Iwasawa-Tate on ζ-functions and L-functions

1. Simplest case: Riemann's zeta [done]
2. Dirichlet L-functions
3. Dedekind zetas of number fields
4. General case: Hecke L-functions

Main part of proof of *analytic continuation* and *functional equation* for Dirichlet L-functions, in modern form.

Repeated for Dedekind zetas number fields.

Repeated for Hecke's (größencharakter) L-functions.

Some issues are postponed: adelic Poisson summation, evaluation of local integrals, ...

Again, a virtue of the modern (Tate-Iwasawa) viewpoint is that *units* and *class numbers* disappear.

Dirichlet L-functions

We prove *analytic continuation* and *functional equation* for Dirichlet L-functions. Few changes are needed.

Dirichlet characters as idele-class characters: For a Dirichlet character χ_d with conductor N. The main adaptation necessary is rewriting χ_d as a character χ on $\mathbb{J}/k^\times$.

Given idele α, by unique factorization in $\mathbb{Z}$, adjust α by $\mathbb{Q}^\times$ to put its local component inside $\mathbb{Z}_v^\times$ at all finite places. Adjust by ± 1 to make the archimedean component *positive*. Thus, an idele-class character is completely determined by its values on

$$U \;=\; \mathbb{R}^+ \cdot \prod_{v<\infty} \mathbb{Z}_v^\times$$

As the diagonal copy of $\mathbb{Q}^\times$ meets U just at $\{1\}$, there is no risk of ill-definedness. Continuity on U implies continuity on $\mathbb{J}$.

At finite places $v \sim p$ *not* dividing N, we declare χ to be trivial on the local units: $\chi(\mathbb{Z}_v^\times) = 1$ for $v \sim p$ not dividing N.

For $v \sim p$ with $N = p^e M$ and $p \not| M$, given $x \in \mathbb{Z}_v^\times$, let $n \in \mathbb{Z}$ such that $n = x \bmod p^e\mathbb{Z}_v$, and $n = 1 \bmod M$, and define $\chi(x) = \chi_d(n)$. Say χ is *unramified* at v when $\chi(\mathbb{Z}_v^\times) = 1$. At finite places v where χ is *non-trivial* on the local units, χ is *ramified.*

Global zeta integrals We consider only idele-class characters χ *trivial* on the copy $\{(t, 1, 1, \ldots, 1) : t > 0\}$ of positive reals inside $\mathbb{J}$. Define **global zeta integrals**

$$Z(s, \chi, f) \;=\; \int_{\mathbb{J}} |x|^s \, \chi(x) \, f(x) \, d^\times x \qquad (f \in \mathscr{S}(\mathbb{A}),\ s \in \mathbb{C},\ \mathrm{Re}\, s > 1)$$

For suitable f, $Z(s, \chi, f)$ is the Dirichlet L-function function with its gamma factor, except for complications at ramified primes. *Every* zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s, \chi, f) \;=\; Z(1 - s, \chi^{-1}, \widehat{f}) \qquad \text{(for arbitrary } f \in \mathscr{S}(\mathbb{A}))$$

Euler products and local zeta integrals

For *monomial* Schwartz functions $f = \bigotimes f_v$, for Re $s > 1$,

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s\,\chi_v(x)\,f_v(x)\,d_v^\times x$$

with χ_v the restriction of χ to $\mathbb{Q}_v^\times$. That is, $Z(s,\chi,f)$ is an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions, in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,\chi_v,f_v) \;=\; \int_{k_v^\times} |x|_v^s\,\chi_v(x)\,f_v(x)\,d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,\chi,f) \;=\; \prod_v Z_v(s,\chi_v,f_v) \qquad\qquad (\text{Re } s > 1, \text{ with } f = \textstyle\bigotimes_v f_v)$$

