat{f}}(x)\,d^\times x$$

The integral of $\theta_{\widehat{f}}$ over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. Thus,

$$Z(s,\chi,f)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} \Big(|x|^s\,\chi(x) \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) + |x|^{1-s}\,\chi^{-1}(x) \sum_{\alpha\in\mathbb{Q}^\times} \widehat{f}(\alpha x)\Big)\,d^\times x$$

The integral is entire, and gives the *analytic continuation*. Further, there is visible symmetry $\chi \leftrightarrow \chi^{-1}$, $s \leftrightarrow 1-s$, $f \leftrightarrow \widehat{f}$, so we have the *functional equation*

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f})$$

Remark: There was no compulsion to track of $|x|^s$ and $\chi(x)$ separately in the above argument. We could rewrite the above to treat an *arbitrary* χ on $\mathbb{J}/\mathbb{Q}^\times$, define

$$Z(\chi, f) \;=\; \int_{\mathbb{J}} \chi(x)\, f(x)\, d^\times x$$

and obtain the slightly cleaner functional equation

$$Z(\chi, f) \;=\; Z(|.|\,\chi^{-1}, \widehat{f})$$

That is, rather than $s \to 1 - s$ and $\chi \to \chi^{-1}$, simply replace χ by $x \to |x| \cdot \chi^{-1}(x)$.