Usual Euler factors, with a complication

We see later that a reasonable choice of f produces the standard factors:

$$Z_v(s, \chi_v, f_v) = \begin{cases} \dfrac{1}{1 - \dfrac{\chi(p)}{p^s}} & (v \sim p,\ p \not| N) \\ \pi^{-\frac{s}{2}} \Gamma\left(\dfrac{s}{2}\right) & (\ v \approx \mathbb{R} \text{ and } \chi_d(-1) = 1) \\ \pi^{-\frac{s+1}{2}} \Gamma\left(\dfrac{s+1}{2}\right) & (\ v \approx \mathbb{R} \text{ and } \chi_d(-1) = -1) \end{cases}$$

There is a complication at finite $v \sim$ with $p|N$: typically there is no Schwartz function f recovering the factor $N^{-s/2}$ in the known functional equations

$$N^{\frac{s}{2}}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) = \varepsilon(\chi)\,N^{(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1})$$

for χ even, and for χ odd

$$N^{\frac{s}{2}}\pi^{-\frac{(s+1)}{2}}\Gamma(\tfrac{s+1}{2})\,L(s,\chi) = \varepsilon(\chi)N^{\frac{(1-s)}{2}}\pi^{-\frac{(2-s)}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1})$$

Nevertheless, a reasonable choice will produce $Z(s,\chi,f)$ and $Z(s,\chi^{-1},\widehat{f})$ such that, letting $\Lambda(s,\chi)$ be the L-function with its gamma factor and *with* factor of $N^{s/2}$,

$$Z(s,\chi,f) = N^{-\frac{s}{2}}\Lambda(s,\chi) \qquad Z(1-s,\chi^{-1},\widehat{f}) = \varepsilon N^{-\frac{s}{2}}\Lambda(1-s,\chi^{-1})$$

with $|\varepsilon| = 1$. Thus, from $Z(s,\chi,f) = Z(1-s,\chi^{-1},f)$ the symmetrical functional equation can be obtained.

Functional equation of a theta function As before, the *theta function* attached to a Schwartz function f is

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \qquad\qquad (\text{for } x\in\mathbb{J},\ f\in\mathscr{S}(\mathbb{A}))$$

Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in k}\widehat{f}\Big(\frac{\alpha}{x}\Big) \;=\; \frac{1}{|x|}\,\theta_{\widehat{f}}\Big(\frac{1}{x}\Big)$$

Main argument: analytic continuation and functional equation of global zeta integrals

Again, analytic continuation and functional equation arise from *winding up*, breaking the integral into two pieces, and applying the functional equation of θ, as in the classical scenario.

For non-trivial χ, the Schwartz function f can be taken so that

$$f(0) = 0 \qquad \text{and} \qquad \widehat{f}(0) = 0$$

relieving us of tracking those values, and giving the simpler presentation

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \qquad \text{(for } x \in \mathbb{J} \text{ and } f \in \mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula and $\mathbb{Q}^\times$-invariance of χ, and break the integral into two pieces:

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha\in\mathbb{Q}^\times} |\alpha x|^s\,\chi(\alpha x)\,f(\alpha x)\,d^\times(\alpha x)$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x) \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\,d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x \;+\; \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

The integral over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/\mathbb{Q}^\times$ into an integral over $\mathbb{J}^+/\mathbb{Q}^\times$. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s \, \chi(x) \, \theta_f(x) \, d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s \, \chi(1/x) \, \theta_f(1/x) \, d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s} \, \chi^{-1}(x) |x| \theta_{\widehat{f}}(x) \, d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s} \, \chi^{-1}(x) \, \theta_{\widehat{f}}(x) \, d^\times x$$

The integral of $\theta_{\widehat{f}}$ over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. Thus,

$$Z(s,\chi,f)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} \Big(|x|^s \, \chi(x) \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \, \chi^{-1}(x) \sum_{\alpha \in \mathbb{Q}^\times} \widehat{f}(\alpha x) \Big) \, d^\times x$$

The integral is entire, and gives the *analytic continuation*. Further, there is visible symmetry $\chi \leftrightarrow \chi^{-1}$, $s \leftrightarrow 1-s$, $f \leftrightarrow \widehat{f}$, so we have the *functional equation*

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f})$$

Remark: There was no compulsion to track of $|x|^s$ and $\chi(x)$ separately in the above argument. We could rewrite the above to treat an *arbitrary* χ on $\mathbb{J}/\mathbb{Q}^\times$, define

$$Z(\chi, f) \;=\; \int_{\mathbb{J}} \chi(x)\, f(x)\, d^\times x$$

and obtain the slightly cleaner functional equation

$$Z(\chi, f) \;=\; Z(|.|\,\chi^{-1}, \widehat{f})$$

That is, rather than $s \to 1 - s$ and $\chi \to \chi^{-1}$, simply replace χ by $x \to |x| \cdot \chi^{-1}(x)$.

Dedekind zetas of number fields

The argument is repeated, proving *analytic continuation* and *functional equation* for Dedekind zetas of number fields.

Global zeta integrals

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \qquad\qquad (f \in \mathscr{S}(\mathbb{A}),\ s \in \mathbb{C},\ \mathrm{Re}\ s > 1)$$

We will see that, for suitable choice of f, the zeta integral is the Dedekind zeta function with its gamma factors. Just below, we prove that *every* such global zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s,f) \;=\; Z(1-s,\widehat{f}) \qquad\qquad (\text{for arbitrary } f \in \mathscr{S}(\mathbb{A}))$$

Euler products and local zeta integrals For *monomial* Schwartz functions $f = \bigotimes f_v$, for Re $s > 1$, the zeta integral factors over primes as a product of local integrals

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s\; f_v(x)\; d_v^\times x$$

Letting

$$Z_v(s,f_v) \;=\; \int_{k_v^\times} |x|_v^s\; f_v(x)\; d_v^\times x$$

and without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,f) \;=\; \prod_v Z_v(s,f_v) \qquad\qquad (\text{ Re } s > 1, \text{ monomial } f = \textstyle\bigotimes_v f_v)$$

the usual Euler factors, with a complication We see later that a reasonable choice of f (and measures $d_v^\times x$) produces the standard factors at *all but finitely-many primes*: with q_v the cardinality of the residue field for non-archimedean v,

$$Z_v(s, f_v) \;=\; \begin{cases} \dfrac{1}{1 - \dfrac{1}{q_v^s}} & \text{(for } k_v \text{ unramified over } \mathbb{Q}_w\text{)} \\ \pi^{-\frac{s}{2}}\Gamma\left(\dfrac{s}{2}\right) & \text{(for } v \approx \mathbb{R}\text{)} \\ (2\pi)^{-s}\Gamma(s) & \text{(for } v \approx \mathbb{C}\text{)} \end{cases}$$

However, there is a complication due to finite v with $k_v/\mathbb{Q}_w$ *ramified.* The Dedekind zeta function of k is

$$\zeta_k(s) \;=\; \prod_{v<\infty} \frac{1}{1 - \dfrac{1}{q_v^s}}$$

Let

$$\Gamma_{\mathbb{R}}(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big) \qquad\qquad \Gamma_{\mathbb{C}}(s) \;=\; (2\pi)^{-s}\,\Gamma(s)$$

and let r_1,r_2 be the number or real and complex places. Hecke found that the functional equation of the Dedekind zeta function $\zeta_k(s)$ involves the *discriminant* D_k of $\mathfrak{o}_k$ over $\mathbb{Z}$, with symmetrical form

$$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\cdot|D_k|^{-\frac{s}{2}}\cdot\zeta_k(s)$$

$$=\;\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\cdot|D_k|^{-\frac{1-s}{2}}\cdot\zeta_k(1-s)$$

The discriminant is

$$D_k \;=\; \mathrm{vol}\,(k\otimes_{\mathbb{Q}}\mathbb{R}/\mathfrak{o})^2 \;=\; |\det\begin{pmatrix}\sigma_1(\alpha_1) & \sigma_2(\alpha_1) & \ldots & \sigma_r(\alpha_1)\\ \vdots & & & \vdots\\ \sigma_1(\alpha_r) & \sigma_2(\alpha_r) & \ldots & \sigma_r(\alpha_r)\end{pmatrix}|^2$$

where σ_j are the topologically distinct imbeddings $k\to\mathbb{C}$.

The factor $|D_k|^{-\frac{s}{2}}$ *is* a product of local contributions, as follows. The absolute value of the discriminant is the ideal-norm of the *absolute different*

$$\mathfrak{d}_{\mathfrak{o}/\mathbb{Z}} \;=\; \{\alpha \in k \;:\; \mathrm{tr}^k_{\mathbb{Q}}(\alpha\mathfrak{o}) \subset \mathbb{Z}\}^{-1} \qquad \text{(fractional ideal inverse)}$$

This is essentially the product of *local* differents

$$\mathfrak{d}_v \;=\; \mathfrak{d}_{\mathfrak{o}_v/\mathbb{Z}_w} \;=\; \{\alpha \in k_v \;:\; \mathrm{tr}^{k_v}_{\mathbb{Q}_w}(\alpha\mathfrak{o}_v) \subset \mathbb{Z}_v\}^{-1}$$

Thus, to have the functional equation, the local factor at ramified v should be

$$\frac{[\mathfrak{o}_v : \mathfrak{d}_{\mathfrak{o}_v/\mathbb{Z}-w}]^{-\frac{s}{2}}}{1 - \dfrac{1}{q_v^s}}$$

However, typically, there is no choice of f or local component f_v to produce this Euler factor as a local zeta integral!

In fact, typically, there is no choice of f such that $\widehat{f} = f$, because, typically, at ramified v there is no $f_v \in \mathscr{S}(k_v)$ with $\widehat{f}_v = f_v$. That is, there is no choice of Schwartz function to make the local zeta functions $Z_v(s, f_v)$ and $Z_v(s, \widehat{f}_v)$ the same.

That is, while the functional equation

$$Z(s, f) \;=\; Z(1-s, \widehat{f})$$

holds, there is simply no choice of f to make the functional equation obviously relate a zeta integral to itself.

However, there are other options. A reasonable choice of $f = \bigotimes_v f_v$ will produce the expected factors at archimedean and unramified finite places, and at ramified finite v will produce

$$Z_v(s, f_v) \;=\; \frac{[\mathfrak{o}_v^* : \mathfrak{o}_v]^{-\frac{1}{2}}}{1 - \dfrac{1}{q^s}} \qquad\qquad Z_v(s, \widehat{f}_v) \;=\; \frac{[\mathfrak{o}_v^* : \mathfrak{o}_v]^{s-\frac{1}{2}}}{1 - \dfrac{1}{q^s}}$$

Thus,

$$Z(s,f) \;=\; |D_k|^{\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s)$$

$$Z(s,\widehat{f}) \;=\; |D_k|^{s-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s)$$

From $Z(s,f) = Z(1-s,\widehat{f})$,

$$|D_k|^{\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{R}}(s)^{r_2}\,\zeta_k(s)$$

$$=\; |D_k|^{(1-s)-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{R}}(1-s)^{r_2} \cdot \zeta_k(1-s)$$

Divide through by $|D_k|^{s/2}$ to obtain the symmetrical form of the functional equation for $\zeta_k(s)$.

Remark: Asymmetry in zeta integrals cannot be avoided, in general. Thus, zeta *functions*, including optimized gamma factors and powers of discriminants, are *not exactly* given by zeta *integrals*. Nevertheless, the zeta integrals *are* inevitably correct at all but finitely-many places.

Functional equation of a theta function

The analogue of the *theta function* appearing in Riemann's and Hecke's classical arguments is

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \qquad\qquad (\text{for } x \in \mathbb{J},\; f \in \mathscr{S}(\mathbb{A}))$$

Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in k} \widehat{f}\Big(\frac{\alpha}{x}\Big) \;=\; \frac{1}{|x|}\,\theta_{\widehat{f}}\Big(\frac{1}{x}\Big)$$

Analytic continuation and functional equation arise from *winding up*, and breaking the integral into two pieces, and applying the functional equation of θ's.

Notation for θ_f with its constant removed:

$$\theta_f^*(x) \;=\; \theta_f(x) \;-\; f(0) \;=\; \sum_{\alpha \in k^\times} f(\alpha x) \qquad\qquad (x \in \mathbb{J},\; f \in \mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula, and break the integral into two pieces:

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \int_{\mathbb{J}/k^\times} \sum_{\alpha \in k^\times} |\alpha x|^s\, f(\alpha x)\, d^\times(\alpha x)$$

$$= \int_{\mathbb{J}/k^\times} |x|^s \sum_{\alpha \in k^\times} f(\alpha x)\, d^\times x \;=\; \int_{\mathbb{J}/k^\times} |x|^s\, \theta_f^*(x)\, d^\times x$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^s\, \theta_f^*(x)\, d^\times x \;+\; \int_{\mathbb{J}^-/k^\times} |x|^s\, \theta_f^*(x)\, d^\times x$$

The integral over $\mathbb{J}^+/k^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/k^\times$ into an integral over $\mathbb{J}^+/k^\times$ plus two elementary terms describing the poles.

Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/k^\times} |x|^s \, \theta_f^*(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/k^\times} |1/x|^s \, \theta_f^*(1/x)\, d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^{-s} \cdot \Big[|x|\theta_{\widehat{f}}(x) - f(0)\Big]\, d^\times x \;=\; \int_{\mathbb{J}^+/k^\times} |x|^{1-s} \, \theta_{\widehat{f}}^*(x)\, d^\times x$$

$$+ \widehat{f}(0) \int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x \;-\; f(0) \int_{\mathbb{J}^+/k^\times} |x|^{-s}\, d^\times x$$

The integral of $\theta_{\widehat{f}}^*$ over $\mathbb{J}^+/k^\times$ is *entire*. The elementary integrals can be evaluated:

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x \;=\; \text{meas}\,(\mathbb{J}^1/k^\times) \cdot \int_1^\infty x^{1-s}\, \frac{dx}{x} \;=\; \frac{|\mathbb{J}^1/k^\times|}{s-1}$$

In this case, the natural measure of $\mathbb{J}^1/k^\times$ is

$$\text{meas}\,(\mathbb{J}^1/k^\times) \;=\; \frac{2^{r_1}\,(2\pi)^{r_2}\,h\,R}{|D_k|^{\frac{1}{2}}\,w}$$

where r_1, r_2 are the numbers of real and complex places, respectively, h is the class number of $\mathfrak{o}$, R is the *regulator*

$$R \;=\; \text{vol}\Big(\{\alpha \in k \otimes_{\mathbb{Q}} \mathbb{R} \;:\; \prod_{v|\infty} |\alpha|_v = 1\}\Big/\mathfrak{o}^\times\Big)$$

D_k is the *discriminant*, and w is the number of roots of unity in k. Thus,

$$Z(s,f) \;=\; \int_{\mathbb{J}^+/k^\times} \Big(|x|^s \sum_{\alpha\in k^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha \in k^\times} \widehat{f}(\alpha x)\Big)\,d^\times x$$

$$+\; \frac{|J^1/k^\times|\cdot \widehat{f}(0)}{s-1} \;-\; \frac{|J^1/k^\times|\cdot f(0)}{s}$$

The integral is entire, so the latter expression gives the *analytic continuation*. Further, there is visible symmetry under $s \longleftrightarrow 1-s$ and $f \longleftrightarrow \widehat{f}$ and so we have the *functional equation*

$$Z(s,f) \;=\; Z(1-s,\widehat{f})$$
